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THE OCTOBER MEETING IN COLLEGE PARK

The five hundred seventeenth meeting of the American Mathematical Society was held at the University of Maryland, College Park, Maryland, on Saturday, October 22, 1955. The meeting was attended by about 175 persons, including 145 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings Professor Paul Olum of Cornell University delivered an address entitled *The fundamental problem of topology for 2-manifolds* at a general session presided over by Professor L. W. Cohen. Sessions for contributed papers were held in the morning and afternoon, presided over by Professors I. N. Herstein, G. S. S. Ludford, and C. O. Oakley.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, that author whose name is followed by "(p)" presented it.

Algebra and Theory of Numbers

1. W. E. Baxter: *Lie simplicity of a special class of associative rings.*

In a forthcoming paper I. N. Herstein proves that if A is a simple ring of characteristic different from 2 and 3, then the only proper Lie ideals of $[A, A]$ are contained in the center of A. In the present paper, the situations when A has characteristic 2 and characteristic 3 are studied and Herstein's results are extended to these cases. In fact it is proved that the proper Lie ideals of $[A, A]$ are contained in the center except for the case $A = 2\times 2$ matrices over a field of characteristic 2. These results are then applied to extend results of Hattori and Iwahori on invariant subrings of simple rings with descending chain conditions. (Received September 2, 1955.)

2*t*. H. E. Campbell: *On the Casimir operator.*

Let A be either an associative, an alternative or a Lie algebra over an arbitrary field F and let $t(R_x)$ be the trace of the right multiplication R_x for x in A. It is proved that if the bilinear form $t(R_xR_y)$ is nondegenerate then the Casimir operator of the mapping $x \rightarrow R_x$ is the identity transformation. If A is associative and $x \rightarrow S_x$ is a representation of A and the bilinear form $t(S_xS_y)$ is nondegenerate, then A is known to be semi-simple and the representation is completely reducible. If A is also simple and none of the irreducible components of the representation are zero then it is proved that the Casimir operator of $x \rightarrow S_x$ is either the identity transformation or zero according as $t(R_xR_y)$ is nondegenerate or not. Hence if A is not simple the Casimir operator can be taken to have the form diag $\{I, 0\}$. Similar results are obtained for the case where A is alternative and the characteristic of F does not divide the dimension of A. For Lie algebras of prime characteristic the Casimir operator is used together with the enveloping associative algebra of the R_x to obtain the Levi theorem for the special case where $AN = 0$ (N the radical of A), and A/N has nondegenerate Killing form. Research sponsored by U. S. Air Force through the Office of Scientific Research. (Received August 31, 1955.)

3*t*. Eckford Cohen: *Simultaneous pairs of linear and quadratic equations in a Galois field.*

Let F be a Galois field of order p^f, p an odd prime, and let $N_s(m, n)$ represent the number of simultaneous solutions in F of the equations $m=\alpha_1x_1^2+\cdots+\alpha_sx_s^2$, $n=\beta_1x_1+\cdots+\beta_sx_s$, where α_i, β_i $(i=1, \cdots, s)$ are nonzero elements of F. In this paper explicit formulas for $N_s(m, n)$ are determined using elementary properties of the generalized Cauchy-Gauss sum. A precise solvability criterion is derived, from which it follows that $s=4$ is the minimum value of s for which $N_s(m, n)>0$ for all odd p. (Received September 6, 1955.)

4. Trevor Evans: *Some remarks on a paper by R. H. Bruck.*

In a recent paper (*Analogues of the ring of rational integers*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 50–58) R. H. Bruck has introduced the concept of right neoring and discussed some properties of these systems. This paper is essentially a commentary on Bruck's paper and some of his results are generalized as follows. The construction of the universal right neoring in Bruck's paper is applied to the free monogenic V-loop in any subvariety V of the variety of loops and a complete analogue of Bruck's Theorem 4.1 is obtained for any one of these subvarieties. This construction yields uncountably many right neorings with an identity which generates the additive loop of the right neoring. Conversely, every right neoring with an identity which generates its additive loop can be obtained from a free monogenic V-loop by this construction. One possible answer is given to the question raised by Bruck concerning the existence of universal right neorings with free additive loop of arbitrary rank. A brief proof is given of the cancellation properties of the monogenic universal right neoring. Finally, the relationship between right neorings and the logarithmetics of Etherington is discussed. (Received September 6, 1955.)

5*t*. Bruno Harris: *Commuting self-adjoint linear transformations.* Preliminary report.

Let V be a finite dimensional vector space over a field of characteristic not 2, $B(x, y)$ a hermitian or skew-symmetric nondegenerate scalar product in V, and A a self-adjoint linear transformation on V. Then: 1. If B is hermitian, every self-adjoint linear transformation commuting with A is a polynomial in A and 1 if and only if the minimum polynomial of A has degree equal to the dimension of V. 2. If B is skew-symmetric, every self-adjoint linear transformation commuting with A is a polynomial in A and 1 if and only if the minimum polynomial of A has degree exactly half the dimension of V. These results are analogues of the theorem that a matrix is nonderogatory if and only if every matrix commuting with it is a polynomial in it. The method is to decompose V into a direct sum of mutually orthogonal nonisotropic subspaces which are either cyclic for A (hermitian case) or a sum of two cyclic subspaces (skew-symmetric case). (Received September 6, 1955.)

6*t*. Bruno Harris: *Centralizers in special Jordan algebras.* Preliminary report.

Let J be a special Jordan algebra, assumed always of characteristic not 2, K a subalgebra. Let $C(K)$ be the set of elements of J commuting with every element of K (in the associative multiplication)—for the algebras considered this is also the set of elements of J that operator-commute with every element of K. For $x\in J$, let

(x) be the subalgebra of J generated by x and the identity element (if it exists). Then if J is special and finite dimensional central simple, $C(C((\chi))) = (\chi)$ for all $\chi \in J$. $C(C((x))) = (x)$ also holds when: 1. A is an involutorial simple associative ring with minimum condition, J is the Jordan algebra (over the subfield of self-adjoint elements of the center) of self-adjoint elements of A, and x is nilpotent. 2. V is a vector space of countable dimension over a field, $B(x, y)$ is a nondegenerate hermitian or alternate scalar product in V, J is the Jordan algebra of self-adjoint linear transformations, and χ is an algebraic linear transformation. The proofs are based on the preceding abstract. (Received September 6, 1955.)

7. Juris Hartmanis: *Two embedding theorems for finite lattices.*

One of the important unsolved problems of lattice theory is the problem of embedding every finite lattice in a finite partition lattice. If a geometry on a set S is defined as a collection of subsets (called lines), such that any two distinct elements of S are contained in one and only one subset and every subset contains at least two distinct elements, then the partition lattice can be considered as the lattice of subspaces of such a geometry. The author shows that every finite lattice can be embedded in the lattice of subspaces of a finite geometry. Using this result it is shown that every finite lattice can be embedded in the lattice of all geometries on a finite set. The lattice of all geometries on a set is shown to be again a lattice of subspaces of a geometry and its structure is similar to that of a partition lattice. This reduces the above mentioned problem to the problem of embedding every finite lattice of geometries in a finite partition lattice. (Received August 29, 1955.)

8t. D. R. Hughes: *Partial difference sets.*

Let Π be a finite projective plane and G a nontrivial group of collineations of Π. Let Π_0 be the set of points and lines of Π which are fixed by every element of G. A point of Π which is not in Π_0 but is on a line of Π_0 is called a tangent point; a point which is not a tangent point and is not in Π_0 is called an ordinary point. Dually, tangent line and ordinary line are defined. Suppose there are some ordinary points and lines, and G is transitive and regular on both ordinary points and ordinary lines. Then Π_0 must be one of five types of degenerate sub-planes of Π, none of which contains more than four points or four lines. Examples of each type exist; for three of the types, Π can be either Desarguesian or non-Desarguesian; for one type, the only known finite examples are Desarguesian, and for the other type there is a Desarguesian example of order four, while any larger examples (if they exist) must be non-Desarguesian of order 2^k+1. For each type, the existence of the plane is equivalent to the existence of a "partial difference set" D for the group G: every element of G excepting the elements in certain distinguished subgroups is represented in a unique way as $d_1d_2^{-1}$ and as $d_3^{-1}d_4$, where the d_i are in D. (Received August 29, 1955.)

9. W. E. Jenner: *On the class number of nonmaximal orders in p-adic division algebras.*

Let D be a division algebra of finite dimension over a p-adic field and let O be an arbitrary order of D (cf. Compositio Math. vol. 11 (1953) pp. 187–203). Two left (right) ideals a and b of O are said to be equivalent if $a = b \cdot \lambda$ $(a = \lambda \cdot b)$ for some $\lambda \in D$. The number of equivalence classes of left ideals and of right ideals is the same and is finite. This is applied to show that there exist only a finite number of unimodularly inequivalent integral p-adic representations of given degree of a finite group, a result which follows also from a theorem of Maranda (Canadian Journal of Mathematics vol. 5 (1953) pp. 344–355). (Received September 7, 1955.)

10*t*. M. D. Marcus and J. M. McGregor: *Extremal properties of hermitian matrices.*

Let A be a non-negative hermitian (n.n.h) matrix with eigenvalues $\lambda(1) \leq \cdots \leq \lambda(n)$. For $r \leq k \leq n$ and $x(1), \cdots, x(k)$ a set of orthonormal (o. n) vectors let $\Delta(r, k)$ and $E(r, k)$ denote the rth elementary symmetric functions (e.s.f) of the $\lambda(i)$ and $(Ax(i), x(i))$, $i=1, \cdots, k$, respectively. Theorem: $\min E(r, k) = \Delta(r, k)$, $\max E(r, k) = C_{k,r}(\sum_{j=1}^{k} \lambda(n-j+1)/k)^r$ *where both* min *and* max *are taken over all sets of k* o.n. *vectors* $x(1), \cdots, x(k)$. This extreme value result is applied to obtain inequalities for the e.s.f of the eigenvalues of a sum and product of n.n.h matrices. Results on the distribution of the diagonal elements of A with respect to the $\lambda(j)$ as well as a lower bound on the Euclidean norm of a product of arbitrary matrices in terms of the norms and singular values of the factors are also obtained. (Received August 3, 1955.)

11. Morris Newman: *The coefficients of modular forms.*

If n is a non-negative integer, define $p_r(n)$ as the coefficient of x^n in $\prod_{n=1}^{\infty} (1-x^n)^r$; otherwise define $p_r(n)$ as zero. The following theorem is proved using the elliptic modular functions: Let r be an integer, and p a prime >3 such that $r(p+1) \equiv 0 \pmod{24}$. Put $\Delta = r(p^2-1)/24$. Then the identity $p_r(np+\Delta) = (-p)^{r/2-1} p_r(n/p)$ is valid if and only if $r=2, 4, 6, 8, 10, 14, 26$. This identity furnishes a simple, uniform proof of the three Ramanujan congruences for partitions modulo 5, 7, 11. Identities to prime power moduli of the same type are also obtained. In connection with this work tables of $p_r(n)$ were computed on SEAC, the High Speed Electronic Digital Computer of the National Bureau of Standards at Washington, D. C. (Received September 29, 1955.)

12. M. H. Pearl: *On Cayley's parameterization.* I.

Let A be a complex Hermitian (real symmetric) matrix. If Q is a skew-Hermitian (real skew-symmetric) matrix such that $A+Q$ is nonsingular, then $P=(A+Q)^{-1} \cdot (A-Q)$ is a cogredient automorph of A such that $I+P$ and A span the same row spaces (and $d(P) = +1$). Conversely, if P is a cogredient automorph of A such that A and $I+P$ span the same row spaces (and if $d(P) = +1$), then there exists a skew-Hermitian (real skew-symmetric) matrix Q such that $A+Q$ is nonsingular and $P=(A+Q)^{-1}(A-Q)$. This result extends the Cayley parameterization to a class of singular matrices. It is also shown that the theorem as stated above does not hold for matrices over a field of characteristic 2. (Received September 12, 1955.)

13. J. J. Price: *Some duality theorems.* Preliminary report.

Let $S=(X, F, m)$ be a measure space and R a countable family of matrix-valued functions defined on S. Sufficient conditions are found in order that R be essentially a complete system of irreducible representations of a compact group G whose Haar measure is essentially m. This result generalizes some known theorems. When the functions in R are one-dimensional, G turns out to be abelian. The result is then equivalent to a theorem of Fine (Pacific Journal of Mathematics vol. 5, pp. 51–59). When S is taken to be a compact group and R a complete system of irreducible representations of S, the Tannaka duality theorem is obtained. Several analogous results for finite groups are also found. (Received April 26, 1955.)

14t. Irving Reiner: *Integral Maschke representations.*

Let A be an algebra over the ring $\mathfrak{o}$ of integers in an algebraic number field k. For a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, let $\bar{k} = \mathfrak{o}/\mathfrak{p}$ be the residue class field, and set $\bar{A} = A/\mathfrak{p}A$. Following Gaschütz (Math. Zeit. vol. 56 (1952) pp. 376–387) and Ikeda (Osaka Math. J. vol. 5 (1953) pp. 53–58), define a Maschke representation of an algebra (over a ring) as a representation which can be split off as direct component from any representation of which it is a constituent. It is shown here that if Γ is a representation of A in $\mathfrak{o}$, and if for each $\mathfrak{p}$ the induced representation $\bar{\Gamma}$ of $\bar{A}$ in $\bar{k}$ is a Maschke representation, then Γ is also a Maschke representation. This reduces the problem of determining whether Γ is a Maschke representation over the ring $\mathfrak{o}$, to that of deciding (for those $\mathfrak{p}$ for which $\bar{A}$ is not semi-simple) whether $\bar{\Gamma}$ is a Maschke representation over the field $\bar{k}$. The converse of the above result is also proved, for the case where A is a Frobenius algebra. (Received September 8, 1955.)

ANALYSIS

15t. F. F. Bonsall: *Extreme maximal ideals of a partially ordered vector space.*

An *extreme* maximal ideal of a partially ordered vector space V with an order unit e is one that corresponds to an extreme point of the set of normalized positive linear functionals. The importance of extreme maximal ideals in representation theory has been demonstrated by Kadison [*A representation theory for commutative topological algebra*, Memoirs of the American Mathematical Society, no. 7, 1951]. An ideal J of V is said to be *perfect* if, whenever $x \in J$ and $\epsilon > 0$, there exists an element $j_\epsilon \in J$ with $-(j_\epsilon + \epsilon e) \leq x \leq j_\epsilon + \epsilon e$. It is proved that a maximal ideal is extreme if and only if it is perfect. The set of all perfect ideals behaves, in some ways, like the set of all ideals. In particular, if the only proper perfect ideal of V is (0), then V is one-dimensional. This implies that each proper perfect ideal is contained in a perfect maximal ideal. (Received August 1, 1955.)

16. F. E. Browder: *Smoothing processes in the theory of elliptic and hyperbolic differential equations.*

If μ is a bounded measurable function on E^n, $k_\epsilon(x) = (2\epsilon)^{-n} \prod_{i=1}^{n} \exp\{-|\epsilon^{-1}x_i|\}$, let $K_\epsilon(u) = k * u$. Using various special properties of this smoothing operator, one obtains strengthened and simplified forms of various proofs in the general theory of partial differential equations. A proof is obtained that a weak solution of a strongly elliptic system of equations is a strict solution using only L^2 energy arguments without assuming that u is $(m-1)$-times strongly-differentiable, where $2m$ is the order of the system. A simplified form is given of Leray's proof for the existence and properties of solutions of the Cauchy problem for hyperbolic equations. (Received September 8, 1955.)

17. R. B. Davis: *Asymptotic solutions of the first boundary value problem for a fourth-order elliptic partial differential equation.*

Let $L_\epsilon(u_\epsilon) = (\epsilon^2\Delta + a)(\Delta + b)u_\epsilon = 0$, where a and b are negative constants. Prescribe boundary values for u_ϵ and $\partial u_\epsilon/\partial n$. Define u_0 in the identical fashion, but relaxing the prescribed value for the normal derivative, and setting $\epsilon = 0$ in the differential equation. Then $u_\epsilon = u_0 + \tilde{\Psi} + \Omega$, where $\tilde{\Psi}$ is a boundary layer term for which an explicit expression is given, and Ω and its first derivatives are $O(\epsilon)$ in the sense of a Hilbert space norm. The method follows Levinson. Use of the integral norm follows a suggestion of Peter Lax and Paul Berg. (Received June 13, 1955.)

18. T. P. Higgins and W. T. Guy, Jr. (p): *On the Hankel transform of higher derivatives.*

A formula is developed which expresses the Hankel transform of the nth derivative of a function in terms of a finite series of Hankel transforms of the function. (Received September 8, 1955.)

19t. Eugene Leimanis: *On a theorem of Poincaré and Malkin.*

The theorem in question is concerned with the existence of periodic solutions of a nonautonomous dynamical system of order n, containing a small parameter μ and being such that the generating system admits an infinity of periodic solutions depending upon $1 \leq k < n$ parameters $h_1, h_2, \cdots, h_k$ [H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, vol. 1, p. 84, Paris, Gauthier-Villars, 1892, and I. G. Malkin, *The methods of Lyapunov and Poincaré in the theory of nonlinear oscillations*, p. 17, Gos. Teh. Teor. Lit., Leningrad-Moscow, 1949 (Russian)]. In this paper an exceptional case, when certain equations $P_j(h_1, h_2, \cdots, h_k) = 0$, $j = 1, 2, \cdots, k$, (derived from the necessary conditions of periodicity) are satisfied identically with respect to the parameters, is considered and a generalization of the above mentioned basic theorem is obtained. This exceptional case is of importance in some applications to the theory of nonlinear oscillations. (Received August 1, 1955.)

20t. Robert McKelvey: *Solution about a regular singular point of a second order linear ordinary differential equation involving a large parameter.*

This is an investigation of the differential equation $u'' - [\lambda^2\phi^2(z) + \lambda z^{-1}\psi(z, \lambda) + z^{-2}\tau]u = 0$, $|\lambda| > N$, in a bounded domain R about the origin of z, under the assumption that $\phi(z)$ does not vanish in R. Both z and λ are complex, $\phi(z)$ and $\psi(z, \lambda)$ are analytic, and τ is constant. $\phi(0)$ is adjusted to 1. Formal solutions $u = A(z, \lambda)W_{k(\lambda),m}(\xi) + B(z, \lambda)W_{1+k(\lambda),m}(\xi)$ are obtained, where $W(\xi)$ is the Whittaker function, ξ denotes $2\lambda\int\phi(z)dz$, and $A(z, \lambda)$, $B(z, \lambda)$, and $k(\lambda)$ are formal series $\sum_{j=0}^{\infty} \alpha_j(z)/\lambda^j$, $\sum_{j=0}^{\infty} \beta_j(z)/\lambda^j$, and $\sum_{j=0}^{\infty} k_j/\lambda^j$. It is shown that, for specific ranges of ξ, the expressions obtained from this formal one by replacing $A(z, \lambda)$, $B(z, \lambda)$, $K(\lambda)$ by finite segments are asymptotic approximations to true solutions. (Received August 17, 1955.)

21. W. R. Mann (p), C. L. Bradshaw, and J. G. Cox: *Improved approximations to differential equations using low order differences.*

The purpose of this paper is to present a method for approximating differential equations with difference equations more closely than is possible with the usual finite difference approaches, but without introducing higher order differences. Though not applicable to all differential equations, it can be used very effectively on some which are of great practical interest. This paper is largely taken up with an illustration of the usefulness of the method in the solution of a two-region, two-dimensional reactor problem which is simple enough to be solved analytically. Numerical data, obtained on the ORACLE of the Oak Ridge National Laboratory, enable a comparison of the analytical solution for the buckling with values obtained by the ordinary difference methods and those obtained by the refined process presented in this paper. The comparison shows that the refined process gives considerably greater accuracy than the ordinary difference methods, and the cost in increased labor is comparatively small. (Received August 31, 1955.)

22. Lawrence Markus (p) and R. A. Moore: *Disconjugate solutions of linear differential equations with almost periodic coefficients.*

One considers the differential equation (L) $y''+(-a+bp(x))y=0$, where a, b are real parameters and $p(x)$ is a real almost periodic function on $-\infty<x<\infty$. Either every solution of (L) is oscillatory or else every (nontrivial) solution is disconjugate (has at most one zero). The set of pairs (a, b) for which (L) is disconjugate is a closed convex set D in the (a, b)-plane. Interior to D each equation has a solution basis of the form $y_i(x)=\exp\left[(-1)^i\alpha x+\int_0^x\phi_i(s)ds\right]$ where the $\phi_i(x)$ are almost periodic functions of mean zero. The characteristic exponent α is continuous in the interior of D and when defined by lim $\sup_{x\to\infty}(1/x)\log|y(x)|=\alpha$, it is zero on the boundary of D. The domain D is the same for every function $p^*(x)$ in the closed hull of $p(x)$. For some such $p^*(x)$, the resulting equation (L*) on the boundary of D has a positive bounded solution. The domain D varies with the choice of the almost periodic function $p(x)$ in a continuous manner. (Received September 7, 1955.)

23. O. G. Owens: *Polynomial solutions of the characteristic value problem for the cylindrical wave equation.*

There is a known integral representation for that solution $u(x, y, t)$ (existence assumed) of the cylindrical wave equation, $u_{tt}=u_{xx}+u_{yy}$, which assumes the given boundary values $u(x, y, (x^2+y^2)^{1/2})=\psi(x, y)$ on the characteristic cone $t^2=x^2+y^2$ $(t>0)$. Regarding this formula, the following facts must be established: the partial derivatives exist, are bounded on the characteristic cone, have explicit integral representations, and satisfy the wave equation. A direct deduction from the formula of these facts seems to involve substantial difficulties. However, these complexities are avoidable by means of an indirect analysis based upon the principal theorem of the paper. The theorem asserts that a polynomial solution $u(x, y, t)$ of the characteristic value problem exists whenever the prescribed boundary function $\psi(x, y)$ is a polynomial. (Received September 9, 1955.)

24. G. O. Peters: *Boole polynomials of higher and negative orders.*

Charles Jordan in his book *Calculus of finite differences* (Budapest, 1939) defines Boole polynomials of order one. The author uses different definitions for the Boole polynomials and Boole numbers, definitions consistent with those for the Bernoulli and Euler polynomials and numbers found in Nörlund's *Vorlesungen über Differenzenrechnung* (Berlin, 1924). The author defines the Boole polynomials of degree ν and order n, as the polynomial $\zeta_\nu^{(n)}(x)$ satisfying the equation $\nabla^n\zeta_\nu^{(n)}(x)=x^{(\nu)}$, where $x^{(\nu)}=x(x-1)(x-2)\cdots(x-\nu+1)$ and ∇ is the mean operator, $\nabla=1+\Delta/2$, $\nabla f(x)=[f(x+1)+f(x)]/2$. The Boole polynomial of negative order is defined as the polynomial $\zeta_\nu^{(-n)}(x)$, of degree ν, and order $-n$, satisfying the equation $\zeta_\nu^{(-n)}(x)=\nabla^n x^{(\nu)}$. The Boole numbers are then defined by the equation $\zeta_\nu^{(n)}(x)=\sum_{i=0}^{\nu}{}_\nu C_i(\zeta_i^{(n)}/2^i)x^{(\nu-i)}=(\zeta^{(n)}/2+x)^{(\nu)}$, where n can be either positive or negative. Some properties of these Appell polynomials are found and the values of the Boole numbers are determined. The symmetry of the polynomials is also found. An interesting polynomial relation is found relating the Euler, Boole, and Bernoulli polynomials of the first kind. (Received October 7, 1955.)

25. P. P. Saworotnow: *Involutions on H^*-algebra.* Preliminary report.

Let $\mathfrak{A}$ be a simple proper H^*-algebra and let $x \to x^\sigma$ be any involution such that $x^\sigma = 0$ implies $x = 0$ and such that $(x^\sigma, x^*) \geqq 0$ for every $x \in \mathfrak{A}$ where x^* is the adjoint of x (i.e., such that $(xy, z) = (y, x^*z)$ and $(yx, z) = (y, zx^*)$). It is shown that under the above conditions the following is true: (1) There exists a scalar product $[\,,\,]$ on $\mathfrak{A}$ defining the original topology such that $[xy, z] = [y, x^\sigma z]$ and $[yx, z] = [y, zx^\sigma]$. (2) There exists a scalar product $\{\,,\,\}$ on $\mathfrak{A}$ defining the same topology such that $\{xy, z\} = \{y, x^\sigma z\}$ and $\{yx, z\} = \{y, zx^*\}$. (Received August 22, 1955.)

26. C. T. Taam: *On the solutions of nonlinear differential equations.*

Let $p(t)$ and $q(t)$ be real-valued Lebesgue integrable functions on $[0, a]$ for every $a > 0$. The oscillatory property of the solutions of (A): $u'' + pu + qu^3 = 0$ is investigated by a comparison method. Various results concerning the zeros and amplitudes of the solutions are obtained. These results are applied to study the periodic solutions of (A). The following result, among others, is established: Let p and q be even periodic functions with a common period L and having positive lower and upper bounds; then, for each positive integer n, there exists a periodic oscillatory solution of (A) with period nL. A boundary value problem is also solved. Set $p = A + f$, $q = 2(B + g)$. It is shown that if (i) A and B are positive constants, (ii) f and g are of BV on $[0, \infty)$, (iii) tf and tg are Lebesgue integrable on $[0, \infty)$, then every solution u of (A) is bounded on $[0, \infty)$ and has an asymptotic representation $u = A^{1/2}B^{-1/2}\phi(A^{1/2}t + C) + \epsilon(t)$, where C is a constant, $\epsilon(t) \to 0$ as $t \to \infty$, and $\phi(t)$ is a periodic function defined by $v'' + v + 2v^3 = 0$, $v(0) = 0$, $v'(0) = E$ for some $E \neq 0$. This work was supported by the Office of Ordnance Research. (Received September 6, 1955.)

27. H. F. Weinberger: *An upper bound for the free membrane.*

The eigenvalue problem $\Delta u + \mu u = 0$ on a region R with $\partial u/\partial n = 0$ on the boundary of R is considered. The lowest eigenvalue is always zero, and it is the second eigenvalue μ_2 that is of interest. Szegö [Journal of Rational Mechanics and Analysis vol. 3 (1954) p. 343] has shown by means of conformal mapping that if R is two-dimensional and simply connected, its μ_2 is at most that of a circle of equal area. An alternate proof of this theorem depending only on a real variables argument is given. It is not necessary to assume that R is simply connected. Furthermore, the proof is easily extended to show that if R is an N-dimensional region, its μ_2 is at most equal to the μ_2 of the N-sphere of equal volume. (Received September 9, 1955.)

APPLIED MATHEMATICS

28. G. E. Forsythe (p) and Louise W. Straus: *The Souriau-Frame characteristic equation algorithm on a digital computer.*

A is a regular square matrix of order n. Let $F(\lambda) = |A - \lambda I| = (-1)^n(\lambda^n - c_1\lambda^{n-1} - \cdots - c_n)$. Let λ_1, λ_n be the least, greatest in modulus of the zeros of F. Let $P = |\lambda_n|/|\lambda_1|$. In the Souriau-Frame algorithm to get $F(\lambda)$ [see P. Dwyer, *Linear computations*, Wiley, 1951] one computes $A_1 = A$, $c_1 = \text{trace } A$, $B_1 = A_1 - c_1 I, \cdots, A_k = AB_{k-1}$, $c_k = \text{trace } A_k/k$, $B_k = A_k - c_k I, \cdots$. Theoretically $c_k = 0$ $(k \geqq n+1)$. Let a machine word length be s. The authors have prepared a single-precision SWAC code $(s = 36)$ for the algorithm, using scaled vectors. Tests with several matrices A $(n \leqq 10)$ confirm for $s = 36$ the heuristic argument that the round-off error Δc_k in c_k

behaves roughly thus: (*) $|\Delta c_k| \doteq 2^{-s+k}|\lambda_n|^k$. When (*) holds, it is shown that b bits of relative accuracy may be expected in c_n if and only if $n \log_2 P \leqq 2s-2b+10$, a rather severe restriction. A paper will appear in J. Math. Physics. (Received August 3, 1955.)

29. Patrick Leehey: *The Hilbert problem for an airfoil in unsteady motion.*

Birnbaum's linearization of the equations of motion for unsteady incompressible flow about a thin airfoil in arbitrary accelerated motion leads to a particular Hilbert problem with the complex perturbation velocity Φ representing a sectionally holomorphic function which vanishes at infinity and satisfies a condition $\Phi^+ = G\Phi^- + g$ on the real axis. The functions G and g have discontinuities at points corresponding to the leading and trailing edges of the airfoil and to the end point of its wake. Following Keldysch and Sedov (cf. Muskhelishvili, *Singular integral equations*, Moscow, 1946) an expression for Φ is determined under the condition that Φ remain bounded in a neighborhood of the trailing edge. This, in turn, yields the well-known singular integral equation of unsteady lifting theory and its inversion. The "continuity" condition of Söhngen (Luftfahrtforschung vol. 17 (1940) p. 402) is shown to be a consequence of the boundedness condition. With the aid of certain results concerning the interchange of integration processes for contours and arcs, Kelvin's circulation theorem is employed to relate the wake vortex distribution to prescribed vertical motions of the airfoil in the form of Wagner's integral equation. (Received September 2, 1955.)

30. H. D. Mills: *Sensitivity of linear programs.*

Let $A = \|a_{pq}\|$ and $H = \|h_{pq}\|$ be real $m+1$ by $n+1$ matrices ($p = 0, 1, \cdots, m$; $q = 0, 1, \cdots, n$). Let $V(A)$ be the maximum (value) of the linear function $a_{00} + \sum x_i a_{i0}$ constrained by the $m+n$ linear inequalities $a_{0j} + \sum x_i a_{ij} \geqq 0$, $x_i \geqq 0$ ($i = 1, \cdots, m$; $j = 1, \cdots, n$). Let A^* be a matrix identical with A except that $a^*_{00} = a_{00} - V(A)$. Then if $V(A+\alpha H)$ exists for $0 \leqq \alpha <$ some α^0, $\lim_{\alpha \to 0+} [V(A+\alpha H) - V(A)]/\alpha = V\left(\left\|\begin{matrix} H & A^* \\ A^* & 0 \end{matrix}\right\|\right)$. That is, the one-sided directional derivative of the value of the $m+1$ by $n+1$ program A with respect to H is the value of a $2m+2$ by $2n+2$ program constructed from A^* and H as indicated. (Received September 9, 1955.)

31*t*. L. E. Payne: *On certain multi-valued functions in generalised axially symmetric potential theory.*

To each solution ϕ_p of the equation of generalized axially symmetric potential theory (1) $\partial^2\phi_p/\partial x^2 + \partial^2\phi_p/\partial y^2 + (p/y)\partial\phi_p/\partial y = 0$ (p any positive real number) there exists a conjugate or stream function ψ_p defined up to an additive constant by the relations (2) $y^p \partial\phi_p/\partial x = \partial\psi_p/\partial y$, $y^p \partial\phi_p/\partial y = -\partial\psi_p/\partial x$. By introducing toroidal coordinates one can represent certain multi-valued conjugate functions in forms which clearly exhibit both their cyclic nature and analytic character. In particular the expressions obtained for the function conjugate to the Neumann's function for the half plane and that for the stream function for a source disk are analytically more transparent than the previously known Bessel function representations (A. Weinstein, Bull. Amer. Math. Soc. vol. 59 (1953)). (Received September 9, 1955.)

32*t*. Raymond Sedney: *Some properties of the three-dimensional boundary layer equations.*

The equations for laminar flow in the boundary layer over a general surface in three dimensions are analyzed in a normal coordinate system. These are not tensor equations, but they do have the invariance properties of subtensors. It is found that the flow is essentially two-dimensional if the external flow streamlines are geodesics on the surface. Physically this is equivalent to a criterion for no "secondary flow." (Received September 12, 1955.)

33. Domina E. Spencer: *Electrodynamic field theory.*

Previous papers (for example, Moon and Spencer, J. Franklin Inst. vol. 257 (1954) pp. 369–382) have presented an electrodynamic equation for the force between two moving charges. This single equation is relativistic in the Galilean sense and may be employed to replace the four Maxwell equations. The present paper shows how the new electrodynamics can be expressed as a *field*. Equations are developed for scalar and vector potentials, and the new field theory is compared with the electromagnetic theory of Maxwell. (Received September 7, 1955.)

GEOMETRY

34. Fred Supnick: *Optimal closed paths through certain point sets.*

It is shown in Theorem I that if a finite set of points in E_r can be ordered $p_1, \cdots, p_n$ so that for every set of integers a, b, c, d for which $1 \leqq a < b < c < d \leqq n$ ("$d(\ )$" indicates "distance") $d(p_ap_b)+d(p_cp_d) \leqq d(p_ap_c)+d(p_bp_d) \leqq d(p_ap_d)+d(p_bp_c)$, then the shortest and longest (closed, rectilinear) polygons having these and only these points as vertices are $(\cdots p_7p_5p_3p_1p_2p_4p_6 \cdots)$ and $(\cdots p_{n-5}p_5p_{n-3}p_3p_{n-1}p_1 \cdot p_np_2p_{n-2}p_4p_{n-4}p_6 \cdots)$ respectively (edges are the "joins" of adjacent vertices, first and last being considered adjacent). Realizations of the above criterion are constructed. Thus e.g. let n rays $r_1, \cdots, r_n$ emanating from a point 0 be such that the angles $r_ir_j (<180°)$ are all equal for $i \neq j$. Let a point q_i be selected on r_i $(i=1, \cdots, n)$. Let $p_1, \cdots, p_n$ be that rearrangement of $q_1, \cdots, q_n$ so that $d(0p_1) \leqq d(0p_2) \leqq \cdots \leqq d(0p_n)$, then $p_1, \cdots, p_n$ satisfies the conditions of the above theorem. It is shown however that not every finite point set in E_r satisfies the conditions of Theorem I. In Theorem II conditions less stringent than those of Theorem I are established, but applicable only for finding the shortest polygon. This permits a wider class of realizations for the minimal case than does Theorem I. The same is done for the maximal case in Theorem III. (Received September 9, 1955.)

TOPOLOGY

35. Eldon Dyer: *Some examples of quasi-complexes.*

A *chainable* continuum is a compact (f.o.c.) Hausdorff space having a cofinal family Ω of finite open coverings such that the elements of each member of Ω can be linearly ordered in such a way that two of them intersect if and only if they are adjacent in the ordering. The metric chainable continua have been called *snake-like* by R. H. Bing and include arcs, pseudo-arcs, the closure of the graph of $y=\sin(1/x)$, $0<x\leqq 1$, and certain other continua. Any linearly ordered, connected, compact Hausdorff space is also a chainable continuum. In this note it is shown that the cartesian product of the elements of any finite set of chainable continua is a quasi-complex. The term quasi-complex has been defined by S. Lefschetz in *Algebraic topology*, vol. 27 of the Colloquium Publications of the American Mathematical Society. Lefschetz also shows in the same book that each zero-cyclic quasi-complex has the fixed point

property. Thus the cartesian product of the elements of any collection of chainable continua has the fixed point property. This generalizes a result announced earlier by the author. (Received September 7, 1955.)

36. E. E. Floyd: *Examples of fixed point sets of periodic maps.* II.

Y will denote a space and T a periodic map on Y of period 6, with fixed point set F. An example is given in which Y is a contractible finite complex, and F is not connected. The author also gives an example in which Y is a compact absolute retract, and F is not locally connected. Other examples are given, all showing the eccentricities of fixed point sets of periodic maps of composite period. (Received September 6, 1955.)

37*t*. L. F. McAauley: *A note on naturally ordered sets in semi-metric spaces.*

G. T. Whyburn and Zarankiewicz have proved that if G is any naturally ordered collection of disjoint sets in a separable metric space, then at most a countable number of elements g of G can contain a point which is not a condensation point both of the collection P_g of all predecessors of g in G and the collection F_g of all successors of g in G. This note shows that the analogue of such a lemma holds in a hereditarily separable semi-metric topological space S; furthermore, a theorem due to G. T. Whyburn [*Analytic topology*, p. 45 (2.2)] related to this lemma is also true in S. Thus, certain theorems which depend upon these fundamental results of Whyburn and Zarankiewicz may be established for spaces which are more general than separable metric spaces. (Received September 9, 1955.)

38*t*. S. K. B. Stein: *Local connectedness and closed continuous maps.*

Let X and Y be a topological space (satisfying the usual 3 axiom on open sets), and f a continuous map from X onto Y. Theorem: If X is locally-connected (l.c.) and f is closed, then Y is l.c. The proof consists of showing first that Y is weakly locally connected (w.l.c.) and then employing the classical result that w.l.c. implies l.c. This enables one to state the Hahn-Mazurkiewicz theorem without any restriction (such as Hausdorff, or metric) on the space Y, but instead with the (usually superfluous) restriction that f be closed. Specifically, a necessary and sufficient condition that a space Y be the image of the closed unit interval, I, under a closed continuous function is that Y be t_1, t_4, connected, l.c., compact, and have a countable base (t_4 does not assume t_2). This theorem suggests defining an equivalence relation r between spaces: XrY iff these are continuous closed onto maps $f\colon X\rightarrow Y$, $g\colon Y\rightarrow X$. Employing the H.-M. theorem, one shows easily that the spaces equivalent to I are precisely the Peano spaces (i.e. spaces satisfying the above six conditions) containing at least two points. Also, two trianguable spaces are equivalent iff they have the same number of components and the same number of isolated points. For finite spaces the relation r reduces to "homeomorphic." (Received August 8, 1955.)

39. G. R. Strohl, Jr.: *Strongly cyclic Peano spaces.*

A cyclic Peano space M is defined to be strongly cyclic at a point p if, for every infinite sequence of points of M converging to p, there is in M a simple closed curve containing infinitely many points of the sequence. M is strongly cyclic if it is strongly cyclic at each of its points. This extends the concept of strongly arcwise connected spaces as studied by Hall and Puckett, Amer. J. Math. vol. 63 (1941) pp. 554–562.

Results obtained are as follows: (1) a necessary and sufficient condition that M be strongly cyclic is that for every infinite sequence of open sets in M, there is in M a simple closed curve intersecting infinitely many of these sets; (2) if M is strongly arcwise connected but not strongly cyclic at a point p, there is in M a closed set D and a separation $M-D=(\bigcup K_i)\bigcup N$ where the K_i form an infinite collection of distinct components of $M-D$ converging to p, and $F(K_i)$ consists of exactly two points of D. As a corollary, if no two points cut M, then M is strongly cyclic. (Received August 9, 1955.)

R. D. SCHAFER,
Associate Secretary

THE NOVEMBER MEETING IN LOS ANGELES

The five hundred eighteenth meeting of the American Mathematical Society was held at the University of Southern California, Los Angeles, California, on Saturday, November 12, 1955. Attendance was approximately one hundred, including 88 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Paul Garabedian delivered an address on *The mathematical theory of three-dimensional cavities and jets*. He was introduced by Professor J. W. Green. Professors R. S. Phillips and L. J. Paige presided at the sessions for contributed papers.

Following are the abstracts of papers presented at the meeting, those whose numbers are followed by "*t*" having been given by title. Mr. Steger was introduced by Professor A. V. Martin, Dr. Nitsche by Professor Stefan Bergman, and Dr. Banaschewski by Professor D. B. Summer.

ALGEBRA AND THEORY OF NUMBERS

40. D. L. Boyer: *Enumeration theorems in infinite Abelian groups.*

W. R. Scott [Amer. J. Math. vol. 74 (1952) pp. 187–197] has proved that an Abelian group of order $A > \aleph_0$ has 2^A subgroups of order A and the intersection of all the subgroups is the identity. He has proved in the same paper that the intersection of all the infinite subgroups of a countable Abelian group G is the identity unless $G = Z(p^\infty) \oplus F$, where F is finite. The present paper extends the remaining parts of the above theorem to include countable Abelian groups. It is also pointed out that the above theorem is valid for modules over a principal ideal ring provided the order of the ring is less than the order of the module. Finally it is shown that the order of the automorphism group of a countable torsion Abelian group is $2^{\aleph_0}$. (Received September 26, 1955.)

41. D. H. Lehmer: *On the roots of the Riemann zeta function.*

Calculations are described which show that the first 10,000 roots of the Riemann zeta function in the upper half plane have real parts of 1/2. The calculations were done on the SWAC in a few hours time. The basic computation consisted in evaluating a real multiple $[\exp(\pi i\vartheta)]\zeta(s)$ of the zeta function at the first 10,000 Gram points. Later steps were taken to dispose of doubtful situations by a refined mesh. Use was made of a suitably modified formula of Titchmarsh. Finally two near misses of the Riemann Hypothesis had to be examined by the Euler-Maclaurin formula using 2000 terms of $\sum n^{-s}$ (35 seconds) to obtain a small error. Two pairs of nearly coincident roots with serial numbers 4763, 4764, 6707, 6708 were computed. The latter pair have imaginary parts of 7005.0629 and 7005.1006. The zeta function exhibits a disregard for Gram's Law which increases with t. Over 800 violations were found. In five cases there are three roots between two consecutive Gram points. (Received October 6, 1955.)

42. T. S. Motzkin: *Simple and equi-element finite semigroups.* Preliminary report.

A finite semigroup is equi-element (i.e., has a transitive automorphism group) only if it is the direct product of a factor in which $ab=a$, for any elements a and b, and a factor in which $cd=d$ for all c and d. Simple semigroups (with no nontrivial homomorphism) of order n, besides the simple groups, exist certainly for $n=rs+1$, $r\leqq s<2^r$; the number of nonisomorphic simple semigroups of a given order is unbounded. For $n=3, 4, 5$ there are no other simple semigroups, as the list mentioned in the following abstract shows. (Received November 7, 1955.)

43. T. S. Motzkin and J. L. Selfridge (p): *Semigroups of order five.*

After theoretical exclusion of impossible first rows, the lexicographically arranged multiplication tables of all nonisomorphic semigroups of order 5 and (relying on the first result of the preceding abstract) their automorphism groups have been listed, using the electronic computer SWAC. There exist 183732 semigroups of order 5, of which 1915 are nonisomorphic. Of these, 325 are commutative and 80 others are selfdual (antiisomorphic to themselves). If antiisomorphic semigroups are identified the number of types is thus 1160. (Received November 7, 1955.)

44t. R. S. Pierce: *Distributivity in Boolean algebras.* Preliminary report.

Let α and β be infinite cardinals. An α-complete Boolean algebra B is called (α, β)-distributive if the identity: (*) $\prod_{\sigma\in S}(\sum_{\tau\in T} a_{\sigma\tau})=\sum_{\phi\in F}(\prod_{\sigma\in S} a_{\sigma\phi(\sigma)})$, where $F=T^S$, holds in B whenever $\overline{\overline{S}}\leqq\alpha$ and $\overline{\overline{T}}\leqq\beta$. B is strongly (α, β)-distributive if the above equality prevails and $\prod_{\sigma\in S} a_{\sigma\phi(\sigma)}=0$ for all $\phi\in F$ except a subset of cardinality $\leqq\alpha$. Finally, B is weakly (α, β)-distributive if (*) holds in the lattice of dual ideals of B, whenever the $a_{\sigma\tau}$ are principal dual ideals and $\sum_{\tau\in T} a_{\sigma\tau}$ is the unit ideal for all $\sigma\in S$. Sample results are: B is weakly (α, β)-distributive for all α iff B is (isomorphic to) a β-field of sets; B is weakly (α, α)-distributive iff there is an α-field F, an α-homomorphism $h:F\rightarrow B$ and a homomorphism $i:B\rightarrow F$ such that $h\circ i$ is the identity map of B; B is (α, α)-distributive iff B is isomorphic to F/I, where F is an α-field and I is an α-ideal closed under certain 2^α-joins (if F is 2^α-complete, I must be a 2^α-ideal); B is strongly (α, α)-distributive iff B is the limit of an α-directed system of algebras isomorphic to 2^X, where $\overline{\overline{X}}\leqq\alpha$; B is $(\aleph_0, \aleph_1)$-distributive iff every real-valued, continuous function on the Boolean space of B is locally constant on a dense subset of the space. (Received October 4, 1955.)

45t. R. S. Pierce: *Products of Boolean algebras.* Preliminary report.

Let α be an infinite cardinal. An algebraic definition is given for the α-direct product $(\alpha)\prod_{\sigma\in S} B_\sigma$ of a set of Boolean algebras B. It is shown that the definition is equivalent to one given by Sikorski (Fund. Math. vol. 39 (1952)) in the case $\alpha=\aleph_0$. For each $\sigma\in S$, there is an α-isomorphism $i_\sigma:B_\sigma\rightarrow(\alpha)\prod_{\sigma\in S} B$. Main theorem: If α is a regular cardinal and $h_\sigma/B_\sigma\rightarrow B$ is an α-homomorphism, all B_σ are strongly (α, α)-distributive and B is weakly (α, α)-distributive, then there is a unique α-homomorphism $h:(\alpha)\prod_{\sigma\in S} B_\sigma\rightarrow B$ such that $h\circ i_\sigma=h_\sigma$. Corollary: The free weakly (α, α)-distributive Boolean algebra with β generators is the α-product of β replicas of the four element Boolean algebra. (Received October 4, 1955.)

46. Arthur Steger: *Direct decomposition of finitely idempotent rings.*

A. L. Foster showed that if R is a commutative ring with unit and if R^* is the set of idempotents in R then, under a suitable addition and ordinary ring multiplication, R^* is a Boolean ring. Hence, if R^* is finite it contains exactly 2^t elements. If $t=1$, R will be called minimally idempotent. The main result of this paper is the following: If R^* is finite with 2^t elements then R is the direct sum of t minimally idempotent rings. Since a commutative ring with unit is directly indecomposable if and only if it is minimally idempotent, it follows that the direct summands are uniquely determined. This theorem yields as special cases the result of McCoy and Montgomery on the decomposition of finite p-rings and the result of Harary on the decomposition of finite Boolean-like rings. The noncommutative case (with unit) yields to the same approach provided the role played by R^* is now assumed by the set of idempotents in the center of R. (Received September 26, 1955.)

47. Morgan Ward: *Linear divisibility sequences of order three.*

The form of all integral cubic linear divisibility sequences is determined where characteristic polynomial does not have three distinct integral roots. With trivial exceptions, every such sequence is the square of a Lehmer sequence. (Received October 3, 1955.)

ANALYSIS

48. M. G. Arsove: *Proper bases and automorphisms in the space of entire functions.*

A sequence $\{\alpha_n\}_{n=0}^{\infty}$ in the space Γ of entire functions is a *basis* if every $f\in\Gamma$ can be represented uniquely as $f=\sum_{n=0}^{\infty}c_n\alpha_n$, where the c_n's are complex constants and the series converges uniformly on compact sets. If, further, $\sum_{n=0}^{\infty}|c_n\alpha_n|$ converges uniformly on compact sets, the basis is *absolute*. A basis in which $|c_n|^{1/n}\to 0$ for all f is *(Iyer)-proper*, i.e. proper in the sense of Iyer [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 874–883]. Iyer's conjecture that all bases are (Iyer)-proper is observed to be false (e.g. $\alpha_n(z)=z^n/n!$): Modifying Iyer's terminology, the author calls $\{\alpha_n\}$ a *proper basis* provided it is an absolute basis and $\sum_{n=0}^{\infty}|c_n\alpha_n|$ converges uniformly on compact sets if and only if $|c_n|^{1/n}\to 0$. Proper bases are characterized in terms of $M_n(R)=\max_{|z|=R}|\alpha_n(z)|$ as those absolute bases for which $\limsup_{n\to\infty}[M_n(R)]^{1/n}<+\infty$ for each $R>0$ and $\lim_{R\to\infty}\{\liminf_{n\to\infty}[M_n(R)]^{1/n}\}=+\infty$. Theorem: the linearly homeomorphic images of Γ in itself coincide with the closed subspaces Γ_0 admitting proper bases. For the case of automorphisms ($\Gamma_0=\Gamma$) this resolves certain questions left open by Iyer. It is shown how the successive remainders of suitably restricted functions of exponential type generate proper bases, and inversion yields a class of proper bases in which each α_n is a polynomial of degree $n+1$. (Received September 30, 1955.)

49. W. G. Bade (p) and J. T. Schwartz: *On abstract eigenfunction expansions.*

Let T be a self-adjoint operator in $L_2(S, \Sigma, \nu)$ where (S, Σ, ν) is a positive measure space. Let E be the resolution of the identity for T. Assume there exists an increasing sequence $\{S_n\}$ of sets covering S, $\nu(S_n)<\infty$, such that for each bounded Borel set e and $f\in L_2(S, \Sigma, \nu)$, ν-ess. sup. $|(E(e)f)(s)|<\infty$ on S_n, $n=1, 2, \cdots$. Then T has an abstract eigenfunction expansion in the sense of Mautner (Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 49–53); i.e., there exist Borel measures μ_α on the real line and functions $W_\alpha(s, \lambda)$ defined and measurable with respect to $\nu\times\mu_\alpha$ on $S\times$reals and a linear isometry U of $L_2(S, \Sigma, \nu)$ onto $\Sigma_\alpha L_2(\mu_\alpha)$ which diagonalizes T such that

$(Uf)_\alpha(\lambda) = \int_S f(s)\overline{W_\alpha(s,\lambda)}\nu(ds)$ and $f(\cdot) = \sum_\alpha \int_{-\infty}^{\infty}(Uf)_\alpha(\lambda)W_\alpha(\cdot, \lambda)\mu_\alpha(d\lambda)$, the integrals existing in the mean square sense. Moreover, ν-ess. sup. $\int_e |W_\alpha(s, \lambda)|^2\mu_\alpha(d\lambda) < \infty$ on S_n for each bounded Borel set e and $n = 1, 2, \cdots$. If $\mu_{\alpha_1} \geq \mu_{\alpha_2} \geq \cdots$, the functions $W_{\alpha_1}(\cdot, \lambda), \cdots, W_{\alpha_n}(\cdot, \lambda)$ are linearly independent for μ_{α_n}-almost all λ. The results may be applied to simplify recent work of Gårding and Browder on eigenfunction expansions for elliptic operators. If T is an ordinary differential operator of order n, the spectral multiplicity of T is $\leq n$. The kernels $W(s, \lambda)$ satisfy the boundary conditions at any fixed end point. (Received October 7, 1955.)

50. Seymour Ginsburg: *On mappings from the family of well ordered subsets of a set.*

For a simply ordered set E let ωE denote the family of all nonempty well ordered subsets of E ordered as follows: $W_1 <_k W_2$ if W_1 is a proper initial segment of W_2. A function f from ωE to E is called a k-function if $W_1 <_k W_2$ implies $f(W_1) < f(W_2)$. If there exists a k-function on E, then E is called a k-set. The existence of k-sets is studied. For example, each simply ordered set E is similar to a terminal segment of some k-set $F(E)$. It is not true that each simply ordered set E is similar to an initial segment of some k-set $F(E)$. Finally it is shown that no infinite, simply ordered group is a k-set. (Received September 26, 1955.)

51. C. J. A. Halberg, Jr.: *Remarks concerning the spectra of bounded linear operators on the sequence spaces.* Preliminary report.

Let $[l_p]$ denote the algebra of all bounded linear operators mapping the sequence space l_p into itself. Given an infinite matrix A, let A_p and A'_p denote the operators defined on l_p by the matrix and its transpose respectively, and let $\|A_p\|$ denote the norm of A_p as an operator on l_p. By the use of a convexity theorem of M. Riesz the validity of the inequality $\|A_{1/\alpha_2}\|^{(\alpha_3-\alpha_1)} \leq \|A_{1/\alpha_1}\|^{(\alpha_3-\alpha_2)}\|A_{1/\alpha_3}\|^{(\alpha_2-\alpha_1)}$ for $0<\alpha_1<\alpha_2<\alpha_3$ is established. Let $\sigma(A_p)$ and $|\sigma(A_p)|$ denote the spectrum and the spectral radius, respectively, of the operator A_p. The above inequality is used to derive some results of the following types: *Theorem* 1. *If* $A_{1/\alpha} \in [l_{1/\alpha}]$, *then* $\log |\sigma(A_{1/\alpha})|$ *is a convex function of* α *for* $0 \leq \alpha \leq 1$. *Theorem* 2. *If both* T_p *and* T'_p *belong to* $[l_p]$, $1 \leq p \leq 2$, *and* $|\sigma(T_p)| \neq |\sigma(T'_p)|$, *then either* $|\sigma(T_p)| > |\sigma(T_2)|$ *or* $|\sigma(T'_p)| > |\sigma(T'_2)|$, *according as* $|\sigma(T_p)| > |\sigma(T'_p)|$ *or* $|\sigma(T'_p)| > |\sigma(T_p)|$. *Theorem* 3. *Suppose that both* T_p *and* T'_p *belong to* $[l_p]$ *and* $1 \leq p \leq q \leq 2$. *Then* (a) $\sigma(T_q) \subset \sigma(T_p) \cup \sigma(T'_p)$; (b) *if* C *is any component of* $\sigma(T_q)$, *then the set* $C \cap (\sigma(T_p) \cap \sigma(T'_p))$ *is nonvoid.* (Received October 3, 1955.)

52t. D. G. Kendall: *Some further pathological examples in the theory of denumerable Markov processes.*

(i) An outstanding uniqueness problem concerning the Kolmogorov differential equations is answered in the negative by the construction of two distinct honest Markov processes (transition semigroups of operators on l) having the same (finite) values for the limits $q_{ij} = \lim [p_{ij}(t) - \delta_{ij}]/t \downarrow (t\ 0)$, with $\sum_{j=0}^{\infty} q_{ij} = 0$, and satisfying both the "backward" and the "forward" equations. This improves an earlier result of Ledermann and Reuter; they also found two distinct solutions, but one of these was a dishonest "quasi-process" (contraction semigroup). (ii) Kolmogorov has given examples in which (a) $q_{00} = -\infty$, and (b) $\sum_{j\neq 0} q_{0j} < -q_{00}$, and he has remarked that the extreme pathology, (c) $q_{00} = -\infty$ and $q_{0j} = 0$ (all $j \neq 0$), could be realized by similar methods. Here a transition semigroup of operators on l is constructed for which the associated process has property (c). Informally one can say of the system that "it leaves state 0 with infinite velocity to go—nowhere in particular." (Received August 22, 1955.)

53*t*. D. G. Kendall and G. E. H. Reuter: *Some ergodic theorems for semigroups of operators.*

Let $G=\{T_t : t\geqq 0\}$ be a semigroup of operators on a Banach space X, strongly continuous for $t>0$. Call $x\in X$ ergodic if T_tx has some kind of generalized limit as $t\to\infty$, and call G ergodic if each $x\in X$ is ergodic. With suitable restrictions on G: (1) Abel or (C, k) summability, and weak or strong convergence, give four equivalent definitions of ergodicity. (2) G and the discrete semigroup $\{(\lambda J_\lambda)^n : n=0, 1, 2, \cdots\}$, J_λ being the resolvent operator for any one $\lambda>0$, have similar ergodic properties. (3) Suitable compactness properties of J_λ imply strong or uniform ergodicity of G; reflexive semigroups are strongly ergodic. (4) The subspace of ergodic elements can be calculated from the infinitesimal generator of G. (5) Ergodic properties of G for general weak topologies are studied. (6) Strong ergodicity of the adjoint semigroup of G implies strong ergodicity of G, if X is weakly sequentially complete. (Received September 27, 1955.)

54. V. A. Kramer: *Asymptotic inverse series.*

Let $H(t)$ be the operator defined on a Hilbert space by the series (possibly finite), $H(t)=H_0+tH_1+t^2H_2+\cdots$. Suppose that $\mathfrak{D}$, the domain of $H(t)$, is dense, that H_0 is self-adjoint and bounded below by 1, that H_i for $i>0$ is symmetric and bounded below by 0, and that $t\geqq 0$. Then $H(t)$ has a Friedrichs extension, $\tilde{H}(t)$. Suppose that $\tilde{H}_0$ is the Friedrichs extension of its own contraction to $\mathfrak{D}$. The extended operator $\tilde{H}(t)$ has a bounded inverse with the formal expansion $\tilde{H}^{-1}\phi=A_0\phi+tA_1\phi+t^2A_2\phi+\cdots$, where the A's are obtained by an identification of coefficients. If (for fixed N) $\phi\in\mathfrak{D}(A_i)$ for $i\leqq N$ and $A_i\phi\in\mathfrak{D}$ for $i<N$, then $\lim_{t\to 0} t^{-N}\|\tilde{H}^{-1}\phi-\sum_{i=0}^{N} t^iA_i\phi\|=0$. If in addition $A_N\phi$ belongs to $\mathfrak{D}^{1/2}$, the domain of the square root of $\tilde{H}(t)$, then an error estimate can be given. If ϕ, ψ satisfy the first set of conditions above for N, M respectively, then $(\tilde{H}^{-1}\phi, \psi)$ can be expanded to order $N+M$. If in addition $A_N\phi$ and $A_M\psi$ belong to $\mathfrak{D}^{1/2}$ then $(\tilde{H}^{-1}\phi, \psi)$ can be expanded to order $N+M+1$. These results have applications to perturbation theory. (Received October 3, 1955.)

55. L. Kuipers: *Note on the location of zeros of polynomials* (*a converse of Jensen's theorem concerning real polynomials*).

In the theory of the geometry of the zeros of polynomials $f(z)$ much attention has been paid to the problem of the location of the zeros of the derivative $f'(z)$, the zeros of $f(z)$ being known (Lucas's theorem, Jensen's theorem). The converse of the above problem, namely to give information on the location of the zeros of the primitives $\int^z f(\zeta)d\zeta$ of a given polynomial $f(z)$ is less often studied. See Chamberlin and Wolfe: *Note on a converse of Lucas's theorem*, Proc. Amer. Math. Soc. vol. 5 (1954) p. 203. Now let $\mathfrak{H}_{a,b}$ denote the orthogonal hyperbola with $a+bi$ and $a-bi$ as vertices. Then we have: In the closed interior of any $\mathfrak{H}_{a,b}$ of a real polynomial $f(z)$ vanishing for $z=a\pm bi$ lies at least one pair of conjugate complex zeros of the real primitive $\int_0^z f(\zeta)d\zeta+C$ (C real). This follows from Jensen's theorem. By repetition of the argument we have furthermore: If the real polynomial $f(z)$ has complex zeros $a\pm bi$, then the polynomial $\int_0^z(z-t)^nf(t)dt+c_0z^n+c_1z^{n-1}+\cdots+c_n$ (all c_i real) has in the closed interior of the hyperbola $(1/(n+1))(x-a)^2+b^2=y^2$ $(n=2, 3, \cdots)$ a pair of conjugate zeros. Other similar results can be derived. (Received October 3, 1955.)

56t. R. M. Redheffer: *Characteristic values, zeros, and comparison.*

In a closed bounded region R let k be a characteristic value for the problem $z_{xx}+z_{yy}+kz=0$, $z=0$ on the boundary. If $w\in C^{(2)}$ and $w\neq 0$ in R, $-k\leqq \max\,(w_{xx}+w_{yy})/w$. Hence, if the boundary is smooth, $-k=\min_w \max_{(x,y)}(w_{xx}+w_{yy})/w$. More generally, let $Az_{xx}+2Bz_{xy}+Cz_{yy}+Dz=0$, where A, B, C, D are arbitrary but well-defined functions of $(z_x/z, z_y/z, x, y)$, with $A\geqq 0$, $C\geqq 0$, $AC\geqq B^2$. Assume z continuous in R, $z=0$ on the boundary, $z\neq 0$. If $Aw_{xx}+2Bw_{xy}+Cw_{yy}+D^*w=0$ with $D^*>D$, then w vanishes somewhere in R. Similar results in one dimension yield an extension of Sturm's theorem on separation of zeros. This extension embraces nonlinear equations and the classical linear equation $z''+a(x)z'+b(x)z=0$ with no continuity or measurability hypothesis on a, b. (Received October 12, 1955.)

57. H. L. Royden: *Rings of meromorphic functions.*

Let D_1 and D_2 be two plane domains each having the property that for each boundary point there is a bounded analytic function in the domain with that point as an essential singularity. Kakutani and Chevalley have shown that if the rings B_1 and B_2 of bounded analytic functions on these two domains are algebraically isomorphic then the domains are conformally equivalent. In this talk the following generalization is given: Let R_1 and R_2 be any two rings of meromorphic functions on D_1 and D_2 which contain the rings B_1 and B_2 respectively. Then if R_1 and R_2 are algebraically isomorphic, the domains D_1 and D_2 are conformally equivalent. The proof is effected by characterizing algebraically the bounded functions in R_1 and R_2 and then applying the Kakutani-Chevalley theorem. (Received October 4, 1955.)

58. O. K. Smith: *Discontinuous oscillations.*

The system (*) $\dot{x}-k\dot{y}-ky=0$, $G(\dot{y})+x=0$ ($\dot{x}=dx/dt$, $\dot{y}=dy/dt$) is investigated by considering the phase portrait in the cylindrical surface $G(y)+x=0$ in Cartesian $(x, y, \dot{y})$ space rather than in a phase plane. On the surface, lines where $G'(\dot{y})=0$, $G''(\dot{y})\neq 0$ form boundaries across which trajectories cannot be extended continuously with t increasing, therefore ***discontinuous solutions*** are defined. It is shown that if $k>0$ and G is subjected to appropriate restrictions, for instance if $G(v)=v-2\arctan v$, then there exists a unique discontinuous periodic solution and this solution is orbitally stable. Similar results are obtained for the system (**) $\dot{x}+\dot{y}-G(\dot{y})+x=0$, $\dot{x}+\dot{y}-G(\dot{x})+y=0$ by considering a phase surface in $(x, y, \dot{x}, \dot{y})$ space. Both (*) and (**) arise from vacuum tube circuits and are discussed, without proofs, by Andronow and Chaikin in *Theory of oscillations* (Princeton University Press, 1949). (Received October 3, 1955.)

59. G. T. Thompson: *On Bateman's method for solving linear integral equations.*

The integral equations considered here are those of Fredholm type and second kind, that is, of the form $x(s)-\lambda\int_0^1 K(s,t)x(t)dt=y(s)$, $0\leqq s\leqq 1$, where $y(s)$ and $K(s, t)$ are given and $x(s)$ is to be found. If $K(s, t)$ is a bilinear form of $2n$ functions $\phi_1(s), \cdots, \phi_n(s); \Psi_1(t), \cdots, \Psi_n(t)$, the equation above reduces to a system of n algebraic equations. In this case a kernel is said to be of "finite rank" or "degenerate." Ordinarily $K(s, t)$ is not of finite rank; however, one may replace $K(s, t)$ by an approximation of finite rank and solve the resulting equations. Bateman accomplished this systematically with determinantal formulas which are derived outside the framework of Fredholm's equations. In this paper Bateman's method is analyzed in detail for the physically important case in which $K(s, t)$ is the Green's function for the self-adjoint second order linear differential operator $L(u)=(pu')'-qu$ with homogeneous boundary conditions. An error bound is derived which is $O(\epsilon_n/n^{1/2})$ where ϵ_n is a measure of the error incurred in the approximation to $K(s, t)$ by $K_n(s, t)$. (Received September 19, 1955.)

APPLIED MATHEMATICS

60. R. G. Stoneham: *The propagation of waves in a semi-infinite nonhomogeneous elastic solid.*

Let an inhomogeneity be introduced into an elastic half-space by assuming that in the neighborhood of the free surface $z=0$ the Lamé constants are functions of the depth. The equations of motion for such an elastic medium are derived. Typically, one has $(\lambda+\mu)\partial e/\partial z+\mu\nabla^2 w+e\partial\lambda/\partial z+2\partial\mu/\partial z\cdot\partial w/\partial z=\rho\partial^2 w/\partial t^2$ for the z-component of the displacement. Reasonable interpretation of the expected variation of the physical constants with depth may be introduced by assumptions on the character of $\lambda(z)$ and $\mu(z)$. In this paper is presented an exact solution in the form of complex contour integrals for the displacements produced by a continuous harmonic point source disturbance. The solution is based on the method of images wherein the point source is constructed from a Hankel integral transform of a one-dimensional Green's function which has the proper discontinuity as well as Sommerfeld radiation condition. Later further work will be presented on the evaluation of the contour integrals for the displacements in order to determine whether physically sensible waves forms are exhibited which are not given by a purely elastic model. (Received September 29, 1955.)

61*t*. A. C. Sugar: *Definitional axiom systems for thermodynamics.*

A substantial step in the clarification of thermodynamics was taken in an earlier paper by the writer (Bull. Amer. Math. Soc. Abstract 60-6-756). Here, for the first time, an important branch of thermodynamics was axiomatized. An examination shows that all of the axioms are definitional in character. One can classify this set of axioms as being two-dimensional. In addition to analyzing the general notion of a definitional axiom system, this paper is also concerned with axiomatizing a system of one-dimensional systems of thermodynamics. An example of a one-dimensional system is a mass of water under atmospheric pressure between 0° and 100°C. Clearly such a system has only one thermodynamic coordinate. For example, by a differentiable one-dimensional classical reversible system of thermodynamics we mean an ordered quadruple, $\Sigma_1=\langle c, T, \theta, q\rangle$, which satisfies axioms B1–B4. B1. c is a positive real number. B2. T is a closed interval of real numbers. B3. θ is a differentiable function whose domain is T and whose value is a real number. B4. $q=c\theta$. If we isolate two realizations of these one-dimensional systems at different temperatures, in juxtaposition, we obtain an irreversible two-component system. More generally, the irreversible phenomenon of calorimetry can be axiomatized in the following fashion. The ordered set $\sum_3=\langle c, \sigma, \tau, m, \theta, \Delta q, s\rangle$ which satisfies axioms C1–C7 is called a calorimetric system of thermodynamics. C1. c is a positive real number. C2. σ is a set of n elements, $w_1, \cdots, w_n$. C3. τ is a set consisting of two real numbers t_1 and t_2. C4. m is a positive real valued function on $\sigma\times\tau$ such that $m(\sigma_i, t_1)=m(\sigma_i, t_2)$. C5. θ is a function on $\sigma\times\tau$ such that $\theta(w_i, t_2)=[1/\sum_{i=1}^n m(w_i)]\sum_{i=1}^n m(w_i)\theta_i(t_1)$, $i=1, \cdots, n$. C6. Δq is a real valued sequence such that $\Delta q_i=cm(w_i)[\theta(w_i, t_2)-\theta(w_i, t_1)]$. C7. $s=m\ln\theta$. Finally, it is observed that the latter two axiom systems are also definitional. (Received October 5, 1955.)

62t. C. H. Wilcox: *An expansion theorem for electromagnetic fields. I.*

Let D be an exterior domain bounded internally by a regular surface S. Let A be of class C^2 in the closure of D and let it satisfy (1) $\nabla\times(\nabla\times A)=k^2A$ in D and (2) $\lim_{R\to\infty}\int_{r=R}|r\times(\nabla\times A)+ikA|^2dS=0$ (the vector radiation condition). Here k^2 is an arbitrary nonzero complex number and Im $k\geqq 0$. It is shown that (I) A can be represented as an integral over S involving the values of A and $\nabla\times A$ on S, and (II) if S is contained in the sphere $r=c$, where (r, θ, ϕ) are spherical coordinates, then $A=1/r\cdot\exp(ikr)\sum_{n=0}^{\infty}(1/r^n)A_n(\theta, \phi)$. The series converges in $r>c$ and converges absolutely and uniformly in the variables r, θ, and ϕ in any region $r\geqq c+\epsilon>c$. The series can be differentiated termwise with respect to r, θ, and ϕ any number of times and the resulting series all converge absolutely and uniformly. (Received October 6, 1955.)

63t. C. H. Wilcox: *An expansion theorem for electromagnetic fields. II.*

Let A satisfy the hypotheses of the preceding abstract. It is shown that (I) $A_0(\theta, \phi)$ is tangent to any sphere $r=r_0$ at the point (r_0, θ, ϕ), and (II) A_0 determines $A_n=A_n^1\hat{r}+A_n^2\hat{\theta}+A_n^3\hat{\phi}$ through the recursion relations $ikA_1^1=(-1/\sin\theta)[\partial(\sin\theta A_0^2)/\partial\theta+\partial A_0^3/\partial\phi]$, $2iknA_{n+1}^1=n(n-1)A_n^1+DA_n^1$, and $2iknA_n^2=n(n-1)A_{n-1}^2+DA_{n-1}^2+D_\theta A_{n-1}$, $2iknA_n^3=n(n-1)A_{n-1}^3+DA_{n-1}^3+D_\phi A_{n-1}$ where $Df=(1/\sin\theta)\partial(\sin\theta\partial f/\partial\theta)/\partial\theta+(1/\sin^2\theta)\partial^2f/\partial\phi^2$ is Beltrami's operator for the sphere, while $D_\theta F=2\partial F^1/\partial\theta-(1/\sin^2\theta)F^2-(2\cos\theta/\sin^2\theta)\partial F^3/\partial\phi$, $D_\phi F=(2/\sin\theta)\partial F^1/\partial\phi+(2\cos\theta/\sin^2\theta)\partial F^2/\partial\phi-(1/\sin^2\theta)F^3$. A field $A(r)$ that is of class C^2 and that satisfies $\nabla\times(\nabla\times A=k^2A$ and $\lim_{R\to\infty}\int_{r=R}|r\times(\nabla\times A)+ikA|^2dS=0$ in an exterior domain D is called a *vector radiation function* for D. Similarly a scalar field $u(r)$ that is of class C^2 and satisfies $\nabla^2u+k^2u=0$ and $\lim_{R\to\infty}\int_{r=R}|\partial u/\partial r-iku|^2dS=0$ is called a *scalar radiation function* for D. The expansion theorem is used to demonstrate the following relationships between vector and scalar radiation functions: (III) A is a vector radiation function for D if and only if the Cartesian components of A are scalar radiation functions for D and, where $r=r\hat{r}$, (a) $\lim_{r\to\infty}\exp(\text{Im } kr)r\cdot A(r)=0$; (IV) The last result also holds if (a) is replaced by (b) $\nabla\cdot A=0$ on the boundary of D. (Received October 6, 1955.)

64t. C. H. Wilcox: *On the representation of electromagnetic fields by Debye potentials.*

A vector field A is a *vector wave function* for a domain V if $A\in C^2$ and $\nabla\times(\nabla\times A)=k^2A$ in V. Similarly, a scalar field u is a *scalar wave function* for V if $u\in C^2$ and $\nabla^2u+k^2u=0$ in V. A vector (scalar) wave function $A(u)$ is a *vector* (*scalar*) *radiation function* for V if V is an exterior domain and $A(u)$ satisfies the vector (scalar) radiation condition (see preceding abstract). Time-harmonic electromagnetic and acoustic radiation fields are important examples of vector and scalar radiation functions, respectively. It is well known that if u is a scalar wave function and $\Pi=ur$, where r is a position vector, then $\nabla\times\Pi$ and $\nabla\times(\nabla\times\Pi)$ are vector wave functions. The principal results of this paper are (I) Let A be a vector wave function for a domain V defined by $a<r<b$ where (r, θ, ϕ) are spherical coordinates. Then there exist functions u, v such that (1) u and v are scalar wave functions for V, (2) $\int_\Omega u(r\hat{r})d\Omega=\int_\Omega v(r\hat{r})d\Omega=0$ where Ω is the unit sphere $r=1$, $\hat{r}\in\Omega$, and $d\Omega=\sin\theta d\theta d\phi$ is the element of area on Ω, (3) $A=\nabla\times(\nabla\times\Pi)+k\nabla\times\Pi'$ in V, where $\Pi=ur$, $\Pi'=vr$. u and v are uniquely determined by (1), (2), and (3). They are called the electric and magnetic Debye potentials, respectively, for A. (II) u is related to the radial component $A_1=A\cdot\hat{r}$ by the reciprocal

formulas $Du = -rA_1$ where $Df = (1/\sin\theta)\partial(\sin\theta\partial f/\partial\theta)/\partial\theta + (1/\sin^2\theta)\partial^2 f/\partial\phi^2$ and $u(r\hat{r}) = -(r/2\pi)\int_\Omega(\log\sin\rho/2)A_1(r\hat{r}')d\Omega'$ where $\rho(\hat{r}, \hat{r}')$ is the geodesic distance from $\hat{r}$ to $\hat{r}'$ on Ω. v is similarly related to $\nabla x A\cdot\hat{r}$. (III) A is a vector *radiation* function for V if and only if the corresponding Debye potentials u and v are scalar *radiation* functions for V. (Received September 29, 1955.)

65. C. H. Wilcox: *On isolated singularities of electromagnetic fields.*

Time-harmonic electromagnetic fields are described by vector wave functions (defined in the preceding abstract). In this paper the behavior of such functions near isolated singularities is investigated. Results obtained include the following. (I) Let A have an isolated singularity at P_0. Then $A = A' + A''$ where A' is regular at P_0 and A'' is a vector *radiation* function for the domain whose complement is P_0. The radiation functions $H_n = k\nabla x h_n(kr)S_n(\theta, \phi)r$ and $E_n = \nabla x\nabla x h_n(kr)S_n(\theta, \phi)r$ are called electromagnetic multipoles of order n and of magnetic and electric type, respectively. (Here $h_n(x) = (\pi/2x)^{1/2}H^{(1)}_{n+1/2}(x)$ and $S_n(\theta, \phi)$ is a spherical harmonic of order n.) (II) Let A be a radiation function for the domain $r > c \geq 0$. Then $A = E_1 + E_2 + \cdots + H_1 + H_2 + \cdots$. The series converges in $r > c$ and converges uniformly in $r \geq c + \epsilon > c$. The isolated singularity at P_0 ($r = 0$) has finite order $\mu \geq 0$ if $A(P) = O(1/r^\mu)$, $r \to 0$. (III) If A has an isolated singularity of order μ at $r = 0$ then $A = A' + A''$ where A' is regular at $r = 0$ and A'' consists of a finite number of electromagnetic multipoles. The orders of those of electric (magnetic) type do not exceed $[\mu - 2]([\mu - 1])$. Simple corollaries of this result are (IV) if A has order $\mu < 2$ at P_0 then the singularity at P_0 is removable and (V) if A has order $\mu < 3$ at $P_0 (r = 0)$ then $A = A' + \nabla x(e^{ikr}/r)u$ where A' is regular at P_0 and u is a constant vector. (Received September 29, 1955.)

GEOMETRY

66t. J. R. Munkres: *The equivalence problem for differentiable manifolds of dimension* 2.

The equivalence problem for differentiable manifolds is the following: "Are two topologically equivalent differentiable manifolds also equivalent in the sense of differential geometry?" The author solves this problem for manifolds of dimension 2. He proves the following approximation theorem: Let M and N be two homeomorphic differentiable manifolds of class C^1 and dimension 2; no compactness assumptions are made. Let h be a homeomorphism of M onto N; let $\phi(x)$ be a positive continuous function defined on M; and let $d(x, y)$ be a metric on N. Then there is a homeomorphism g of M onto N which is of class C^1 and has nonvanishing Jacobian, such that for each x in M, $d(g(x), h(x)) < \phi(x)$. The methods used involve the C^1 triangulations of J. H. C. Whitehead (Ann. of Math. vol. 41 (1940) pp. 809–824) and known facts about approximating an arbitrary homeomorphism between triangulated 2-manifolds by a piecewise-linear homeomorphism. (Received September 29, 1955.)

67. Johannes Nitsche: *A uniqueness theorem on the deformation of surfaces.*

The question of the number of different realizations in R_3 of an abstractly defined $(m+1)$-times connected piece of a surface $\mathfrak{F}$ is discussed. The surface is defined by its metric ds^2 (with positive Gaussian curvature) and for its boundary strip there holds a relation (*) $a(s)\cdot\rho_n + b(s)\cdot\tau_g = c(s)$. ρ_n means here the normal curvature, τ_g the geodetic torsion, and $a(s)$, $b(s)$, $c(s)$ are functions of the arc length on the boundary. The integer n defined by $2\pi in = \oint d\log(a + ib)$ is called the index corresponding to

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the condition (*). This question leads to the consideration of Gauss-Codazzi's equations, which because of the theorema egregium can be understood as a quasilinear system of elliptic differential equations. Employing results of I. N. Vekua (see Mat. Sbornik N.S. vol. 31 (73) (1952) pp. 217–314) there can be established the theorem: Let $\mathfrak{F}$ be a solution of the problem. (i) $n>2(m-1)$. Then $\mathfrak{F}$ is uniquely determined. (ii) $n\leqq m-2$. Then there exists a $[3(m-1)-2n]$-parametric set of deformations of $\mathfrak{F}$. For the purpose of a unique determination of $\mathfrak{F}$ further conditions must be added. The intermediate cases $m-1\leqq n\leqq 2(m-1)$ (for example, the case of a doubly-connected surface for which the spatial curvature of the boundary curve is given) have to be studied in particular (see Joachim Nitsche, Math. Zeit. vol. 62 (1955) pp. 388–401). (Received October 24, 1955.)

TOPOLOGY

68t. Bernhard Banaschewski: *Local connectedness of extension spaces.*

Given an extension E^* of a space E (E dense in E^*), each $u\in E^*-E$ determines on E a filter (in the sense of N. Bourbaki) consisting of all $U\cap E$, U any neighborhood of u in E^*. Many properties of E^* can be related to these so-called trace filters $\gamma(u)$. As shown elsewhere (B. Banaschewski, *Überlagerungen von Erweiterungsräumen*, to appear in Archiv der Mathematik) the following condition for filters $\mathfrak{A}$ is of interest in this respect: (C) If $O\cup P\in\mathfrak{A}$, O and P open disjoint, then either $O\in\mathfrak{A}$ or $P\in\mathfrak{A}$. It is first proved that an extension E^* of a locally connected E for each of whose $\gamma(u)$ (C) holds is locally connected if and only if each $\gamma(u)$ has a basis consisting of connected open sets. From this and the previously obtained results (loc. cit.) that (C) holds for the maximal open, maximal regular and maximal completely regular filters is then deduced: For locally compact spaces E, denumerable at infinity, the Čech compactification βE, Alexandroff's extension $\alpha' E$ and Katětov's maximal Hausdorff extension of E can never be locally connected. (Received October 3, 1955.)

69t. Herbert Federer: *A study of function spaces by spectral sequences.*

Suppose Y is a simple space (i.e. the operations of the fundamental group on all the homotopy groups are trivial), X is a finite-dimensional cellular space, and $v\in Y^X$. For each cell complex K on X, with p-skeletons K^p, the restriction operation defines maps of Y^X into Y^{K^p}; let G_n^p be the kernel of the induced homomorphism of $\pi_n(Y^X, v)$ into $\pi_n(Y^{K^p}, v|K^p)$; also let $G_n^{-1}=\pi_n(Y^X, v)$. Then for $n\geqq 1$ and $p\geqq 0$ the factor group G_n^{p-1}/G_n^p is isomorphic to the limit group $E_\infty^{p,n}$ of a spectral sequence whose term $E_2^{p,n}$ is isomorphic to the p dimensional cohomology group of X with coefficients in $\pi_{p+n}(Y)$. This spectral sequence arises from an exact couple obtained from the homotopy sequences of the fibre maps of Y^{K^p} into $Y^{K^{p-1}}$ defined by the restriction operation. While the exact couple depends on K, its derived couples are homotopy type invariants of X, Y and v. (Received October 5, 1955.)

70. V. L. Klee, Jr.: *Fixed-point sets of periodic homeomorphisms of Hilbert space.*

Considerable space in the literature has been devoted to the following question, for various metric spaces X: If Y is the set of all fixed points of a periodic homeomorphism of X, what topological properties of Y can be deduced from those of X? In the most-studied cases, the currently known results assert (under various additional hypotheses) that if X is E^n [S^n], then Y is in some sense homologically similar to E^k [S^k] for an appropriate $k<n$. The present note establishes the following result, whose contrast with the finite-dimensional situation is striking: *If Y is a compact [closed] subset of an infinite-dimensional Hilbert space H and n is an integer ≥ 2, then H admits a homeomorphism of period n whose fixed-point set is Y [is homeomorphic with Y]*. The basic tools employed are contained in earlier papers by the author [Trans. Amer. Math. Soc. vol. 74 (1953) pp. 10–43 and vol. 78 (1955) pp. 30–45]. (Received October 5, 1955.)

71t. E. A. Michael: *On a theorem of Borsuk.*

The space of nonempty, closed, LC^n subsets of a complete metric space is metrized with a complete metric in such a way that convergence of a sequence of sets in this metric is equivalent to homotopy-n-regular convergence in the sense of M. L. Curtis [Ann. of Math. vol. 57 (1953) pp. 231–247]. This slightly generalizes a result of K. Borsuk [Fund. Math. (1954) pp. 168–202] and K. Kuratowski [Bull. Acad. Polon. Sci. Cl. III vol. 3 (1955) pp. 75–80]. (Received October 5, 1955.)

72t. J. R. Munkres: *The general triangulation problem for dimension 3.*

A separable metric space X is said to be *locally triangulable at x* if there exists a complex K and a homeomorphism h of $|K|$ into X such that x lies in the interior of $h(|K|)$. A locally triangulable space is one which is locally triangulable at each of its points. The general triangulation problem reads as follows: "Can every locally triangulable space be triangulated?" The author solves the problem for spaces of dimension not greater than 3. The method used consists essentially of reducing the problem to the triangulation problem for 3-manifolds with boundary, or rather, to the more general result proved in Theorem 8′ of the paper, *Locally tame sets are tame*, by R. H. Bing (Ann. of Math. vol. 59 (1954) pp. 145–158). (Received September 29, 1955.)

73. L. M. Sonneborn: *A coincidence theorem for functions on product spaces.*

Let X and Y be compact, arc-wise connected topological spaces. Let $f:X\rightarrow R$, $g:Y\rightarrow R$, and $h:X\times Y\rightarrow R$ be real-valued, continuous functions such that the range of h is contained in both the range of f and the range of g. Then there exist $x\in X$ and $y\in Y$ such that $f(x)=g(y)=h(x, y)$. (Received October 5, 1955.)

74. H. C. Wang: *Fundamental groups of solvmanifolds.*

By a solvmanifold is meant a coset space of a connected solvable Lie group. Mostow has proved that the topological type of a solvmanifold M is determined by its fundamental group $\pi_1(M)$. Moreover if M is compact, then $\pi_1(M)$ determines M up to a homeomorphism. It is the aim of this paper to characterize the class of groups $\pi_1(M)$ as abstract groups. Let N be a finitely generated, nilpotent group without torsion, and H a finitely generated, free abelian group. Any extension of N by H is called a special $\mathfrak{S}$-group. It is shown that (i) Any special $\mathfrak{S}$-group is the fundamental group of a solvmanifold and conversely; (ii) Any special $\mathfrak{S}$-group has a normal subgroup T of finite index such that T can be imbedded, as a uniform discrete subgroup, in a connected and simply-connected solvable Lie group.

V. L. Klee, Jr.
Associate Secretary

THE NOVEMBER MEETING IN KNOXVILLE

The five hundred nineteenth meeting of the American Mathematical Society was held at the University of Tennessee in Knoxville, Tennessee on Friday and Saturday, November 18–19, 1955. About 130 persons registered, including 82 members of the Society.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, Professor Sze-Tsen Hu of the University of Georgia addressed the Society Friday evening on the subject *Homotopy groups*, with Professor A. D. Wallace presiding.

Sessions for contributed papers were held Friday afternoon and Saturday morning, Professors Wallace Givens, M. K. Fort, Jr., Tomlinson Fort, A. S. Householder, and S. T. Hu presiding.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Mr. Simon and Mr. Anderson were introduced by Professor A. D. Wallace, Dr. Campbell by Professor J. H. Roberts, and Mr. Kimura by Professor R. J. Koch.

ALGEBRA AND THEORY OF NUMBERS

75. A. T. Brauer: *On the Schnirelmann density of the sum of two sequences of which one has positive density.*

Let A and B be sequences of positive integers, and S the sum of A and B. Assume that the Schnirelmann density α of A is positive and that B is a sequence of order k. It was proved by P. Erdős [Acta Arithm. vol. 1 (1936) pp. 197–200] that the Schnirelmann density γ of S satisfies the inequality $\gamma \geqq \alpha\{1+(1-\alpha)/2k\}$. E. Landau [*Über einige neuere Fortschritte der additiven Zahlentheorie*, Cambridge, 1937] showed that the order k can be replaced by the mean order λ. A. Brauer [Math. Zeit. vol. 44 (1938) pp. 212–232] improved this to $\gamma \geqq \alpha\{1+(1-\alpha^{1/2})/\lambda\}$ and S. Selberg [Archiv for Mathematik og Naturvidenskab vol. 47 (1944) pp. 111–118] to $\gamma \geqq \alpha\{1+3(1-\alpha)/4\lambda\}$. Selberg's result is better for $\alpha > 1/9$ and Brauer's result for $\alpha < 1/9$. In this paper these results will be further improved for the case $1/9 \leqq \alpha \leqq 1/2$. (Received October 13, 1955.)

76*t*. Leonard Carlitz: *Resolvents of certain linear groups in a finite field.*

Let Γ denote the group of linear transformations $x' = (ax+b)/(cx+d)$ with coefficients in $GF(q)$ and of determinant 1. Dickson has proved that every absolute invariant of Γ is a rational function of $J = Q^{(q+1)/2}L^{-(q^2-q)/2}$ $(p>2)$, where $L = x^q - x$, $Q = (x^{q^2}-x)/(x^q-x)$; when $q=2$ the divisor 2 in the exponents is omitted. The equation $J(x) = y$ is normal over $GF[q, x]$ with Galois group Γ. It is easy to construct a resolvent of degree $q+1$. The principal object of the present paper is the construction of resolvents of lower degree when they occur. It is well known that Γ can be represented as a permutation group of degree $\leq q$ only when $q=5, 7, 9, 11$ in which case the degree is 5, 7, 6, 11, respectively. In each case a resolvent of minimum degree is obtained. In the last part of the paper the ternary linear group is discussed briefly. For $q=2$ the group is of order 168 and a resolvent of degree 8 is constructed; the resolvent of degree 7 in this case is easily obtained. (Received September 23, 1955.)

77*t*. Leonard Carlitz: *The number of solutions of a particular equation in a finite field.*

Faircloth (Canadian Journal of Mathematics vol. 4 (1952) pp. 343–351) has determined explicitly the number of nonzero solutions in $GF(q^2)$ of the equation $x_1^m+\cdots+x_s^m=1$, where $m|q+1$. In the present paper the number of solutions in $GF(q^2)$ of $\alpha_1x_1^m+\cdots+\alpha_sx_s^m=\alpha$, where $m|q+1$ and α_i, $\alpha\in GF(q^2)$, is determined. The proof makes use of a result of Stickelberger (Math. Ann. vol. 37 (1890) pp. 321–367). (Received September 23, 1955.)

78*t*. Leonard Carlitz and H. H. Corson: *Some special equations in a finite field.*

Let $N=N(a_1, \cdots, a_r)$ denote the number of solutions of the equation $a_1x_1^{m_1}+\cdots+a_rx_r^{m_r}=c$ in the finite field $GF(q)$, where $m_i|q-1$. Let A denote the average of N for fixed c and fixed m_i and define the variance $\sum\{N(a_1, \cdots, a_r)-A\}^2$ where the sum is over all nonzero a_i. When $c=0$ it is shown that $\sum(N-A)^2=q^{r-2}(q-1)^{r+1}Z$, where Z is a certain function of the m_i. A similar result is obtained in the case $c\neq0$. Necessary and sufficient conditions for the vanishing of Z are determined. More generally similar but more complicated results are obtained for equations of the type $a_1f_1(u_1)+\cdots+a_rf_r(u_r)=c$, where $u_i=(x_{i1}, \cdots, x_{in_i})$. (Received September 23, 1955.)

79. A. H. Clifford and D. D. Miller (p): *Regular $\mathcal{D}$-classes in a semigroup.*

J. A. Green (Ann. of Math. vol. 54 (1951) pp. 163–172) defines the commuting equivalence relations $\mathcal{L}$ and $\mathcal{R}$ in any semigroup S as follows: $a\mathcal{L}b(a\mathcal{R}b)$ if a and b generate the same principal left (right) ideal of S. He discusses also their (relative) product $\mathcal{D}=\mathcal{L}\mathcal{R}=\mathcal{R}\mathcal{L}$ and their intersection $\mathcal{L}\cap\mathcal{R}$. Following von Neumann (Proc. Nat. Acad. Sci. U.S.A. vol. 22 (1936) pp. 707–713) the authors say that an element a of S is *regular* if $axa=a$ for some $x\in S$, and they show that either every element or no element of a $\mathcal{D}$-class is regular. Defining an *inverse* of a to be an element a' such that $aa'a=a$ and $a'aa'=a'$, and setting $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$, an $\mathcal{H}$-class $H=L\cap R$ can contain at most one inverse of a, and contains exactly one if and only if each of the $\mathcal{H}$-classes $L\cap R_a$ and $L_a\cap R$ contains an idempotent. If $a\mathcal{D}b$, $L_a\cap R_b$ contains an idempotent e if and only if $ab\in L_b\cap R_a$. If $L_b\cap R_a$ also contains an idempotent f, and if $H_{11}=\mathcal{H}(e)$, $H_{21}=\mathcal{H}(a)$, $H_{12}=\mathcal{H}(b)$, $H_{22}=\mathcal{H}(f)$, then $H_{ij}H_{jk}=H_{ik}$, and there exist $\mathcal{D}$-classes D_{12} and D_{21} such that $H_{11}H_{22}\subseteq D_{12}$, $H_{22}H_{11}\subseteq D_{21}$. Other results of a similar nature are obtained, including a theorem which reduces to the Rees-Suschkewitsch Theorem when $S=D$ or $S=D\cup\{0\}$. (Received October 13, 1955.)

80*t*. Eckford Cohen: *The characteristic divisors of a polynomial in a matrix argument.*

Let A represent a matrix in an arbitrary field F, and suppose A to have characteristic divisors of the form $Q^k(x)$ where $Q(x)$ is irreducible and separable over F. Letting

$f(x)$ represent a polynomial of $F[x]$, the author determines the characteristic divisors of $f(A)$, originally determined in the general case by McCoy (Amer. J. Math. vol. 57 (1935) pp. 491–502). The method employed involves the theory of representation spaces and avoids any use of formal matrix theory. (Received October 6, 1955.)

81. P. F. Conrad: *On ordered vector spaces.*

Various ways of ordering a vector space V over an ordered division ring D are considered. Each ordering of V determines an embedding of V into a lexicographically ordered function space. For a given ordering of V let C and C' be convex subspaces such that C covers C'. Then C/C' has a natural order, but C/C' is not necessarily o-isomorphic to D (where D is considered as a vector space). V is of rank one or D is isomorphic to the field of reals if and only if for all orderings of V, every C/C' is o-isomorphic to D. (Received October 13, 1955.)

82. F. A. Ficken: *Basis-free proofs of two theorems on vector spaces.*

It is well known that *every subspace of a vector space has a complement.* This result is established here by applying Zorn's lemma to the class of subspaces not intersecting the given subspace. It is also well known that *the range of the transpose of a linear transformation is the subspace "orthogonal" to the nullspace of the transformation.* Let L map X linearly into Y, let L^* be its transpose, mapping Y^* linearly into X^*, and let $Y=LX\oplus C$. If x^* is in the subspace of X^* orthogonal to the nullspace of L, we define $y^*(Lx+c)$ $(c\in C)$ to be $x^*(x)$ and show easily that y^* is well-defined, $y^*\in Y^*$, and $L^*y^*=x^*$. The rest of the proof is trivial. (Received November 14, 1955.)

83. J. R. Garrett: *Reduction of equations to normal form in fields of characteristic p.*

In an earlier paper (Duke Math. J. vol. 18 (1951)) the author reduced the equation of degree 5 with coefficients in a field F of characteristic $p=2, 3, 5$ to *principal* and normal form by means of Tschirnhaus transformations. Dickson (*Modern algebraic theories*, Chicago, 1930) gives similar results for the quintic in case F is of characteristic zero which also hold for F of characteristic $p\geqq 7$. This paper extends the previous results to equations of degree n, $n\geqq 6$, with coefficients in a field F of characteristic $p\geqq 2$. (Received October 13, 1955.)

84. Morikuni Goto: *Lattices of subalgebras of Lie algebras.*

Various results are known about the lattices of subgroups of finite groups. Similar results are obtained in the case of Lie algebras. (Received October 13, 1955.)

85. Nickolas Heerema: *Sums of normal endomorphisms.*

Let N, E, and A represent the sets of normal nilpotent endomorphisms, normal endomorphisms and normal automorphisms, respectively, on a group G. Then: (1) $\eta_1-\eta_2\in E$ if $\eta_1\in A$, $\eta_2\in E$; (2) $\eta_1-\eta_2\in E$; $\eta_1+\eta_2\in E$ if $\eta_1\in E$, $\eta_2\in N$. If G has no direct abelian factor and satisfies both chain conditions for normal subgroups then: (3) $\eta_1+\eta_2\in N$, $\eta_1-\eta_2\in N$ if $\eta_1\in N$, $\eta_2\in N$; (4) $\eta_1-\eta_2\in A$ and $\eta_1+\eta_2\in A$ if $\eta_1\in A$, $\eta_2\in N$, and (5) $\eta_1-\eta_2\in N$ if $\eta_1\in A$, $\eta_2\in A$. Proof is based on the fact that a normal endomorphism is idempotent on the commutator subgroup. It is shown that the normal endomorphisms of a group generate a ring K_G of mappings. If G fulfills the above conditions and is indecomposable then $\eta\in K_G$ if and only if η is normal and for some positive integer n either η or $-\eta$ satisfies (6) and (7) for all x and y in G. (6) $\zeta(x)^{-1}y\zeta(x)$

$=x^{-n}yx^n$; (7) $\zeta(xy)=\zeta(x)\zeta(y)(y^{-n}x^{-n}(xy)^n)$. Related results are also obtained. (Received October 6, 1955.)

86t. J. H. Hodges: *Distribution of bordered matrices in a finite field.*

Let $q=p^f$ and let A, M, U, V denote matrices with elements in $GF(q)$. If A is nonsingular of order m, U is $s\times m$ and V is $m\times t$, put $M=\left[\begin{smallmatrix} A & V \\ U & 0\end{smallmatrix}\right]$. Then the number of pairs U, V such that M has rank $m+r$, $r\leqq s$, t, is determined. The general case where A is an $m\times n$ matrix of rank $k\leqq m$, n is also considered. The results obtained are analogous to those obtained earlier for A symmetric, skew and hermitian (L. Carlitz and J. H. Hodges, *Distribution of bordered symmetric, skew and hermitian matrices in a finite field*, J. Reine Angew. Math., to appear in 1956). (Received October 13, 1955.)

87. Naoki Kimura: *A decomposition of semigroups containing minimal right ideals.*

Let S be a semigroup containing at least one minimal right ideal, and let $\mathfrak{R}=[R_\gamma:\gamma\in\Gamma]$ be the set of all minimal right ideals. Then S is decomposed into the union of mutually disjoint subsets C_ϕ indexed by the transformation semigroup Φ over Γ such that: (1) $C_{\phi\psi}\subset C_\phi C_\psi$; (2) Φ contains every primitive projection, i.e., every transformation which sends all points of Γ into a single point. Similarly a decomposition of S with at least one minimal right ideal and at least one minimal left ideal is obtained, and the relation between this decomposition and the kernel of S is discussed. (Received October 13, 1955.)

88. G. W. Medlin: *On limits of the real characteristic roots of matrices with real elements.*

In a series of papers published over the past few years A. Brauer has obtained bounds for the characteristic roots of an arbitrary matrix. In this paper it is shown that the bounds for the real roots of matrices with real elements can often be improved. (Received October 6, 1955.)

89. W. M. Perel: *Relative primeness of ordinary integers.*

It is proved that *the probability that a set of r positive integers, chosen arbitrarily, be relatively prime is* $1/\zeta(r)$. (ζ is the Riemann Zeta-function.) Define $P_r(N)$ to be the number of ordered sets of the form $(X_1, X_2, \cdots, X_r)$ such that (a) the X_p are integers satisfying $1\leqq X_p\leqq N$, where N is a fixed positive integer and (b) the g.c.d. $(X_1, \cdots, X_n)=1$ for each set. The probability that an arbitrary set of r positive integers, all less than or equal N, be relatively prime is $P_r(N)/N^r$. The theorem is proved by passing to the limit. (Received October 6, 1955.)

90. Irving Reiner: *Matrix roots of algebraic equations.*

Let $f(x)\in Z[x]$ be a monic nth degree polynomial, irreducible over the rational field Q, where Z denotes the ring of rational integers. The problem considered here is that of determining the integral classes of $m\times m$ integral matrices X for which $f(X)=0$. Latimer and MacDuffee (Ann. of Math. vol. 34 (1933), pp. 313–316) showed that there are exactly k classes of nonderogatory integral solutions of $f(X)=0$, where k is the class number of $Z[\theta]$, θ a zero of $f(x)$. Under the hypothesis that $Z[\theta]$ gives all algebraic integers in $Q(\theta)$, it is proved here that the above k classes of solutions of $f(X)=0$ are the only indecomposable classes, so that every integral solution of

$f(X)=0$ is integrally similar to a direct sum of representatives of these classes. The concept of a Maschke representation plays a central role in the proof. (Received September 14, 1955.)

91. Chien Wenjen: *On pseudo-normed* algebras.*

A pseudo-normed* algebra A is an algebra over complex numbers K topologized by a family $\mathcal{V}$ of pseudo-norms V such that $V(x)=0$ for $x\in A$ and all $V\in\mathcal{V}$ only if $x=0$ and with an involution * satisfying $V(xx^*)\geqq k_V V(x)V(x^*)$ for all $V\in\mathcal{V}$ $(k_V>0)$. (1) A commutative complete pseudo-normed* algebra is equivalent to a complete subalgebra of the algebra $C(T, K)$ of all continuous complex-valued functions defined on a completely regular, locally compact Hausdorff space T with k-topology (compact-open). (2) Let $H=\sum_{V\in\mathcal{V}} H_V$ be a Hilbert space defined as the direct sum of the family of Hilbert spaces H_V. A complete pseudo-normed* algebra A with $k_V=1$ for all $V\in\mathcal{V}$ can be isomorphically mapped onto a complete subalgebra A_1 of the algebra of all linear transformations in a Hilbert space $H=\sum_{V\in\mathcal{V}} H_V$ such that if $x\in A$ maps to $X\in A_1$, then X is bounded in each H_V and $V(x)=\|X\|_V$ for each $V\in\mathcal{V}$, where $\|X\|_V$ denotes the norm of X in the space H_V. The spectral theorem for the unbounded self-adjoint linear transformations in a Hilbert space can be proved by means of the first representation theorem. (Received October 6, 1955.)

92. R. L. Wilson: *Concerning the matrix equation $AX-XB=C$.*

Solutions of the matrix equation $AX-XB=C$ have been given which require that both A and B be in Jordan form [D. E. Rutherford, Neder. Akad. Wetensch. vol. 35 (1932)] and which require canonical forms for neither of the coefficient matrices [M. H. Ingraham and H. C. Trimble, Amer. J. Math. vol. 63 (1941)]. This paper gives a solution of this equation in which either A or B, but not both, are in Jordan form. (Received August 16, 1955.)

ANALYSIS

93. R. C. James: *Linear functionals and reflexivity.*

It is shown that if B has an unconditional basis, then B is reflexive if and only if each linear functional defined on a subspace B_0 of B attains its maximum on the unit sphere of B_0. Thus B is reflexive if and only if each hyperplane H of B intersects any sphere in B_0 which is at zero distance from H, where B_0 is any subspace of B. (Received October 13, 1955.)

94. C. W. McArthur: *On relationships amongst certain spaces of sequences in an arbitrary Banach space.*

Let X denote an arbitrary Banach space and X^* denote its adjoint space. A sequence $\{s(i)\}$ in X has an ***invariant sum*** if and only if there is $x\in X$ such that $x=\lim_{n\to\infty}\sum_{i=1}^{n} s(i)$ and such that x is the sum of each of the convergent rearrangements of $\sum_{i=1}^{\infty} s(i)$. The known result that in any Hilbert space there is a sequence $\{s(i)\}$ with an invariant sum such that $\sum_{i=1}^{\infty} s(i)$ is not unconditionally convergent is extended to any Banach space with infinite dimension. Let $B(X)$ stand for the class of sequences $\{s(i)\}$ in X such that $\sum_{i=1}^{\infty}|f(s(i))|<\infty$ for all $f\in X^*$. Let $B_w(X)$ denote the subclass of $B(X)$ for which there is an $x\in X$ with $f(x)=\sum_{i=1}^{\infty} f(s(i))$ for all $f\in X^*$ and let $B_s(X)$ denote the subclass of $B(X)$ for which there is an $x\in X$ with $x=\sum_{i=1}^{\infty} s(i)$. Let $IS(X)$ stand for the sequences in X with invariant sum and let $U(X)$ denote those sequences $\{s(i)\}$ in X such that $\sum_{i=1}^{\infty} s(i)$ converges unconditionally. For any Banach space X, $U(X) \subset B_s(X) = IS(X) \cap B(X) \subset B_w(X) \subset B(X)$, and it is shown that these containments are proper when X is the Banach space, with usual norm, of sequences of real numbers which converge to zero. (Received October 10, 1955.)

95t. J. S. MacNerney: *Determinants of harmonic matrices.*

This paper is concerned with extensions of a theorem by H. S. Wall [Arch. Math. vol. 5 (1954) pp. 160–167]: if M is a 2×2 *harmonic matrix* and F corresponds to M then det $M = 1$ only in case $F_{11} = -F_{22}$. Theorem 1: If M is an $n \times n$ *harmonic matrix* and F corresponds to M then det $M(s, t) = \exp\left(\sum_1^n [F_{pp}(t) - F_{pp}(s)]\right)$. Theorem 2: If M is an $n \times n$ *quasi-harmonic matrix* and F corresponds to M and G is the continuous part of F then det $M(s, t) = \exp\left(\sum_1^n [G_{pp}(t) - G_{pp}(s)]\right)$. For the notion of a *quasi-harmonic matrix*, see Bull. Amer. Math. Soc. Abstract 61-1-99 [or paper, to appear in J. Elisha Mitchell Sci. Soc. vol. 71 (1955)]. (Received October 13, 1955.)

96t. J. A. Nohel and W. K. Ergen: *Real representations of solutions of a linear functional-differential equation.*

Consider the equation (1) $y'(t) = -\omega^2/\theta\int_0^\theta(\theta-h)y(t-h)dh + W(t)$, $('=d/dt)$, $t \geq \theta > 0$, where y, t, and W are real, and ω^2 and θ are real constants. Let $y(t)$ satisfy the given initial condition (2) $y(t) = q(t)$, $0 \leq t \leq \theta$. The following result is established: Let $q(t)$ be continuous on $0 \leq t \leq \theta$. Let there exist constants c_1 and c_2 such that $|W(t)| \leq c_1 \exp(c_2 t)$, $t \geq \theta$; then the solution of (1) satisfying (2) is given by (3) $y(t) = q(\theta) \cdot K(t-\theta) - \omega^2/\theta\int_0^\theta\{\int_0^r(r-\rho)K(t-\theta-\rho)d\rho\}q(r)dr + \int_\theta^t K(t-\tau)W(\tau)d\tau$, $t > \theta$, where (4) $K(t) = \lim_{T\to+\infty}(1/2\pi i)\int_{b-iT}^{b+iT}\exp(ts)1/\{s+\omega^2/s-(\omega^2/\theta s^2)[1-\exp(-\theta s)]\}ds$, b sufficiently large. Moreover, if $\omega \neq 2n\pi/\theta$, $n = \pm 1, \pm 2, \cdots$, there exists a constant $\lambda > 0$, such that $K(t) = O(\exp(-(\lambda/2)t))$ as $t \to +\infty$. This result is needed in the study of stability of solutions of nonlinear equations similar to (1) where $W(t)$ is replaced by a nonlinear "delay" term; see Nohel and Ergen: *Stability of solutions of a nonlinear functional differential equation*, Bull. Amer. Math. Soc. Abstract 62-1-97. Laplace transforms are used throughout and the results are similar to those of R. Bellman for differential-difference equations. (Received October 3, 1955.)

97. J. A. Nohel (p) and W. K. Ergen: *Stability of solutions of a nonlinear functional-differential equation.*

Consider the equation (1) $x'(t) = -\omega^2/\theta\int_0^\theta(\theta-h)x(t-h)dh - c/\theta\int_0^\theta(\theta-h)g[x(t-h)]dh$ $('=d/dt)$, $t \geq \theta > 0$, where t, x, g are real, and ω^2, c, θ are positive constants, and $x(t)$ satisfies the given initial condition (2) $x(t) = q(t)$, $0 \leq t \leq \theta$. The following result is established: Let $q(t)$ be continuous on $0 \leq t \leq \theta$; let $g(x)$ be continuous for all x with $g(x) = o(|x|)$ $(|x| \to 0)$, uniformly in t, $t \geq \theta$; then if $\omega \neq 2n\pi/\theta$, $n = \pm 1, \pm 2, \cdots$, the identically zero solution of (1) is asymptotically stable, hence stable. Note that the concept of stability first has to be defined. In the proof a real representation theorem for linear equations of the type (1), see Nohel and Ergen: *Real representations of solutions of a linear functional-differential equation*, Bull. Amer. Math. Soc. Abstract 62-1-96, is used to convert (1) to a workable integral equation. The situation here is similar to the case of differential-difference equations studied extensively by R. Bellman. A somewhat sharper result requiring the more stringent hypothesis $|g(x_2)-g(x_1)| \leq \epsilon|x_2-x_1|$, for $|x_1|$ and $|x_2|$ sufficiently small and where $\epsilon \to 0$ as $|x_1|$ and $|x_2|$ tend to 0, is also given. (Received October 3, 1955.)

98. B. J. Pettis: *On Moore-Smith limits and filtered functions.*

The purpose of the present paper is to present for set nets in product spaces theorems that yield in topological algebras the usual "limit of the sum (product) is the sum (product) of the limits" theorems for integrands, functions, derivatives, etc. (Received October 13, 1955.)

99t. Marvin Rosenblum: *Perturbation of the continuous spectrum and unitary equivalence.*

Let H be a Hilbert space and let A and B be possibly unbounded self-adjoint operators in H such that $B-A=\sum_{j=1}^{\infty}\lambda_j(\cdot,\phi_j)\phi_j$, where the ϕ_j are orthonormal and $\sum_{j=1}^{\infty}|\lambda_j|<\infty$. Suppose that the spectral measure of A is weakly absolutely continuous. Theorem: B is unitarily equivalent to A if and only if the spectral measure of B is weakly absolutely continuous. This generalizes the result quoted in Bull. Amer. Math. Soc. Abstract 61-1-155. (Received October 6, 1955.)

100. F. B. Wright, Jr.: *Note on a theorem of Hille and Zorn.*

Hille and Zorn (Ann. of Math. vol. 44 (1943) pp. 554–561) characterized the open additive semigroups of the plane E^2 by means of functions f which satisfy the functional relation $f(x+y)\leqq f(x)+f(y)$. Rosenbaum (Duke Math. J. vol. 17 (1950) pp. 227–247) extended this result to Euclidean n-space E^n. Their characterization depends on a suitable but not unique choice of basis in the vector space. In this note a characterization is given which does not depend on a choice of basis and which furthermore extends to any real linear topological space E. Call an extended real-valued function f on E "submodular" if $f(x+y)\leqq\max(f(x),f(y))$. Then there is a one-one correspondence between open semigroups S in E and upper semicontinuous submodular functions f on E, provided f is suitably normalized. This correspondence is given by the relation $S=[x\in E: 0\leqq f(x)<1]$. A necessary and sufficient condition is given that f define an angular semigroup S. (Received October 11, 1955.)

APPLIED MATHEMATICS

101. S. G. Campbell: *Numerical evaluation of certain statistical integrals.*

For numerical purposes it may be desirable to evaluate the function $w=f(s)$ for s in some domain D of a metric space by evaluating the approximating function $w^*=F[g(s^*)]$. Multiple decision procedure determines the proper transformation $s^*=s^*(s)$ which allows $g(s^*)$ to be calculated with optimum speed and accuracy, and also determines the mapping F, thus trading slow arithmetic operations for fast decision procedure. Comparison of such programs is effected by defining the relative computing efficiency $H(f)$ in terms of certain multiple integrals whose integrands turn out, even in simple cases, to be implicit solutions of transcendental equations. Optimal bounds $N_1(k)$ and $N_2(k)$ are constructed such that $N_1(k)\leqq H(f)\leqq N_2(k)$ where the integrand k is a step function related to f; these bounds are difficult to calculate, particularly in programs with variable multiple induction loops, and do not give the desired statistical information. Accordingly, a scheme of statistical integration is defined in which the computing program itself becomes the integrand and the quantities $N_1(k)$, $N(f)$, $N_2(k)$ may be calculated as accurately as desired. This procedure has been applied effectively to several computing programs; examples are given. (Received October 13, 1955.)

102. W. S. Loud: *Nonlinear systems with limiting and large-amplitude forcing.*

Nonlinear systems of the form $\ddot{x}+2b\dot{x}+(a^2+b^2)x=\phi(A\cos t-\alpha x-\beta\dot{x})$ where α, β, a^2, b and A are real, $\phi(u)=1$ if $u\leq -1$, u if $|u|\leq 1$, and -1 if $u\geq 1$, are studied for large values of the amplitude A. It is shown that with minor exceptions that for large A there exists a unique solution of the system, $x_0(t)$, with period 2π. Moreover given an integer n there exists a number A_0 depending on n such that for $A>A_0$ there is no solution of period $2n\pi$ other than $x_0(t)$, so that proper subharmonics do not exist. The exceptional case is the case $b=0$ with $n\alpha$ an integer. Use is made of the substitution $x=y/k$ $(k=1/A)$ and the degenerate system $\ddot{y}+2r\dot{y}+(a^2+b^2)y=-k\cdot\operatorname{sgn}(\cos t-\alpha y-\beta\dot{y})$. This work was supported in part by the Office of Ordnance Research. (Received October 6, 1955.)

103. E. P. Miles, Jr. (p) and Ernest Williams: *The Cauchy problem for linear partial differential equations with restricted boundary conditions.* I.

Consider a linear partial differential equation $^*\Phi(D, x_1, x_2, \cdots, x_n)u+\Psi(D, t)u=0$, Ψ an ordinary differential operator of order s with respect to t. A solution of * for the Cauchy data $u(x_1, x_2, \cdots, x_n, 0)=P(x_1, x_2, \cdots, x_n)$, $\partial^j u(x_1, x_2, \cdots, x_n, 0)/\partial t^j=0$, $j=1, 2, \cdots, s-1$, where P is annihilated by a $(k+1)$-fold iteration of the operator Φ is shown to be $\sum_{j=0}^{k}\Phi^j(P)\cdot u_j$; where the u_j are a set of solutions of the system of ordinary differential equations $^{\dagger}\Psi(u_j)+u_{j-1}=0$, $j=1, 2, \cdots, k$, $\Psi(u_0)=0$ with initial conditions $u_0(0)=1$, $u_j(0)=0$, $j\geq 1$ $d^m u_j(0)/dt^m=0$ for $m=1, 2, \cdots, s-1$ and all j. An analogous solution for * and the Cauchy Data $u(0)=0$, $u_t(0)=Q$, $s=2$ is obtained. These methods lead to the following improved form of the authors' basic set of solutions of the wave equation (Proc. Amer. Math. Soc. vol. 6, pp. 191–194). For each set of non-negative integers a, b, c, d, $a+b+c+d=n$, $d\leq 1$; $W_{a,b,c,d}=\sum_{j=0}^{[n/2]}\nabla^{2j}(x^a y^b z^c)\cdot t^{2j+d}/(2j+d)$. (Received October 11, 1955.)

104. E. P. Miles, Jr. and Ernest Williams (p): *The Cauchy problem for linear partial differential equations with restricted boundary conditions.* II.

The method of the preceding abstract applied to the Euler-Poisson-Darboux equation $^*\nabla^2 u-[u_{tt}+kt^{-1}u_t]=0$ with $u(x, y, z, 0)=P(x, y, z)$ gives, for $k>0$, a basic set of solutions similar to that given for the wave equation with $d=0$ and $t^{2j}/(2j)!$ replaced by $t^{2j}/(1+k)(3+k)\cdots(2j-1+k)\cdot 2^j j!$. If $k<0$ the system † associated with the solution of * has a solution $u_j=\sum_{n=0}^{j}{}' a_n t^{2j-2n+1-k}/2^{j-n}(j-n)!(1-k)\cdots(2j-2n+1-k)+t^{2j}/2^j j!(1+k)(3+k)\cdots(2j-1+k)$, $k\neq -1, -3, \cdots, -(2j-1)$, which is not unique since the a_n are arbitrary. These results for the E.P.D. equation are in agreement with those of A. Weinstein (Proceedings of the Fifth Symposium in Applied Mathematics, McGraw-Hill, 1954). For the vibrating string problem $a^2u_{xx}-u_{tt}=0$, $u(x, 0)=P(x)=\sum_{n=1}^{\infty}b_n\sin(n\pi x/L)$; $u_t(x, 0)=0$; $u(0, t)=u(L, t)=0$ the method leads to a formal solution which, after interchanging the order of summation, takes the usual form $u=\sum_{n=1}^{\infty}b_n\sin(n\pi x/L)\cos(n\pi at/L)$. (Received October 11, 1955.)

105. N. C. Perry and J. C. Morelock (p): *A statistical approach to the propagation of round-off error by division.*

In this paper a detailed derivation is given of the error in a quotient a/b caused by relatively large errors ϵ and ϵ' respectively in the numerator and denominator. a and b are constant and ϵ and ϵ' are assumed to be independent errors distributed in a rectangular distribution on the range from $-1/2$ to $+1/2$. This result is related to the work of Inman [Math. Gazette (1950)], who, however, takes a and b to be random variables which are large relative to ϵ and ϵ', and tends to give results only. (Received October 11, 1955.)

106t. J. P. Roth: *An application of algebraic topology to numerical analysis*, III. *Three theorems on networks.*

1. Let K be a 1-complex, L an isomorphism of the space of 1-chains of K (complex coefficient field) on the dual space of 1-cochains; L is said to be *ohmic* if $v \neq 0$ implies $vLv \neq 0$: the proof (Proc. Nat. Acad. Sci. U.S.A. vol. 41 (1955) pp. 518–521) for the existence of a solution to the "electrical" network problem actually utilized only the condition that L be ohmic, not the stronger, that L be power definite. 2. Proposition. Any linear system of equations can be represented as a network problem. 3. Proposition. Power is invariant under interconnection. The problem of this invariance has plagued the literature for 25 years. It is shown here to be a simple consequence of the transformations induced by the simplicial map which is the interconnection rather than the reverse as was previously attempted. (Received October 13, 1955.)

107. W. C. Royster (p) and S. D. Conte: *Finite difference approximations to a fourth order parabolic differential equation.*

Consider the fourth order parabolic differential equation (*) $u_x^4 + u_{tt} = 0$, $0 < x < 1$, $t > 0$, subject to the conditions $u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0$, $t > 0$, $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$. The exact solution of (*) is $u(x, t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n^2\pi^2 t$, where $a_n = 2\int_0^1 f(x) \sin n\pi x dx$ and $f(x)$ is assumed to satisfy certain continuity conditions. The forward difference approximation of (*) is (**) $\Delta_t^2 u(x, t) = -r^2 \Delta_x^4 u(x, t)$ where Δ_x^4 and Δ_t^2 denote the fourth central difference with respect to x and the second central difference with respect to t, respectively, and r is the mesh ratio defined by $\Delta t = r(\Delta x)^2$. Collatz (ZAMM vol. 31 (1951)) has shown that (**) is stable when $r \leqq 1/2$. It is shown in this paper that the solution of (**) converges for all $r > 0$ to the solution of (*) for certain restrictions on $f(x)$. An implicit difference approximation (***) $\Delta_t^2 u(x, t) = -r^2/4 \Delta_x^4 [u(x, t + \Delta t) + 2u(x, t) + u(x, t - \Delta t)$ is shown to be stable for all r. The solution of (***) converges to the solution of (*) under the same conditions on $f(x)$ as for the forward difference scheme. The numerical solution of (***) requires the solution of a system of $m-1$ equations in $m-1$ unknowns, where $m\Delta x = 1$, which must be solved at each t line in the grid. A procedure which employs a relatively small amount of machine time is given whereby the values of u can be computed. (Received October 13, 1955.)

108. W. S. Snyder: *A method of obtaining bounds for the solution of Volterra integral equations.*

If the kernel of a Volterra integral equation is non-negative and if values of a function f are not increased at any point of the range by iteration in the equation, then f and its iterate are upper bounds for the solution. A similar statement holds for lower bounds of the solution. A procedure is outlined by which such upper and lower bounds can be generated by a digital computer. These bounds may diverge rapidly in certain cases but tend to a common limit as the mesh size used for the integration tends to zero if the kernel is sufficiently smooth. The method seems practical for a wide class of Volterra equations, differential-difference equations, and differential equations. The advantage of the method is that the computing machine in effect does the error analysis, and the round-off errors may be handled to give completely rigorous bounds for the solution. (Received October 20, 1955.)

TOPOLOGY

109. L. W. Anderson: *A theorem on topological chains.*

A topological lattice is a pair of continuous functions $\vee: L\times L\to L$ and $\wedge: L\times L\to L$ where L is a Hausdorff space and $\vee$ and $\wedge$ satisfy the usual conditions stipulated for a lattice. It is shown that a compact connected one-dimensional topological lattice is a chain. Cohomological methods are required to prove this result. It is also shown that a compact connected topological chain is one-dimensional. (Received October 13, 1955.)

110. T. R. Brahana: *Products of quasi-complexes.*

It is shown that the product space $X\times Y$ of two quasi-complexes X and Y is a quasi-complex. Quasi-complexes are spaces defined by Lefschetz [*Algebraic topology*, p. 322], in connection with the theory of fixed points, and as a consequence of this result, the Lefschetz fixed point theorem holds for products of quasi-complexes. The proof consists in showing the constructability of projections and anti-projections among the nerves of product coverings of $X\times Y$. Projections can be constructed canonically. Given two anti-projections $\omega_{\beta\alpha}$ and $\bar{\omega}_{\beta'\alpha'}$ on nerves of coverings of the spaces X and Y, anti-projections are constructed by forming a chain mapping of the chains of the nerve of the product covering onto the chains of the cartesian product [Eilenberg-Steenrod, *Foundations of algebraic topology*, p. 66], then mapping these chains by means of a suitable chain mapping to the subgroup generated by chains of the form $\sigma^p\times\tau^q$ (cartesian product of simplexes), and then applying the map $(\omega_{\beta\alpha}\times\bar{\omega}_{\beta'\alpha'})$ to this subgroup. The composition map is the desired anti-projection. (Received October 13, 1955.)

111. C. E. Capel (p) and W. L. Strother: *A new proof of a fixed-point theorem of Wallace.*

A. D. Wallace [Bull. Amer. Math. Soc. vol. 47 (1941) pp. 757–760] has shown that if T is a tree (in a combinatorial sense) and F is a continuous, multi-valued function on T to itself, such that the image of each point is connected, then F has a fixed point. Using L. E. Ward's characterization of a tree in terms of a partially ordered space [Proc. Amer. Math. Soc. vol. 5 (1954) pp. 992–994], an order-theoretic version of Wallace's theorem is given. (Received October 10, 1955.)

112. C. E. Capel and W. L. Strother (p): *Knaster continua.*

Let X denote a compact connected metric space. A continuous multi-valued point connected function F on the unit interval to a space X will be called an M-arc in X. Utilizing Kelley's results on hyperspaces [*Hyperspaces of a continuum*, Trans. Amer. Math. Soc. vol. 52 (1942)] indecomposability can be characterized as follows: A compact metric space X is indecomposable if and only if there is a pair of connected closed sets A and B in X such that for every M-arc $F: I\to X$ joining A and B there is a point a in I such that $F(a)$ is the whole space X. Also X is a Knaster continuum if and only if M-arcs in X are unique. (Received October 10, 1955.)

113. Haskell Cohen: *Fixed points in products of ordered spaces.*

Let X and Y be compact connected ordered topological spaces and $Z = X \times Y$. It is shown that Z is unicoherent and that if A is a closed set in Z, either some component of A projects onto X or a component of the complement of A projects onto Y. Using these properties it is shown that Z has f.p.p. (the fixed point property; i.e. every continuous function from Z to itself has a fixed point). This last result can be rephrased and strengthened to read: The product of two ordered spaces with f.p.p. has f.p.p. That this is actually a strengthening can be seen by means of an unpublished example due to C. T. Yang of a noncompact ordered space with the fixed point property. (Received October 11, 1955.)

114. H. C. Griffith: *Tamely imbedded curves in three-space.*

A 1-manifold J in Euclidean 3-space R has property $\mathfrak{P}_t$ at the point $x \in J$ provided in each neighborhood of x there is a topological 2-sphere K enclosing x which is locally tame modulo J and meets J in a set of cardinality equal to the order of x in J. Also, J has property $\mathfrak{Q}_t$ at x provided there is a topological disk D which is locally tame modulo J and meets J in the closure of a neighborhood (relative to J) of x. It is shown that having property $\mathfrak{P}_t$ (property $\mathfrak{Q}_t$) at x is equivalent to having property $\mathfrak{P}$ (property $\mathfrak{Q}$) at x, where J has property $\mathfrak{P}$ (property $\mathfrak{Q}$) at x provided the 2-sphere K (the disk D) exists as above and in addition is locally polyhedral modulo J. Thus results involving properties $\mathfrak{P}$ and $\mathfrak{Q}$ remain valid when properties $\mathfrak{P}_t$ and $\mathfrak{Q}_t$ are substituted. For example, a necessary and sufficient condition that an arc or simple closed curve be tamely imbedded in R is that it have both property $\mathfrak{P}_t$ and property $\mathfrak{Q}_t$ at each of its points (see Harrold, Griffith, and Posey, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954)). (Received October 7, 1955.)

115. O. G. Harrold, Jr.: *Some consequences of the recent approximation theorem of Bing.*

By the use of an approximation theorem proved by R. H. Bing (*Approximating wild surfaces with polyhedral ones*) and results of *A characterization of tame curves*, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 12–34, the following results are established. An arc in 3-space is contained in the interior of a 2-cell if and only if it lies on the boundary of some bounded, open, connected, ulc[1] subset of space. If a simple closed curve is polyhedral and bounds a disk in space, it bounds a polyhedral disk. Finally, the requirements that certain sets referred to in the definitions of properties $\mathfrak{P}$ and $\mathfrak{Q}$ discussed in the paper mentioned above be "almost polyhedral" may be dropped completely. (Received October 10, 1955.)

116. J. G. Horne, Jr.; *On semi-groups and compactification theorems.* Preliminary report.

P. S. Alexandrov defined a topology for the set $\gamma(X)$ of completely regular ends in the family of open subsets of a completely regular space X, and showed $\gamma(X)$ homeomorphic to the Stone-Čech compactification $\beta(X)$of X (Rec. Math. [Mat. Sbornik] N.S. vol. 5 (47) (1939) pp. 403–423). For an arbitrary space X we show directly that $\gamma(X)$ is a compact Hausdorff space and that every continuous real function on X has unique extension to $\gamma(X)$. For this, we use the notion of *p-ideal*. A *p-ideal* in a commutative semi-group is an *o-ideal* M of Milgram (Duke Math. J. vol. 16, p. 377) having the additional property: if e is a unit for $f \notin M$ then $ek \in M$ only if $k \in M$. Sufficient conditions are obtained in order that the space of p-ideals be com pact. Equivalence classes of p-ideals are defined and a one-to-one correspondence exhibited between equivalence classes of p-ideals in the bounded continuous functions under multiplication (lattice "cap") and $\gamma(X)$. It follows that if X is completely regular then the bounded continuous on X under multiplication (lattice "cap") characterize $\beta(X)$. This generalizes a theorem of Milgram (this is Blair's generalization of Kaplansky's result (Bull. Amer. Math. Soc. Abstract 61-3-436)). (Received October 13, 1955.)

117. R. J. Koch (p) and A. D. Wallace: *Notes on a paper of J. A. Green.*

A mob S is a Hausdorff topological semigroup. We say that S is stable if (i) $a, b \in S$ and $Sa \subset Sab$ imply $Sa = Sab$ and (ii) $a, b \in S$ and $aS \subset baS$ imply $aS = baS$. Using stability in place of minimum conditions on principal left and right ideals, J. A. Green's Theorem 8 [Ann. of Math. vol. 54 (1951)] is proved, so that in a stable mob each simple element is regular, and d-classes coincide with f-classes. If S has a unit and satisfies the above mentioned minimum conditions then S is shown to be stable. Further, the class of stable mobs is shown to include compact mobs, commutative mobs, mobs which are the union of groups, and mobs in which each orbit closure $(x, x^2, x^3, \cdots)^*$ is compact and each principal ideal has an idempotent generator. (Received October 13, 1955.)

118. Ceslovas Masaitis: *Complement of a finite dendrite in the compactified 3-space.* Preliminary report.

The following result is proved: If a dendrite d can be represented as a finite union of simple arcs in a compactified 3-space R and if d has property P then $R \backslash d$ is an open 3-cell. (The property P is defined in *The enclosing of simple arcs and curves by polyhedra* by O. G. Harrold, Jr., Duke Math. J. vol. 21, pp. 615–622. The technique of the proof of the result above is essentially that of the paper quoted.) Let $\{B_\alpha\}$, $\alpha = 1, 2, \cdots, \alpha'$, be the set of branchpoints of d and $r(B_\alpha)$ be the order (Menger-Urysohn) of B_α in d. The set of disjoint 2-spheres $K(B_\alpha)$ each meeting d in $r(B_\alpha)$ points is constructed. If int $K(B_\alpha)$ denotes the bounded component of $R \backslash K(B_\alpha)$ (the complement of $K(B_\alpha)$ in R) then $a_\alpha = d \cap \operatorname{int} K(B_\alpha)$ and $\{d_{\alpha\beta}\}$, $\beta = 1, 2, \cdots, r(B_\alpha)$, is the set of components of $d \backslash \bigcup_{\alpha=1}^{\alpha'} a_\alpha$ with the end-points on $K(B_\alpha)$. Further, $S(d, \epsilon) = \{A \mid \rho(A, d) < \epsilon\}$. $K_{\tau-1,\sigma-1}$ is a 2-sphere which has the following properties: (i) $K_{\tau-1,\sigma-1} \subset S(d_1 \epsilon)$, (ii) $\bigcup_{\alpha=1}^{\tau-1} (a_\alpha \cup \bigcup_{\beta=1}^{r(B_\alpha)} d_{\alpha\beta}) \cup a \cup \bigcup_{\beta=1}^{\sigma-1} d_\beta \subset K_{\tau-1,\sigma-1}$, (iii) $K_{\tau-1,\sigma-1}$ is locally polyhedral mod d, (iv) $K_{\tau-1,\sigma-1} \cap d$ consists of finite number of points. It is shown by induction on τ and σ: (a) that $K_{\tau-1,\sigma-1}$ exists for every $\tau \leq \alpha'$ and $\sigma \leq r(B_\tau) + 1$, (b) that if $K = K_{\alpha'-1,r(B_{\alpha'})}$ then $d \subset \operatorname{int} K$. Hence $d \cap K = \square$ and therefore K is polyhedral. Repetition of the argument of Theorem 3 in the paper quoted above proves the assertion. (Received October 10, 1955.)

119. P. S. Mostert (p) and A. L. Shields: *On a class of semigroups on E_n.*

Theorem A. Let S be the half-line $[0, \infty)$. Suppose S is a (topological) semigroup with zero at 0 and identity at 1. Then (i) if S contains no other idempotents, its multiplication is the ordinary multiplication of real numbers on $[0, \infty)$; (ii) if S contains an idempotent different from 0 and 1, then it contains a largest (in the sense of the regular order of real numbers) such idempotent e. Moreover, $e < 1$, $[e, \infty)$ is a subsemigroup iseomorphic to $[0, \infty)$ under the usual multiplication of real numbers, and $[0, e]$ is an (I)-semigroup. (The complete structure of (I)-semigroups has been obtained by the authors.) (Iseomorphism = simultaneous isomorphism and homeo morphism.) Theorem B. Let S be a semigroup with identity on E_n, $n>1$, and B a compact connected submanifold of dimension $n-1$. If B is a subsemigroup containing the identity of S, then (i) $n=2$ or 4 and B is a Lie group which is S^1 if $n=2$ and S^3 if $n=4$ (where S^i denotes the i-sphere); (ii) there exists a subsemigroup J contained in the center of S which is isomorphic to a semigroup of the type described in Theorem A; (iii) the subsemigroup J meets each orbit $xB=bx$ of B in exactly one point, and $JB=S$; (iv) if 0 denotes the zero for J, then 0 is a zero for S, and $(J-\{0\})\times B$ is isomorphic to $(J-\{0\})B=S-\{0\}$ in the natural way. (Received October 13, 1955.)

120. P. S. Mostert and A. L. Shields (p): *On continuous multiplications on the two-sphere.*

By a semigroup we mean a topological semigroup, that is, a Hausdorff space with a continuous, associative multiplication. A. D. Wallace has shown that a compact connected manifold which is a semigroup with identity must be a group. Hence the two-sphere cannot be a semigroup with identity. S. T. Hu has raised the question whether the two-sphere can be a semigroup with a circle subgroup. The answer is yes, and one can give both abelian and non-abelian examples. The following theorem can be proven. Theorem: Let S be a semigroup which is topologically the two-sphere, and suppose B is a circle subgroup. Let $S\backslash B=A\cup C$ (the symbol $\backslash$ denotes set-theoretic difference). Then, (i) S has a zero, 0, and $0\notin B$. (ii) If $0\in C$, then $Q=C\cup B$ is an ideal of S, and the elements of Q commute with every element of S. (iii) There exists an element $a\in A$, $a\notin SS$, such that $ax=xa=0$ for every $x\in S$. Remark: Q is a two-cell semigroup in which the boundary is a group. The authors have previously classified all the possible multiplications for Q. (Received October 13, 1955.)

121. A. D. Wallace: *The structure of one-dimensional Peano clans.*

A mob is a Hausdorff space together with a continuous associative multiplication. A clan is a compact connected mob with unit. A Peano space is one which is compact, connected, locally connected and metrizable. It is known that a one-dimensional homogeneous clan is a group. In this note it is shown inter alia that a one-dimensional Peano clan is either a tree ($=$dendron, dendrite, acyclic continuous curve) or contains just one simple closed curve which is the minimal ideal of the clan. It is shown that the well-known dyadic tree will support a continuous associative multiplication so that its set of end points is also the maximal subgroup containing the unit. (Received October 5, 1955.)

122. A. B. Simon: *n-cyclic elements.* Preliminary report.

In this note some of the results of G. T. Whyburn on *Cyclic elements of higher order* are generalized to the case of arbitrary compact Hausdorff spaces and any non-zero coefficient group. The main result obtained is the following: Let (X, X_0) be a compact Hausdorff pair and fix a coefficient group, G. Let $\{K_\delta^{n-1}\}_{\delta\in\Delta}$ be the collection of all $n-1$ cyclic elements (mod X_0) of X. There exists a closed subset, A, of X such that if U is open about A then, for all but a finite number of $\delta\in\Delta$, $K_\delta^{n-1}\subset U$. If this set A is a T^{n-1} (mod X_0), i.e. the $n-1$ groups of A are trivial, then $I^*: H^n(X, X_0)\approx\sum_{\delta\in\Delta} H^n(K_\delta^{n-1}, K_\delta^{n-1}\cap X_0)$ where I^* is the product of the i_δ^* induced by inclusion and $\sum_{\delta\in\Delta}$ is the weak cartesian product. (Received October 13, 1955.)

J. H. Roberts,
Associate Secretary

THE NOVEMBER MEETING IN MILWAUKEE

The 520th meeting of the American Mathematical Society was held at the Milwaukee Branch of the University of Wisconsin on Friday, November 25, and at Marquette University on Saturday, November 26. There were 130 registrations of which 107 were members of the Society. The sessions began at 9:00 A.M. Friday and ended at noon on Saturday.

The Committee to Select Hour Speakers for Western Sectional Meetings had invited Professor E. H. Spanier of the University of Chicago to address the Society. Professor S. S. Chern presided at the lecture, *Aspects of duality in homotopy theory*, which was presented on Friday morning at 11:00 A.M.

Presiding officers at the sessions for contributive papers were Professors W. T. Reid, Morris Marden, and H. P. Pettit.

There was a tea for members of the Society and their guests on Friday afternoon at the Milwaukee Branch of the University of Wisconsin, and on Saturday Marquette University played host at a luncheon celebrating the 75th anniversary of their founding. On Friday evening the Society were guests of the Miller Brewing Company for a tour of the brewery followed by a buffet supper and beer at the Miller High Life Inn. The occasion was probably unique in the long and distinguished annals of the Society.

Abstracts of papers presented at the meeting follow. Those with "*t*" after the abstract number were presented by title. In the case of joint papers, the name of the author who read the paper is followed by (p). Mr. Ornstein was introduced by Professor Irving Kaplansky and Professor L. V. Toralballa by Professor J. W. T. Youngs.

ALGEBRA AND THEORY OF NUMBERS

123*t*. K. T. Chen: *Integration of paths—geometric invariants and a generalized Baker-Hausdorff formula.*

Let the path α in the affine m-space R^m be given by $x_i = \alpha_i(t)$, $i = 1, \cdots, m$, $a \leq t \leq b$. Starting from the line integral $\int_\alpha d(i) = \int_\alpha dx_i$, one defines inductively, for $p \geq 2$, $\int_\alpha d(i_1, \cdots, i_p) = \int_a^b [\int_{\alpha^t} d(i_1, \cdots, i_{p-1})] d\alpha_{i_p}(t)$ where α^t denotes the portion of α with the parameter ranging from a to t. The integral remains unchanged when α undergoes a translation. It is further observed that the totality of $\int_\alpha d(i_1, \cdots, i_p)$, with p fixed, forms a pth order tensor associated with α under linear transformations of R^m. If all $\int_\alpha d(i_1, \cdots, i_p) = 0$ for $p \leq r$, then α is said to be of lower central class r, which is an affine invariant of α. The paths of lower central class 1 are characterized by being closed; and those of lower central class 2, by enclosing zero algebraic area when projected on any plane in R^m. Let $X_1, \cdots, X_m$ be noncommutative indeterminates. Define $\theta(\alpha) = 1 + \sum \int_\alpha d(i) X_i + \sum \int_\alpha d(i, j) X_i X_j + \sum \int_\alpha d(i, j, k) X_i X_j X_k$

$+ \cdots, i, j, k, \cdots = 1, \cdots, m$. Obtain $d \log \theta(\alpha^t)/dt$ from $\log \theta(\alpha^t)$ by differentiating each of the coefficients of the formal power series $\log \theta(\alpha^t)$ with respect to t. By expressing $d \log \theta(\alpha^t)/dt$ in terms of $\sum (d\alpha_i/dt)X_i$ and $\log \theta(\alpha^t)$, one proves that $\log \theta(\alpha)$ is a Lie element. As a corollary follows the Baker-Hausdorff formula. (Received October 10, 1955.)

124. N. J. Divinsky: *Commutative subdirectly irreducible rings.*

It is shown that a commutative subdirectly irreducible ring is either (α) a field; (β) a ring bound to its maximal nilideal with every element being a divisor of zero, and when either the descending or the ascending chain condition is present, it is nilpotent; (γ) a ring bound to its maximal nilideal, possessing elements which are not divisors of zero, and possessing a unity element when either the descending or the ascending chain condition is present. An example of a ring of type (γ) is given which does not possess a unity element, and an unpublished result of McCoy's is included which proves that every subdirectly irreducible ring (not necessarily commutative) without a unity element is contained in one with a unity. The important consequence of these considerations is the fact that every commutative ring is isomorphic to a subdirect sum of fields, and rings bound to their maximal nilideals. (Received October 11, 1955.)

125*t*. Franklin Haimo: *Endomorphisms of some dihedral groups.*

Let H be an abelian group in which the torsion subgroup P is a direct summand. Let G be a semi-direct product of H and I_2, the group of two elements, which forms the obvious generalization of the dihedral groups. The endomorphisms and the automorphisms of G are examined. The endomorphisms fall into eight classes. One of these classes has the two-sided ideal property with respect to another class which is closed under endomorphism composition. The holomorph of H can be injected into the automorphism group of G. (Received October 13, 1955.)

126*t*. Melvin Henriksen: *On the equivalence of the ring, lattice, and semigroup of continuous functions.*

Let $C(X)$, $L(X)$, $S(X)$ denote respectively the ring, lattice, and multiplicative semigroup of all continuous real-valued functions on a topological space X. Then, for any topological spaces X, Y, the following are equivalent: (i) $C(X)$ and $C(Y)$ are isomorphic. (ii) $L(X)$ and $L(Y)$ are isomorphic. (iii) $S(X)$ and $S(Y)$ are isomorphic. The proof follows in a straightforward way from a theorem of T. Shirota (Osaka Math. J. vol. 4 (1952) pp. 121–132). (The author was supported by the National Science Foundation, contract no. NSF-G1129.) (Received October 13, 1955.)

127. Erwin Kleinfeld: *The structure of standard rings.*

In his paper *Power-associative rings* (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593) Albert introduced a class of rings called standard. The defining relations of such rings are (1) $(wx, y, z)+(xz, y, w)+(wz, y, x)=0$, and (2) $(x, y, z)+(y, z, x)-(y, x, z)=0$, where (a, b, c) denotes $(ab)c-a(bc)$. Interchange w and x in (1) and subtract from (1) in order to obtain the identity (3) $(wx-xw, y, z)=0$. In the present paper the author considers a class of rings more general than standard rings, namely those subject to (2) and (3). For these he obtains structure theorems. The principal result is that simple rings satisfy (2) and (3) if and only if they are either associative or commutative rings. This generalizes Albert's result that simple, finite-dimensional algebras are standard if and only if they are either associative or Jordan algebras. (Received August 30, 1955.)

128. Donald Ornstein: *Splittability and modularity in dual vector spaces.*

The author considers two properties of dual vector spaces over arbitrary division rings, splittability and modularity. If the sum of two closed subspaces is closed the space is modular and if every closed subspace admits a closed complement such that the annihilators span the dual the space is splittable. If one space has either property so does its dual. If every subspace of E is closed then obviously E is modular. A sort of converse is proved: if a pair E, F of dual vector spaces is modular then in one space every subspace of countable dimension is closed. With separability conditions the converse is proved. Examples are given of dual vector spaces in which both spaces have nonclosed subspaces. A typical result about splittability is that the direct sum of two splittable spaces need not be splittable. (Received October 13, 1955.)

129. A. M. Yaqub: *On the theory of ring-logics.*

Boolean rings and Boolean logics ($=$ Boolean algebras) are equationally interdefinable in a familiar way (Stone, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 37–111). The theory of ring-logics (Foster, University of California Publications vol. 1 (1951) pp. 385–396) raises this interdefinability and the entire Boolean theory to a more general level. Here a ring R is studied modulo K, where K is an arbitrary transformation group in R. The Boolean theory results on choosing $K=C=$ "Boolean group," generated by $x^*=1-x$ (order 2, $x^{**}=x$). More generally in a commutative ring with identity the *natural group* N, generated by $x^\frown=1+x$ (with $x^\smile=x-1$ as inverse) is of particular interest. Thus specialized to N, a commutative ring with identity, $(R, \times, +)$ is essentially a *ring-logic*, mod N, if the $+$ of the ring is equationally definable in terms of its N-logic $(R, \times, {}^\frown, {}^\smile)$. It was shown (Foster, loc. cit.) that p-rings are ring-logics mod N; p^k-rings were also treated (Foster, Ann. Scu. Norm. Pisa vol. 5 (1951) pp. 279–300). In the present paper it is shown that (A) p^k-rings, (B) periodic rings, (C) the rings of residues mod n (n arbitrary) and others are all ring-logics mod N. The proofs use some elementary number-theoretic results, notably Fermat-Euler's Theorem. (Received July 14, 1955.)

ANALYSIS

130*t*. Stefan Bergman: *An operator generating solutions of a class of partial differential equations.*

In Trans. Amer. Math. Soc. vol. 68 (1950) pp. 461 ff., operators transforming functions of two complex variables into solutions of certain differential equations in three variables are considered. In the present paper an operator P is introduced which transforms expressions $X^m g(Z)$ into (complex) solutions Φ of the equation $L(\Phi)=\Delta_3\Phi+F\Phi=\Phi_{XX}-\Phi_{ZZ^*}+F(Z, Z^*)\Phi=0$. Here $g(Z)$ is an entire function of the complex variable Z, while $F(Z, Z^*)$ is an entire function of Z and Z^*. The operator P is defined as follows: Let $q^{(m,m-2s,s+p)}$, (Z,Z^*) $[q^{(m,m-2s,s+p)}(Z, 0)=0$, except for $s=p=0]$, $m=0, 1, 2, \cdots$, be functions which satisfy the following recursive relations: $q^{(m,m,0)}=1$, $T_{m,s}q^{(m,m-2s+2,s-1)}-q_{Z^*}^{(m,m-2s,s)}=0$, $T_{m,s}q^{(m,m-2s+2,s+p-1)}-q_{Z^*}^{(m,m-2s,s+p)}-q_{ZZ^*}^{(m,m-2s,s+p-1)}+Fq^{(m,m-2s,s+p-1)}=0$, $s=0, 1, 2, \cdots, [m/2]$; $p=1, 2, \cdots$; $T_{m0}=0$, $T_{m,s}=(m-2s+2)\cdot(m-2s+1)$; $[m/2]=m/2$ for an even m, $=(m-1)/2$ for an odd m. Let $\psi^{(m,m)}=g(Z)$

$+\sum_{p=1}^{\infty} q^{(m,m,p)}(Z, Z^*)J_p(Z; g)$; $\psi^{(m,m-2s)} = \sum_{p=0}^{\infty} q^{(m,m-2s,s+p)}(Z, Z^*)J_{s+p}(Z; g)$; $J_m(Z; g) = \int_0^Z \cdots \int_0^{Z_{m-1}} g(Z_m)dZ_m \cdots dZ_1$. Then $\Phi(X, Z, Z^*) = \mathrm{P}[X^m g(Z)] = \sum_{s=0}^{[m/2]} X^{m-2s}\psi(Z, Z^*)^{(m,m-2s)}$ are solutions of $\mathrm{L}(\Phi) = 0$. They possess the property that $\Phi(X, Z, 0) = X^m g(Z)$, $\Phi(X, 0, Z^*) = \text{const } X^m$. (Received November 21, 1955.)

131*t*. H. D. Brunk, G. M. Ewing, and W. R. Utz, Jr.: *Some Helly theorems for monotone functions.*

The principal results of this paper are Helly theorems on the convergence of monotone functions of n real variables. A monotone function, in this paper, means a function monotone in its n individual variables and having all second differences nonnegative or non-positive. A typical theorem asserts that if $\{F_q(x_1, x_2, \cdots, x_n)\}$ is a sequence of monotone functions on an n-dimensional rectangle S and if A is a positive number such that $|F_q(x_1, x_2, \cdots, x_n)| \leq A$ for $q = 0, 1, 2, \cdots$, then there exists a sequence of integers $q_0 < q_1 < q_2 < \cdots$ and a monotone function $F(x_1, x_2, \cdots, x_n)$ bounded by $\pm A$ such that $\lim_{i\to\infty} F_{q_i}(x_1, x_2, \cdots, x_n) = F(x_1, x_2, \cdots, x_n)$ on S. In proving such theorems it is convenient to first discover limitations on the discontinuities of monotone functions. Such limitations are also discovered for functions monotone in their n variables (but without a condition involving second, or higher, differences) and it is shown that for such functions a Helly theorem analogous to the one quoted above is impossible. (Received October 10, 1955.)

132*t*. Evelyn Frank: *A new class of continued fraction expansions for the ratios of generalizations of hypergeometric functions.* Preliminary report.

A new class of continued fraction expansions for the ratios of two generalizations of hypergeometric functions is described in detail. A number of such ratios are expanded, and the convergence of the corresponding continued fractions is determined as well as the functions to which the continued fractions converge in various regions. Recurrence relations between the numerators and denominators of the approximants of these continued fractions are found. Many special relations are obtained between functions, finite continued fractions, and infinite continued fractions. Specialized continued fraction expansions and specialized functions are discussed, and integral representations for such continued fractions are found. (Received October 6, 1955.)

133. R. E. Fullerton: *The Cesari inequality for surfaces defined over a two-dimensional manifold.*

Let M be a compact two-dimensional metric manifold with boundary and S a Fréchet surface defined over M. Let T be a representation of S and let f be a function defined on E_3 having the property that for any two points p_1, p_2 in E_3, $|f(p_1) - f(p_2)| \leq K|p_1 - p_2|$. Let $D^-(t)$ be the subset of M defined by $f(T(p)) < t$ for $p \in [S]$ and let $l(t)$ be the length of the boundary of $D^-(t)$ defined in a generalized sense by use of prime ends in a manner similar to that used by Cesari [*Surface area*, Annals of Mathematics Studies, Princeton, to appear]. It is shown that $l(t)$ is an integrable function over $(-\infty, \infty)$ and that if $L(S)$ is the Lebesgue area of S then $KL(S) \geq \int_{-\infty}^{\infty} l(t)dt$. This generalizes a similar theorem of Cesari [loc. cit.] for the case in which M is a simply connected plane set. (Received October 10, 1955.)

134. F. W. Gehring: *The converse of Fatou's theorem for positive harmonic functions.*

Suppose that $u(z) = u(x, y)$ is non-negative and harmonic in the upper half plane

$y>0$ and let $\partial u(z)/\partial\xi_d = \cos d\, u_x(z) + \sin d\, u_y(z)$. A simple extension of the Wiener tauberian theorem leads to very direct and elementary proofs for results of which the following are special cases. THEOREM 1: *Suppose that* $0<a,\ b<\pi$, *that* $a\neq b$, *and that* $u(te^{ia})\to A$ *and* $u(te^{ib})\to B$ *as* $t\to 0+$. *Then for each* $0<c<\pi$ *and each* d, $t(\partial u/\partial\xi_d)\cdot(te^{ic})\to C\sin(d-c)$ *as* $t\to 0+$ *where* $C=(B-A)/(b-a)$. THEOREM 2: *Suppose that* $0<a,\ c<\pi$, *that* $c\neq d \bmod \pi$, *and that* $u(te^{ia})\to A$ *and* $t(\partial u/\partial\xi_d)(te^{ic})\to C\sin(d-c)$ *as* $t\to 0+$. *Then for each* $0<b<\pi$, $u(te^{ib})\to B$ *as* $t\to 0+$ *where* $B=A+C(b-a)$. Next suppose that $f(z)=u(z)+iv(z)$ is bounded and analytic in the upper half-plane $y>0$. From Theorem 2 and a well known tauberian theorem we can immediately obtain the following variant of a result due to Lindelöf. THEOREM 3: *Suppose that* $0<a,\ b<\pi$ *and that* $u(te^{ia})\to A$ *and* $v(te^{ib})\to B$ *as* $t\to 0+$. *Then for each* $0<c<\pi$, $f(te^{ic})\to A+iB$ *as* $t\to 0+$. (Received October 13, 1955.)

135. P. E. Guenther: *Formulation of a Riemann problem.*

A Riemann problem for a linear homogeneous q-difference system $Y(qx) = A(x)Y(x)$, x, q complex, $|q|>1$, with elements of $A(x)$ rational functions of x was stated and solved by G. D. Birkhoff in the case that the related characteristic equation has finite nonzero roots and that $Y(x)$ is free from logarithmic terms (Proceedings of the American Academy of Arts and Sciences vol. 49 (1913) pp. 559–568). The present note considers the system of order two after reduction to the canonical form developed by G. D. Birkhoff and the author (Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 218–222). For systems of order two there are three distinct canonical forms, one of which corresponds to the case mentioned above for which the Riemann problem has been solved. As a first step in formulating a Riemann problem for the remaining cases the behavior of the solution $Y_0(x)$ about zero and $Y_\infty(x)$ about infinity is shown to depend upon the corresponding behavior of the solution of a typical summation problem. For certain types of such typical solutions the characteristic constants involved in the representation of the q-periodic elements of $P(x)$ in the equation $Y_0(x) = Y_\infty(x)P(x)$ are determined. (Received October 11, 1955.)

136. D. L. Guy: *Marcinkiewicz theorem for Hankel transforms.* Preliminary report.

Let $-1/2<\alpha<1/2$, and $\mathfrak{M}_\alpha(f)^2=\int_0^\infty f(x)^2x^{2\alpha}(1+x^{2\alpha})^{-1}dx$. Let $\mathfrak{M}_\alpha$ be the class of f for which $\mathfrak{M}_\alpha(f)^2<\infty$. Given $k(x,t)$, let $K(f;t)=\int_0^\infty f(x)k(x,t)dt$. Let $\bar{f}_\nu(t)$ be the Hankel transform of $f(x)$ of order $\nu\geqq -1/2$. If $g_\nu(t)=\psi(t)f_\nu(t)$, let $\Psi^{(\nu)}(f(x))=g(x)$. Finally, $\psi(t)\in M_N(C)$ if $\psi(t)$ is bounded by C and the variation of $\psi(t)$ on $[2^{n-1}, 2^n]$ is no more than C for all $n\geqq N$. *Theorem* 1. If $k(x,t)$ is bounded and K is a bounded transformation of L^2 into L^2, and if $f\in\mathfrak{M}_\alpha$, $0<\alpha<1/2$, then $\int_0^\infty K(f;t)^2(1+t)^{-2\alpha}dt \leqq A_\alpha\mathfrak{M}_\alpha(f)^2$ [Hirschman, *A note on orthogonal systems*, to appear in Pacific Journal of Mathematics]. *Theorem* 2. If $f\in\mathfrak{M}_\alpha$, $0\leqq\alpha<1/2$, then there are positive constants $A'(\alpha,\nu)$, $A''(\alpha,\nu)$, such that $A'\leqq\int_0^\infty\int_0^\infty[f_\nu(t)t^{-\nu-1/2}]^2Q_{\alpha,\nu}(t,u)\,dt\,du/\mathfrak{M}_\alpha(f)^2\leqq A''$ where $Q_{\alpha,\nu}(t,u)=\lim_{\epsilon\to 0+}\int_1^\infty\int_0^\infty \exp(\epsilon^2-x^2)x^{1-\nu}J_\nu(xt)J_\nu(xu)J_\nu(xa)a^{-\nu-1-2\alpha}dx\,da$ [Hirschman, *Weighted quadratic norms*, etc., to appear in Canadian Journal of Mathematics]. *Theorem* 3. If $\psi(t)\in M_N(C)$ and $f(x)\in\mathfrak{M}_\alpha$, $1<1/2<\alpha<1/2$, then $\mathfrak{M}_\alpha(\Psi^{(\nu)}(f(x)))\leqq A\mathfrak{M}_\alpha(f(x))$, where A depends only on α, ν, and N [Hirschman, Ibid.; Marcinkiewicz, Studia Math. vol. 8, p. 78]. (Received September 20, 1955.)

137t. W. T. Guy, Jr.: *On a Mathieu integral transform.* II.

A Mathieu integral transform is defined using a Mathieu function as the kernel.

A short table of properties of this transform and transforms of some special functions are developed. (Received October 13, 1955.)

138. W. B. Jurkat and Alexander Peyerimhoff (p): *On a Tauberian theorem for resultants.*

The paper deals with the following problem stated by P. Erdős: For what numbers α can the relation $s_n = n + O(n^{\alpha})$ be derived from the relation $\sum_{\nu=0}^{n} s_{n-\nu} a_\nu = n^2/2 + O(n)$, $a_\nu \geq 0$ ($s_n = \sum_{\nu=0}^{n} a_\nu$). Erdős has shown that $s_n = n + o(n)$ holds whereas $s_n = n + o(n^{1/2})$ cannot be derived. V. G. Avakumovic proved that the relation $s_n = n + O(n^{3^{1/2}-1+\epsilon})$ holds for any $\epsilon > 0$. The paper gives a proof of the relation $s_n = n + O(n^{2/3} \log n^{1/3})$. (Received October 13, 1955.)

139. E. G. Kimme: *A Taylor expansion for a certain class of functionals.*

Let Σ be the set of real functions of bounded variation on the whole real line, continuous from the right and vanishing at $-\infty$. Let $h_s(t)$ denote that element of Σ which is zero for $t < s$ and one for $t \geq s$. If F is a functional of $x(t)$ in Σ, define $D_s F[x] = \lim_{a \to 0} (F[x + a h_s] - F[x])/a$ if the limit exists. Repeated "derivatives" are defined in the usual way: $D_{s_1 s_2} F[x] = D_{s_1}(D_{s_2} F[x])$ and so on. Using techniques involving approximations to functions $x \in \Sigma$, it is shown that under certain rather involved continuity hypotheses on $F[x]$ and $D_s F[x]$ one has $F[x] = F[0] + \int_0^1 \int_{-\infty}^{\infty} D_s F[\mu x]\, dx(s) d\mu$. Iteration of this result leads to a finite series with terms of the form $\int_{-\infty}^{\infty} \cdots (k) \cdots \int_{-\infty}^{\infty} D_{s_1 \cdots s_k} F[0] dx(s_1) \cdots dx(s_k)/k!$ and a remainder, reminiscent of Taylor's theorem and the classical Volterra extension thereof. A sufficient condition for F to be representable as an infinite series of such terms is exhibited. If F is linear, these results reduce to a version of Riesz's theorem on the representation of linear operators on a Hilbert space. If $F[x] = f(x(a_1), \cdots, x(a_N))$ the stated results reduce to the N-dimensional Taylor theorem for f. (Received October 10, 1955.)

140t. Jacob Korevaar: *Notes on differential equations.* I. *Analog of a nonlinear Tauberian theorem of Erdős.*

In connection with his proof of the prime number theorem Erdős obtained the following nonlinear Tauberian theorem (J. Ind. Math. Soc. vol. 13 (1949) pp. 131–147). When (1) $\sum_{k=1}^{n} a_k(s_{n-k} + k) = n^2 + O(n)$ and (2) $a_k \geq 0$, then (3) $s_n = \sum_{1}^{n} a_k = n + O(1)$. Setting $\sum a_k e^{-kx} = y(x)$, relation (1) implies that (4) $y' - y^2 = -2/x^2 + O(1/x)$ as $x \downarrow 0$. Question. Do the solutions of (4) satisfy an estimate just as sharp as (3)? The answer is yes: equation (4) implies that either $y = 1/x + O(1)$ or $y = -2/x + O(1)$ as $x \downarrow 0$. For the proof one uses the familiar substitution $y = -u'/u$ to obtain the linear equation $u'' + \{-2/x^2 + O(1/x)\} u = 0$. Analogs of well-known methods (Bellman, *Stability theory of differential equations*, New York, 1953) now show that either $u = c_1/x + O(1)$ and $u' = -c_1/x^2 + O(1/x)$ with $c_1 \neq 0$, or $u = c_2 x^2 + O(x^3)$ and $u' = 2c_2 x + O(x^2)$ with $c_2 \neq 0$, or $u \equiv 0$. This proves the result. It is conjectured that (1) without (2) implies that either (3) holds, or $s_n = -2n + O(1)$. (Received October 3, 1955.)

141t. Karel deLeeuw: *On a type of convexity in the space of n complex variables.*

Let C^n be the space of n complex variables. For X a bounded subset of C^n denote by $\hat{X}$ the set of all y in C^n that satisfy for every polynomial P, $|P(y)| \leq \operatorname{Sup} |P(x)|$

where x runs over X. Let S be a finitely generated subsemigroup of an abelian group. Denote by Hom (S, C) the set of all multiplicative maps from S into the complex numbers that are one on the unit of S if S contains a unit. Let $x_1, \cdots, x_n$ be generators of S. Imbed Hom (S, C) in C^n by identifying ϕ in Hom (S, C) with the point $(\phi(x_1), \cdots, \phi(x_n))$. The image of Hom (S, C) is a variety in C^n that will be denoted by V. Let T be all $(z_1, \cdots, z_n)$ in V that satisfy $|z_i| \leqq 1$ for all i. X is called *V-circular* if whenever $(w_1, \cdots, w_n)$ is in X and $(z_1, \cdots, z_n)$ is in T, the point $(z_1w_1, \cdots, z_nw_n)$ must be in X. Theorem: If X is a subset of V, is V-circular and contains the point $(1, \cdots, 1)$, then $\hat{X}$ is the intersection of V and all sets of the form $|z_1^a z_2^b \cdots| \leqq r$ that contain X. (Received October 10, 1955.)

142. Karel deLeeuw: *On homogeneous Banach algebras.* I. *Identification of the maximal ideal space.*

Let G be a compact abelian group, G^* its dual. A Banach algebra A of continuous functions on G (with multiplication pointwise and topology stronger than pointwise convergence) is a ***homogeneous algebra*** if it contains the constant functions, separates points of G, is invariant under translations by G, and is generated by the semigroup S_A of characters that it contains. Let Char (S_A), supplied with the compact-open topology, be the set of all not identically zero homomorphisms of S_A into the complex numbers. Since distinct multiplicative functionals on A induce distinct homomorphisms of S into the complex numbers, the maximal ideal space $M(A)$ is imbedded in Char (S_A). Let $\hat{A}$ be the Gelfond representation of A. For $\chi \in S_A$ let $p(\chi)$ be the norm of χ in $\hat{A}$. The result of the above abstract is used for THEOREM: $M(A)$ consists of those $\phi \in$ Char (S_A) satisfying $|\phi(\chi)| \leqq p(\chi)$ for each $\chi \in S_A$. THEOREM: Let S be any subsemigroup of G^* containing the unit and generating G^*. Then the compact subsets of Char (S) that can occur as maximal ideal spaces of homogeneous algebras on G are those that contain $\phi \equiv 1$ and are the intersection of sets of the form $\{\phi : |\phi(\chi)| \leqq r \text{ for fixed } \chi \in S\}$. (Received October 10, 1955.)

143t. L. F. Meyers: *Transformations of Hermitian functionals characterized by an extremum property.*

Let A and B be positive definite Hermitian bilinear functionals on a Hilbert space H. Consider all linear transformations T of H into itself carrying A into B, so that $A(Tu, Tv) = B(u, v)$ for all u, v in H. Loewner [Bull. Amer. Math. Soc. Abstract 52-1-19] showed that among all such transformations, the *one* differing least from the identity transformation I in the sense of absolute norm based on A or B is, in matrix-style notation, $+(A^{-1}B)^{1/2}$, under certain conditions. New results are the following. The same minimal transformation is obtained (although without uniqueness) if $T-I$ is to have smallest bound with respect to A or B, and B is bounded with respect to A. If A and B are joined by a sufficiently smooth curve in the space of positive definite functionals, then a noncommutative product integration of minimizing transformations can be carried out along the curve, an approximation to the integral being $\prod_{j=a}^{b} (C_{j-1}^{-1}C_j)^{1/2}$, where the C_j's are points of the curve. (Received October 10, 1955).

144. W. T. Scott: *Linear Baecklund transformations.*

For a linear Baecklund transformation between two sets, v and y, of m functions in $n (n \geqq 2)$ independent variables, there are determined conditions on the coefficients of the transformation under which the transformation links certain linear systems of partial differential equations in v and in y. The coefficient conditions are shown to be conditions that a certain system be passive in the sense of Riquier. In the case where $n=2$ it is possible to duplicate some results of Loewner for $m=2$, $n=2$, and to characterize completely the transformation coefficients and the linked systems. In the general case it is shown that linked first order linear systems have the same characteristic equation. In the case where $m=1$ it is shown that the type of linked equations is the same, thus duplicating a result of Resch for $m=1$, $n=3$. (Received October 13, 1955.)

145. L. V. Toralballa: *On locally distributive algebras.*

A locally distributive algebra is a system $\Sigma(C, +, x)$ satisfying: 1. $(C, +)$ is a real Banach space. 2. (C, x) is closed. 3. x is continuous with respect to both factors, simultaneously. 4. $|a\times b| \leq |a|\cdot|b|$. 5. $(\alpha a)\times(\beta b)=(\alpha\beta)(a\times b)$. 6. $|a+b|^2+|a-b|^2=2|a|^2+2|b|^2$. Define the angle (a, b) between two non-zero vectors by $\cos(a, b)=(|a|^2+|b|^2-|a-b|^2)/2|a||b|$. Given vectors a and c and a class B of vectors b of the same length, then for every $\epsilon>0$, there exists a $\delta>0$ such that if $(a, b)<\delta$ then $|c(a-b)-(ca-cb)|<\epsilon$. Given vectors a, c then for every $\epsilon>0$ there exists a $\delta>0$ such that if $|a-b|<\delta$ then $|c(a-b)-(ca-cb)|<\epsilon$. If S is a sufficiently smooth rectifiable curve in Σ and f is a function on Σ to Σ continuous over S then $\int_S f(s)ds$ exists. (Received October 11, 1956.)

146. F. M. Wright: *Some formulas related to certain C-fractions.*

Let $\{k_j\}$ and $\{p_j\}$ $(j=0, 1, 2, \cdots)$ be given infinite sequences of integers with $k_0=1$, $p_0=-1$ and $k_j\geq 0$, $p_j\geq 0$ for $j=1, 2, 3, \cdots$. Let $\{\alpha_n\}$ $(n=1, 2, 3, \cdots)$ be the sequence of integers with $\alpha_1=1$ and such that if k is any non-negative integer then: (i) $\alpha_n=1$ for $n=2\cdot\sum_{j=0}^{k}(k_j+p_j)+i$ $(i=2, 3, \cdots, 2k_{k+1}+1)$; (ii) $\alpha_n=2$ for $n=2\cdot\sum_{i=0}^{k}(k_j+p_j)+2k_{k+1}+i$ $(i=2, 3, \cdots, 2p_{k+1}+1)$. Further conditions imposed on the sequences $\{k_j\}$, $\{p_j\}$ assure that the sequence $\{\alpha_n\}$ is infinite. The author considers here C-fractions $K_{(n)}[c_nw^{\alpha_n}/1]$ which are either nonterminating or terminating with $[2\cdot\sum_{j=0}^{m+1}(k_j+p_j)+1]$ partial quotients, where m is some given non-negative integer. Let $C(w)$ denote the above C-fraction or the above C-fraction with a partial quotient $cw/1$ attached according as this C-fraction is non-terminating or terminating. Formulas involving the coefficients of $C(w)$ and the coefficients in its corresponding power series $\sum_{n=0}^{\infty}u_nw^{n+1}$ corresponding, in the most part, to results discovered by W. T. Scott (Ann. of Math. vol. 51 (1950) pp. 56–67) are obtained in a relatively simple and direct manner. These formulas, which contain determinant formulas for the coefficients c_n of $C(w)$ such that $\alpha_n=2$, are then used to obtain determinant formulas for the remaining coefficients c_n and information concerning the determinants $\Delta_{1,n}$. (Received October 10, 1955.)

APPLIED MATHEMATICS

147*t*. C. C. Elgot: *Complex least squares.*

The least square solution of a set of linear equations with complex coefficients is often accomplished by applying theorems and techniques expounded by T. Banachiewicz to the set of equivalent real equations. Not only are the theorems and the techniques extendible to the complex case, but less than half the number of real operations is required to compute the solution of the normal equations within the complex field rather than within the real field. A useful relationship among the weights of the real unknowns, apparently not previously observed, becomes obvious. The technique in the complex case is as readily adaptable to high speed machine computation as in the real case. (Received October 10, 1955.)

148. A. C. Eringen: *A transform technique for boundary value problems in 4th order partial differential equations.*

Let $u(x, \lambda_n)$ be the eigenfunction of any fourth order linear differential equation $Lu+\lambda u=0$, $Lu=u^{(iv)}+(pu')'+qu$ $(a\leqq x\leqq b)$, and four homogeneous uncoupled boundary conditions at two ends $x=a$ and $x=b$. The system is assumed to be self-adjoint. Define T-transform by $f_T(\lambda_n)=N_n^{-2}\int_a^b f(x)u(x, \lambda_n)dx$ where N_n is a normalization factor. One has the inversion theorem: $f(x)=\sum_n f_T(\lambda_n)u(x, \lambda_n)$. It is shown that $\{Lf\}_T=\lambda_n f_T(\lambda_n)+B\{f, u(x, \lambda_n)\}$ where B contains boundary values. With the use of this result one can solve two-point boundary-value problems concerning inhomogeneous boundary conditions in fourth order partial differential equations. An example is a vibrating beam with time dependent boundary conditions and space dependent initial and final conditions. (Received September 19, 1955.)

149. P. C. Hammer (p) and A. H. Stroud: *Numerical integration over simplexes.*

In this paper two sets of formulas are given for numerical evaluation of integrals over the n-simplex. One formula holds precisely for the general quadratic polynomial and is based on $n+1$ evaluation points. The other formula holds for the cubic polynomial and is based on $n+2$ evaluation points. These formulas are affine invariant. (Received October 11, 1955.)

150*t*. J. P. Roth: *An application of algebraic topology to numerical analysis*, IV. *Tearing versus partitioning, case histories.*

The first example is a 40 branch network of the type used for solving Poisson's equation (Kron. J. of App. Phys. vol. 24 (1953) pp. 965–980). Number of multiplications required for the general solution: by tearing 250, Maxwell's mesh method coupled with (a) partitioning 1170, (b) Gauss elimination 4000. "Multiplying-out" operations were also less for tearing. Storage requirements for the general solution were considerably less, a change in the original data necessitates relatively few changes in the solution. The second example is a 56 branch network with a symmetric impedance tensor. In this example tearing exceeds standard methods even more, but for the third example tearing is useless. (Received October 13, 1955.)

GEOMETRY

151*t*. M. M. Day: *Some criteria for inner product spaces.*

One result given has as a special case the following generalization of results of Day (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 320–337), Schoenberg (Proc. Amer. Math. Soc. vol. 3 (1952) pp. 961–964), and Kasahara (Proc. Japan. Acad. vol. 30 (1955) pp. 846–848). Let E be a normed linear space and let $\sim$ be one of the relations $=$, $\geqq$, or $\leqq$ (the same one throughout E). Then any one of the following conditions is necessary and sufficient that E be an inner product space: For each pair x, y of points of E of norm one there is a λ depending on x and y such that $0<\lambda<1$ and $1\sim\lambda(1-\lambda)\|x-y\|^2+\|\lambda x+(1-\lambda)y\|^2$. Another result extends a characterization of L. M. Blumenthal (Pacific Journal of Mathematics vol. 5 (1955) pp. 161–167). Let E be a complete, externally convex, metric (or a normed linear) space; then E is isometric with an inner product space if and only if E has the queasy euclidean four-point property: If p and $r \in E$, there is a q metrically between p and r such that for every $s \in E$ the quadruple p, q, r, s is isometrically embeddable in a euclidean plane. (Received October 10, 1955.)

STATISTICS AND PROBABILITY

152. G. E. Baxter: *A Wiener process distribution of the arc-sine law type.*

Let $\{x(t), 0 \leq t < \infty\}$ denote the Wiener process, i.e. a Gaussian stochastic process with $E\{x(t)\} = 0$, $E\{x(t)x(s)\} = \min\{s, t\}$, and with $x(t)$ a real-valued continuous function of t. Furthermore, let $m\{A\}$ denote the Lebesgue measure of the t-set A. For any $\epsilon > 0$, define the random variable $T(x, \epsilon)$ to be the least upper bound (possibly infinite) of the T-set for which $\lim_{t \to T^-} \operatorname{sgn}[x(t) - \epsilon t] = 1$. It is shown that the joint probability density of the distribution function $P\{m(t : x(t) > \epsilon t) \leq u, T(x, \epsilon) \leq v\}$ is $(\epsilon/(2\pi v^3)^{1/2}) \exp\{-\epsilon^2 v/2\}$, $v > u \geq 0$, zero otherwise. The corresponding marginal distributions are of particular interest. $P\{m(t : x(t) > \epsilon t) \leq u\} = \int_0^u \int_w^\infty (\epsilon/(2\pi v^3)^{1/2}) \cdot \exp\{-\epsilon^2 v/2\} dv dw$ is a result along the line of the famous arc-sine law of Erdös and Kac [Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1011–1020]. On the other hand $P\{T(x, \epsilon) \leq v\} = \int_0^v (\epsilon/(2\pi w^3)^{1/2}) \exp\{-\epsilon^2 w/2\} dw$ trivially implies the strong law of large numbers for the Wiener process. (Received October 13, 1955.)

153. R. J. Buehler: *Estimation of a product of two binomial parameters by confidence intervals.*

The product P_1P_2 of two binomial parameters arises when two elements operate in parallel (either must succeed) or in series (each must succeed). If both elements come from the same population, P_1P_2 becomes P^2, and (Neyman) confidence intervals for estimation of P^2 are given by a trivial application of well known formulas for estimating P. Otherwise a nuisance parameter appears, and many more-or-less reasonable systems of intervals can be obtained. Conditions leading to uniqueness are studied after specializing to the case of highly reliable parallel elements (P_1 and $P_2 \ll 1$). The conventional solution for estimating P suggests three conditions, "symmetry," "regularity," and "shortness"; and these eliminate most of the less reasonable solutions. One more condition is added, making the solution unique; and a table of intervals is obtained, based on a Poisson approximation, which is valid when P_1 and $P_2 \ll 1$ and the sample sizes are moderate or large. (Received October 6, 1955.)

154. M. Z. Krzywoblocki: *On the application of Kampé de Fériet's relation: Hilbert spaces and random functions to turbulence in compressible fluids.*

Kampé de Fériet was the first to apply the results of the theory of the random functions to the theory of the homogeneous turbulence. Suppose that we have a probability measure defined on an abstract space Ω; let $T_t\omega$ be a group of measure preserving transformations of the space Ω onto itself, possessing the property $T_{s+t} = T_sT_t$. Let $f(\omega)$ denote a real measurable function defined on the measure space (Ω, S, μ); $F(t, \omega) = f(T_t\omega)$ defines a stationary random function of the variable t. The harmonic analysis of that process can be obtained by means of the harmonic analysis of the group of unitary transformations of the Hilbert space L_2 onto itself, defined by $U_tg(\omega) = g(T_t\omega)$. Kampé de Fériet applied this relation to the theory of turbulence in an incompressible fluid. The author of the present paper applied this relation to the theory of turbulence in compressible fluids. (Received September 30, 1955.)

155. J. V. Talacko: *Some conditional distributions of the Brownian plane area.*

In the study of Brownian motion from the point of stochastic processes, the Wiener's Gaussian stochastic variables may be used. Consider a Brownian plane area, bounded by an arc and its chord, given in parametric form by $x = X(t)$, $y = Y(t)$; $0 \leq t \leq 2\pi$; where $X(t) = X(t_0) + \xi(t - t_0)^{1/2}$, $t > t_0$, t is a continuous parameter and ξ is the Gaussian random variable. This area may be evaluated by a stochastic integral with an almost sure limit: $A = \sum_{n=1}^{\infty} [\xi_n(\eta_n' - \eta' 2^{1/2}) - \eta_n(\xi_n' - \xi' 2^{1/2})]$ and with the characteristic function: $C_A(x; \sigma, \rho) = \pi v/\sinh \pi v \exp \{\rho^2/2 - (\rho^2/2)\pi v \cosh \pi v/\sinh \pi v\}$. It is non-Gaussian. The area depends on non-Gaussian distributions and, in particular, some conditional distributions are also non-Gaussian, but known as the Perks' functions: $\phi(x) = c(e^x + 2 \cos \psi + e^{-x})^{-1}$, $0 \leq \psi \leq \pi/2$, with the characteristic function $C(x, v) = (\pi/\psi) \sinh (\psi v)/\sinh (\pi v)$. (Received October 10, 1955.)

TOPOLOGY

156*t*. R. H. Bing: *A simple closed curve which pierces no disk.*

A simple closed curve J is said to pierce a disk D if J links the boundary of D and $J \cdot D$ is a single point. An example is given of a simple closed curve in E^3 which links no disk in E^3. Essential use is made of the result that if M is a 2-manifold in E^3 and f is a non-negative continuous function defined on M, then there is a 2-manifold M' and a homeomorphism h of M onto M' such that the M' is piecewise linear where fh^{-1} is positive and the distance between x and $h(x)$ is less than $f(x)$. (Received October 10, 1955.)

157. R. H. Bing: *Periodic homeomorphisms of E^3 which are not orthogonal.*

It is shown that for each integer n there is a homeomorphism of period n of E^3 onto itself such that the set of fixed points of the homeomorphism is a wild line. (Received October 10, 1955.)

158. L. J. Heider: *A lattice characterization of compact Hausdorff continuous function spaces.*

Let L denote a lattice within which a sublattice R, lattice-isomorphic to the lattice of all real numbers, has been designated with element r of R corresponding to real number ρ. Describe as a real lattice homomorphism of L every real-valued function ϕ defined on all of L and such that: $\phi(f \vee g) = \phi(f) \vee \phi(g)$, $\phi(f \wedge g) = \phi(f) \wedge \phi(g)$, $\phi(r) = \rho$ for all f, g in L and all r in R. In particular if $L = C_s^*(X)$ where $C_s^*(X)$ is a sublattice of the lattice $C^*(X)$ of all real-valued functions defined and bounded on a set X and $C_s^*(X)$ includes all constant functions $r(x) = \rho$, let these constant functions form the designated sublattice R. For such lattices the functions ϕ_x with $\phi_x(f) = f(x)$ determined by points x in the set X may be called point real lattice homomorphisms. One may now prove: a necessary and sufficient condition that an abstract lattice L be lattice-isomorphic to a lattice $C^*(X^T)$ of all real-valued continuous functions on a compact Hausdorff space X^T is that L contain a sublattice R lattice-isomorphic to the real numbers such that if $X = \{x, y, \cdots\}$ be an index set for the set $\Phi = \{\phi_x, \phi_y, \cdots\}$ of all real lattice homomorphisms of L as containing R and $C^*(X)$ be the lattice of all real-valued bounded functions defined on X, then, under the definitions $f(x) = \phi_x(f)$, the elements f of L each determine distinct elements of $C^*(X)$ while every sublattice of $C^*(X)$ containing properly all functions thus determined has a real lattice homomorphism other than the point real lattice homomorphisms. (G. Birkhoff, *Lattice theory*, p. 176, problem 81.) (Received October 6, 1955.)

159. Melvin Henriksen (p) and J. R. Isbell: *Completions of uniformisable spaces.*

X always denotes a uniformisable space, $C(X)$ the algebra of all continuous real-valued functions on X, and $S(X)$ a subalgebra with unit of $C(X)$ such that for every $x \in X$, and every neighborhood U of X, there is an $f \in S(X)$ such that $f(x) = 0$, $f(X-U) = 1$. If X has nonmeasurable character, then every completion of X is a subspace $K(S)$ of the space of maximal ideals (in the Stone topology) of $S(X)$. Conversely, for any such $S(X)$, $K(S)$ is the completion of X in the coarsest structure making every element of $S(X)$ uniformly continuous. If, in addition, $f \in S(X)$ implies $|f| \in S(X)$, and $f \in S(X)$, $f^{-1} \in C(X)$ implies $f^{-1} \in S(X)$, then $K(S)$ is υX, the completion of X in the coarsest structure making every element of $C(X)$ uniformly continuous. Hence, if X, Y are Q-spaces (i.e., complete in this structure), and $S(X)$, $S(Y)$ satisfy the above hypotheses, and are isomorphic, then X and Y are homeomorphic. If $D(X)$ denotes the number of distinct completions of X, then $D(X) \geqq \max (|\beta X - \upsilon X|, 1)$ ($\beta X =$ Stone-Čech compactification). If there is an unbounded $f \in C(X)$, then $D(X) \geqq \exp \exp \aleph_0$. If X has non-measurable character, then $D(X) \leqq \exp |C(X)|$. (The authors were supported in part by the National Science Foundation.) (Received October 13, 1955.)

160. J. E. Kelley: *Concerning a problem of Sikorski.* Preliminary report.

Sikorski (Colloquium Mathematicum vol. 1 (1948)) announced that in spaces satisfying the Kuratowski axioms certain implications, and no others, hold in general between the following properties: B, second countability; $M(N)$ every well ordered increasing (decreasing) sequence of open sets is countable; I, every isolated set is countable; D, separability; S, every collection of mutually exclusive open sets is countable. Sikorski asked whether additional implications hold in general for normal spaces. Using results of Marczewski (Fund. Math. vol. 2 (1921) pp. 172–183) and of Alexandroff and Urysohn (Verhandelingen Der Koninklijke Akademie van Wetenschappen te Amsterdam, Eerste Sectie No. 1 (1929–1931)) one can construct normal spaces which are counter-examples for twelve of the eighteen possible additional implications. Every compact perfectly normal space has property M. (Received October 10, 1955.)

161t. I. S. Krule: *On connected subsets of a continuum endowed with a struct.*

If X is a Hausdorff space and L is a nonvoid, closed, transitive subset of $X \times X$, then L is called a *struct* on X. (For other terminology see Bull. Amer. Math. Soc. Abstracts 61-1-123 (the author) and 61-1-127 (A. D. Wallace).) If L is a struct on X, let K_L be the set of all L-minimal elements and let $G_L = \{x \in X : L(x) = X\}$. THEOREM: *If L is a continuous struct on the continuum X and if K_L is a proper connected subset of X, then no closed subset of G_L cuts X.* (Cf. R. J. Koch and A. D. Wallace, *Maximal ideals in compact semigroups*, Duke Math. J. vol. 21 (1954) pp. 681–685, LEMMA 2.) THEOREM: *If L is a monotone continuous struct on the continuum X, if $L(X) = X$, and if J is a maximal proper L-ideal containing K_L, then no subset of $X - J$ cuts X.* The conclusion of this theorem does not hold if L is not monotone even though L may be reflexive. As a corollary to this theorem there is the following theorem of Wallace (*Struct ideals*, Proc. Amer. Math. Soc. vol. 6 (1953) pp. 634–638, THEOREM 5): *Let L be a continuous monotone struct on the continuum X. Suppose that $a \in X - K_L$ and let C be the component of $X - L_a$ containing K_L. Then $L_a = a \cup (C^* - C)$ and if a is L-reflexive, then $a \in C^* - C$.* (Received October 11, 1955.)

162*t*. G. O. Sabidussi: *On a multiplication of graphs.*

Let $\{X_\alpha\}$, $\alpha \in A$, be an indexed set of graphs X_α given by $V(X_\alpha)$ and $E(X_\alpha)$, the sets of vertices and edges of X_α. By the product $\prod_{\alpha \in A} X_\alpha$ is meant the following graph Z: $V(Z) = \prod_{\alpha \in A} V(X_\alpha)$ (Cartesian product); $[P, Q] \in E(Z) (P, Q \in V(Z))$ if, for exactly one $\alpha_0 \in A$, $[P_{\alpha_0}, Q_{\alpha_0}] \in E(X_{\alpha_0})$ and $P_\alpha = Q_\alpha$ otherwise (here P_α, Q_α denote the αth component of P, Q respectively). It is shown that for all $\aleph_\alpha$ this multiplication together with graph addition turns the set $\mathfrak{X}_{\aleph_\alpha}$ of all nonisomorphic graphs X with cardinal $V(X) < \aleph_\alpha$ into a commutative semiring with zero and unit elements. This semiring $\mathfrak{X}_{\aleph_\alpha}$ does not possess a Euclidean algorithm; however, the decomposition of graphs $\in \mathfrak{X}_{\aleph_\alpha}$ into prime factors is unique. This implies that $\mathfrak{X}_{\aleph_\alpha}$ is a cancellation semigroup under graph multiplication. As an application of the unique prime factor decomposition it is shown that the automorphism group of a graph X is isomorphic to the automorphism group of the graph $\sum X_\alpha$ where the X_α are the prime factors of X. (Received October 10, 1955.)

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BOOK REVIEWS

Limit distributions for sums of independent random variables. By B. V. Gnedenko and A. N. Kolmogorov. Trans. and annotated by K. L. Chung. Cambridge, Addison-Wesley, 1954. 264+9 pp. $7.50.

This book is a compact and beautifully presented account of the subject matter of its title. Some of the material is either new or previously available only in the Russian language. The authors have strived with good success to make the development complete and definitive and at the same time direct and simple. The main principle guiding them is to attain an exhaustive treatment of results within their framework at the expense of a broader coverage of subjects (such as error terms in the non-equidistributed cases, multi-dimensional limit theorems, generalizations to dependent random variables, etc.) for which complete results await future research. The authors certainly have much that is pertinent and interesting to say about such matters but have refrained from disrupting the unity and conciseness of their exposition by doing so.

Perhaps the most striking feature of the book lies in the incisive simplicity of its organization and exposition. The material is very cleanly and economically motivated—there is no attempt to overwrite or "sell" the subject, just as there is no apology when the authors heuristically discuss here and there connections between some of the theorems and the theory of additive processes, etc. Indeed there may be a risk here, as with any work which is extremely elegant, that the uninitiate, while able to carry through the proofs, may not appreciate nor even understand the theorems.

The following is a brief outline. The book is divided into three parts: I Introduction, II General limit theorems, III Identically distributed summands. Each of these parts consists of three chapters and there are two appendices, one by the translator and one by J. L. Doob. In a short preface sketching the development and history of the subject the authors set forth their main problem—viz., that of delineating the class of all limit laws for sequences of sums of independent and individually negligible random variables and the refinements of these results when the basic random variables are specialized in various ways.

The first two chapters give a compact account of measure and probability theory, convergence of measures and distributions and the basic theorems on Fourier transforms. Here the authors adopt a very inclusive axiomatic framework rather than the usual exposition of probability measure in n-dimensional Euclidean space as would suffice here where the theorems are concerned exclusively with convergence in distribution (and where the notions of conditional probabilities and expectations do not arise). This treatment, evidently a precursor of a forthcoming book by Kolmogorov on measure theory, is simplified in various ways, e.g. by the introduction of "perfect" measures, and some of the results here and in the succeeding chapters are stated in terms of more general (signed) measures. In the first appendix J. L. Doob has elaborated on this measure theoretic setting.

The introduction closes with the third chapter on infinitely divisible distributions which are defined as those which can be factored into the convolution of an arbitrary number of identical components. The canonical representation for these distributions is obtained quite easily from this definition, and this representation, together with theorems on the convergence of sequences of infinitely divisible distributions, is the cornerstone of all the subsequent development.

In chapter 4, which opens part II, the basic problem of establishing necessary and sufficient conditions for a sequence of sums of independent and individually negligible random variables to converge in distribution and showing that when the conditions are met the limiting distribution is necessarily infinitely divisible, is solved. From these results there flow, in a natural way, the various forms for the central limit theorems (uniform negligibility of the summands, truncated moments and other variants of the Lindeberg condition, etc.) convergence to the Poisson law, and the general limit laws for cumulative sums.

The last part deals with sums of identically distributed random variables. A stable law is defined as one which is invariant, within a linear transformation, under compositions with itself. After determining the canonical form of the stable laws, the necessary and sufficient conditions that a random variable belong to the domain of attraction of one of them are deduced, and the domains of partial attraction of the infinitely divisible laws are discussed. The penultimate chapter deals with improvements of convergence to the normal distribution: asymptotic expansions, the Berry-Esseen theorems, convergence of densities and the extremal nature of the lattice variables. The final chapter treats lattice valued identically distributed random variables in the domain of attraction of the stable laws and the corresponding convergence of the discrete mass points.

The authors have in certain major respects altered and inverted the historical development of the topics. For example, the various forms of the central limit theorem are viewed from a subordinate, and in the reviewer's opinion, proper perspective. There are numerous simple examples and counter examples which neatly point up the text.

One obtains the impression on reading the book that the creation and execution of the subject is almost exclusively Russian with an occasional interloper here and there. There are only one or two major results which are unequivocally attributed to outsiders. And while it is perhaps pointless here to press these matters it seems equally profitless for everyone concerned to obtain the authors' view that Schwartz' inequality, Hermite polynomials, etc. ought to bear Russian names.

Though there is a slight nonuniformity in the printing, due to the photo reproduction of the displayed formulas, the main effect is quite pleasing. There are surprisingly few misprints and only a very few rough spots in the analysis, in the latter sections. (The definition of the symbol $A\backslash B$ in the footnote on page 16 should read "the set of points in A, but not in B.")

In closing a word of commendation on the translation. This is much more than a pedestrian transliteration of the Russian text, and the translator has really made a critical analysis, correcting errors, emending the text in numerous places, bridging lacunae, etc. In appendix II he has disproved and analyzed an erroneous theorem of the original. In addition he seems to have caught the flavor of the authors' trenchant style, and the book is pleasant to skim through for a surface taste of the topics. It is, in short, eminently readable.

It seems clear that this book will serve as an authoritative model of clarity, simplicity and definitiveness for some time to come.

DONALD A. DARLING

Methods of theoretical physics. By P. M. Morse and H. Feshbach. New York, McGraw-Hill, 1953. Part I, 22+998+40 pp.; Part II, 18+979 pp. $30.00 a set.

The present two-volume book is a gigantic compendium of methods of mathematical physics. It is truly staggering in scope and one cannot but admire the authors for accomplishing a task of this magnitude. The two most notable features are: 1. an excellent account of the Wiener-Hopf technique with many important applications; 2. a systematic use of the Green's function technique in dealing with differential equations of physics.

The arrangement of material follows a "handbook" pattern, i.e. the methods and techniques are not necessarily arranged according to logical interconnections but rather according to their specific use in physical theories. Thus mathematics is strongly subordinated to physical application and, because of this, methods tend to appear as "tricks" and the book as a collection of the latter rather than the former. This subordination, which was doubtless intentional, tends to emphasize the "handbook" character of the work and makes it less effective as a textbook. It also makes the book much more useful than inspiring. But if usefulness was the purpose of the authors they have achieved it to such a degree that the book will, for many years to come, be an indispensable addition to the bookshelf of everyone who might be even mildly concerned with mathematical physics.

A specific topical review of a book of nearly 2000 pages is clearly impossible. Even the table of contents is too lengthy for reproduction. Suffice it to say that all standard tools of both classical and modern physics, with the exception of group theory and probability theory, are at least touched upon. (Theory of matrices is apparently presupposed because there is no systematic presentation, but matrices are used throughout the book.) The principal emphasis is on differential equations. In some cases the authors have striven to completeness beyond and above the call of duty, such as the coverage of both the tensor and dyadic notations or the scrupulously detailed discussion of separation of variables in less common coordinate systems. The style is lucid and the level of mathematical rigour quite adequate for a work of this kind.

Each chapter ends with a list of extremely well selected problems and an excellent bibliography. Many chapters have been appended by summarizing tables and lists. For instance, Chapter II ends with a list of standard forms of some of the partial differential equations of theoretical physics, Chapter III with a tabulation of variational methods and Chapter XI with a short table of Laplace transforms.

There are curious omissions and inadequacies. For instance, the Huyghens principle is discussed briefly (one page!) in the chapter on Green's functions but never mentioned in an exhaustive (250 pages!) chapter on the wave equation. Nowhere is there even a hint that the principle is wrong in even-dimensional spaces. The method of steepest descent, so important in statistical physics, is treated briefly in the chapter on complex variables with a relatively simple example. It is then applied twice or three times in even simpler situations. Neither the example nor the applications contain a warning as to how mathematically delicate the method becomes as soon as one leaves the elementary realm. The whole question of continuous spectra is dealt with in a most cursory manner and in general one feels that the authors have tried to stay away from situations where conceptual difficulties exceeded the purely technical ones.

Since the book is destined to play an important part in the mathematical education of future physicists it is perhaps not inappropriate to take issue with the authors on the "philosophy" underlying their approach. Throughout history mathematics has served physics in dual capacity: (a) as a tool for obtaining numerical answers, (b) as an aid to deeper understanding and logical evaluation of the generalizations and abstractions one tends to draw from experiment and experience. Although no sharp line can be drawn between (a) and (b) the two aspects are clearly distinguishable especially if pushed to extremes. The authors have concentrated on (a) to such an extent that (b) is all but absent or lost. Thus in spite of the richness of material and the meticulous care taken in its presentation the reviewer feels that the work is inadequate as a treatise or even a textbook on mathematical methods of physics. It is a "super handbook" and an invaluable reference source, but with all its excellent features it somehow lacks cohesiveness and structure. If this book should become the "Riemann-Weber" of the twentieth century the reviewer will regretfully and nostalgically look back to the nineteenth.

MARK KAC

Primzahlen. By E. Trost. Basel, Birkhäuser, 1953. 95 pp. Paperbound, 13.50 Swiss Fr.

This little book is devoted to the elementary or nonanalytic theory of primes. Within this scope, the author has managed to cover many topics in an interesting way. There are no problems.

Among the things which are also found in almost any elementary book on the theory of numbers are the unique factorization theorem, the Fermat-Euler theorem, primitive roots, the Legendre symbol and its reciprocity law, the Möbius inversion formula, residue characters and Wilson's theorem. Also treated in some detail are the partial converse of Fermat's theorem, the representation of prime numbers by certain special forms of the kind x^2+dy^2, the Mersenne and Fermat numbers, prime representing functions and the calculation of $\pi(x)$ by Meissel's method.

Estimates for $\sum_{p\leq x} 1/p$ and the Tchebychef inequalities

$$0 < C_1 < \pi(x) \div (x/\log x) < C_2$$

are given as well as the corresponding inequalities for the functions $\vartheta(x)$ and $\psi(x)$. Also proved is Bertrand's "postulate" that there is a prime between n and $2n$; the following stronger result is also proved:

$$0 < C_3 < \{\pi(2n) - \pi(n)\} + \{n/\log n\} < C_4.$$

A number of sums of the type $\sum_{n\le x} f(n)$, $\sum_{n\le x} 1/f(n)$ for positive, increasing $f(u)$ are estimated. These, together with Abel's partial summation, enable the author to obtain the estimate

$$\sum_{n\le x} \frac{\Lambda(n)}{n} \log n = \frac{1}{2} \log^2 x + O(\log x)$$

and others of a similar, but more complicated, kind. These estimates prepare the way for an elementary proof of the prime number theorem. To this end, Selberg's estimate is obtained in the form

$$\psi(x) \log x + \sum_{n\le x} \psi(x/n)\Lambda(n) = 2x \log x + O(x).$$

The subsequent proof would have been clearer if the following frequently used result had been explicitly mentioned and proved: If $a_n \ge 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{x} o(a_n) = o(\sum_{n=1}^{x} a_n)$.

The proof of the prime number theorem which is given here is in the form due to Selberg. Inasmuch as Nagell's book gave the Erdös-van der Corput form, both proofs are now available for students. It must be stated, however, that neither of these proofs (each of which starts from Selberg's estimate) is likely to appear very transparent to beginning students. On the one hand, Selberg's argument rests on numerous partial summations (or integrations by parts), two initial estimates for $R(x) = \psi(x) - x$, and an involved deduction which leads to the self-improving inequality $|R(x)| < \alpha_n x$ if $x > \beta_n$; here, $\alpha_{n+1} = \alpha_n(1 - c\alpha_n^2) \to 0$. On the other hand, Erdös' argument makes use of the limits superior and inferior, and some rather complicated splitting of summation ranges and choices of various parameters. Although one can motivate the various steps in both proofs, these are still likely to seem unnatural to the novice. Some readers will prefer the version of Selberg's proof, given in the third edition of Hardy and Wright's book, which uses integrals in place of sums.

Trost also gives the elementary proof of H. N. Shapiro that

$$\sum_{p\le x,\, p\equiv a(m)} \frac{\log p}{p} = \frac{1}{\phi(m)} \log x + O(1)$$

provided $(a, m) = 1$. From this it follows at once that the arithmetic progression a, $a+m$, $a+2m$, $a+3m$, $\cdots$ contains an infinity of primes.

Brun's sieve and some of its applications are given. Although only the upper bound estimate is given or used, the author nevertheless seems to prefer Brun's sieve to Selberg's (which he mentions) because one can readily obtain lower bound estimates by Brun's method, whereas there are, to date, no ***published*** proofs for lower bounds resulting from Selberg's sieve. The author's exposition of the Brun sieve, given for the two-residue case, is one of his less successful endeavors. By means of the sieve, the convergence of the series of the reciprocals of the twin primes is proved. So also is the Schnirelmann result that each integer is the sum of a bounded number of primes.

By and large, the exposition is of a high order. This is particularly true of those sections in which the author gives references and résumés of known results. Such sections go beyond the usual textbook treatment of such matters and are to be commended.

On the other hand, there are typographical difficulties and more than the usual number of inaccuracies and falsehoods. The first of these arises from the author's style which frequently makes symbols appear in juxtaposition; for example, on page 40 one finds

$$F_n = p\ 3^{(p-1)/2} \equiv (3/p) = -1 \pmod{p}$$

where, at the very least,

$$F_n = p, \quad 3^{(p-1)/2} \equiv (3/p) = -1 \pmod{p}$$

would have been better. Many inaccuracies arise from the use of $<$ in place of $\leqq$.

As for the errors, we cite the worst. Theorems 11–14 are incorrect; for example, although Theorem 11 states that an odd number of the form $4m+1$ is a prime if and only if it is representable in exactly one way as the sum of two relatively prime squares, the simple counter-example $4\cdot 6+1=3^2+4^2$ shows that the "if" part of this theorem is false. On page 33, the author deduces from $x_1^2+dy_1^2=x_2^2+dy_2^2$ that $x_1-x_2=dut$ for some integers u and t; thus he claims that d divides x_1-x_2 whereas the example $10^2+12\cdot 1^2=2^2+12\cdot 3^2$, in which $d=12$ and $x_1-x_2=8$, disproves the claim. On page 56, the following statements, "proved" by the author, are incorrect:

$$p_1p_2\cdots p_{\pi(n)} \sim e^n, \quad \{1, 2, 3, \cdots, n\} \sim e^n, \quad (p_1p_2\cdots p_{\pi(n)})^{1/\pi(n)} \sim n.$$

Indeed, the first is equivalent to $\vartheta(n)=n+o(1)$ and hence is false; a like remark applies to the second. The correct form of the third result has the right side replaced by n/e and can be deduced from $\vartheta(n)=\pi(n)\{\log n-1+o(1)\}$ or from $\pi(n)=n/\log n+n/\log^2 n+o(n/\log^2 n)$ but not from $\vartheta(x)\sim\pi(x)\log x$ as the author states. On page 74, the statement (108) is false as it stands but it can easily be made correct. And on page 90, the result of Shapiro and Varga is incorrectly quoted as holding for all integers instead of for all sufficiently large integers. Finally, in Hilfssatz C on page 71, one cannot conclude that $(y, e^{\delta/2}y)$ is a subinterval of $(x, \rho x)$ since one only knows that $x \leqq y \leqq \rho x$.

Because this book is well conceived and gives a good picture of the elementary results, it is unfortunate that the execution suffers from the defects noted above. For the more mature student who reads with a skeptical eye, these defects are not of major importance but one cannot recommend this work to the unwary.

LOWELL SCHOENFELD

Entire functions. By R. P. Boas, Jr. (Pure and Applied Mathematics, vol. 5.) New York, Academic Press, 1954. 12+276 pp. $6.00.

This is the first book which is devoted primarily to the theory of analytic functions of exponential type, and it gives a practically complete account of the subject. A function $f(z)$ is said to be of exponential type when it is regular in an unbounded set, often the whole plane, and satisfies an inequality $|f(z)| \leqq Ae^{c|z|}$ on the set. Besides the theory of these functions the book contains a large chapter on entire functions in general, a chapter on the minimum modulus of an entire function, and a chapter on applications of functions of exponential type. The author and his publisher have rendered a great service by making so much material available in a single volume of moderate size and price.

One may regard Pólya, Valiron, Paley and Wiener, and S. Bernstein as the founders of the present theory of functions of exponential type. An unusually large number of mathematicians contributed to its further development (the bibliography lists close to 400 papers!). Of these, the author has probably contributed to more aspects of the theory than anybody else. Moreover, because of his past and present association with Mathematical Reviews the author was in a unique position to become thoroughly familiar with the extensive literature on the subject, including the Russian literature. In addition to these qualifications the author possesses considerable skill as an expositor, and he has thus produced a book which is both extremely useful and very readable.

The author's job has not been easy. Faced with an overwhelming amount of material and striving to be complete he had the tremendous task of ordering a veritable chaos. In the opinion of the reviewer he has succeeded exceedingly well. A good example is furnished by Theorem 8.4.16 for whose proof Levinson (1940) needed 14 pages. In the present book the theorem is just one result in an integrated theory and its proof takes only a little more than half a page. In this and other places the author's method of dealing with a class of related theorems is noteworthy. He begins by proving a simple result, which gives him an opportunity to exhibit the essential ideas of the proofs. Next he either proves the most general result of its kind, or he states the best generalizations that are known without proof, but with reference to the literature. (No use is made of these starred theorems whose proof is omitted.) The above features should make the book very attractive both as an advanced text and as a handbook for nonspecialists who are looking for applicable results. For the student interested in research in the field the reviewer would have welcomed a list of unsolved problems which the author deems worthy of attention.

A description of the content of each chapter follows.

Chapter 1 (7 pages) deals briefly with some of the tools that are frequently used throughout the book: Jensen's and Carleman's formulas, Poisson's formula for a semi-circle, Phragmén-Lindelöf theory, and various kinds of density.

Chapter 2. General properties of entire functions of finite order (31 pages). The author calls an entire function $f(z)$ of growth (ρ, τ) when it is of order $\leqq \rho$ and of type $\leqq \tau$ when of order ρ. An entire function of exponential type τ is then defined as a function of growth $(1, \tau)$; similarly for non-entire functions. For entire functions a great many connections are established between growth and zeros—many more than in any other book. Here one finds Borel's proof of Hadamard's factorization theorem and Lindelöf's very useful theorems relating type and zeros, results of Valiron and the more recent work of Shah.

Chapter 3. The minimum modulus (16 pages). Since Borel compared $\log m(r)$, the minimum of $\log |f(z)|$ on $|z| = r$, with $-r^{\rho+\epsilon}$, considerable progress has been made. On the one hand one finds here the results which show that, for entire functions of small order, $\log m(r)$ is occasionally quite large—for example, as large as $r^{\rho-\epsilon}$, when $\rho < 1/2$ (Wiman). On the other hand there are the much more recent weaker inequalities which hold on large sets. Typical of the latter is a result of Chebotarëv and Meiman for entire functions $f(z)$ of growth (ρ, τ), $\tau < \infty$, which shows that $\log |f(z)|$ is practically everywhere greater than $-Ar^{\rho}$. Applications of these results are made in later chapters where the author has to divide by given entire functions. The chapte[illegible]lso deals with a large class of results in which $m(r)$ is compared with the maximum modulus $M(r)$. Hayman's recent work on the minimum modulus is briefly mentioned in chapter 2.

Chapter 4 (11 pages) deals with Valiron's theorem on entire functions $f(z)$ of order ρ, $0 < \rho < 1$, all of whose zeros are real and negative.

When $n(r)$ denotes the number of zeros of $f(z)$ in $|z| \leq r$ the theorem says that the assertions $n(r) \sim cr^\rho$ $(c>0)$ and $\log f(r) \sim \pi c\ (\csc \pi\rho) r^\rho$ are equivalent. Various generalizations are treated, among them one of the author in which $\log |f(-r)|$ also plays a role, and one due to Pfluger which allows a much more general distribution of zeros.

Chapter 5 (16 pages) roughly follows Pólya's paper (Math. Zeit. vol. 29) in its exposition of the properties of the Phragmén-Lindelöf indicator function

$$h(\theta) = \limsup r^{-1} \log |f(re^{i\theta})| \qquad (r \to \infty)$$

and the indicator diagram. One finds here in particular the two equivalent definitions of the conjugate indicator diagram D of the entire function $f(z) = \sum a_n z^n$ of exponential type: it is the bounded closed convex set whose supporting function is $h(-\theta)$, and also the smallest closed convex set outside which the Borel transform $F(z) = \sum n! a_n / z^{n+1}$ of $f(z)$ is regular. Pólya's representation

$$f(z) = (2\pi i)^{-1} \int_C F(w) e^{zw} dw$$

(where C contains D) is proved as a tool for later chapters, and so are Macintyre's extensions of these results to functions which are of exponential type in an angle.

Chapters 6, 7. Functions of exponential type, restricted on a line (51 pages). Among the three most important results are two representation theorems for functions $f(z)$ which are regular and of exponential type for $y \geqq 0$, and "restricted" along the real axis. Let $h(\pi/2) = c$. (i) When $f(x) \in L^2(-\infty, \infty)$ one has Paley and Wiener's representation

$$f(z) = \operatorname*{l.i.m.}_{A \to \infty} \int_{-A}^{A} e^{izt} \phi(t) dt, \qquad \phi(t) \in L^2(-c, \infty),$$

which becomes particularly simple and useful in the case of an entire function of exponential type. (ii) When $\int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ |f(x)| dx < \infty$ one has Nevanlinna's representation

$$\log |f(z)| = \log |B(z)| + y\pi^{-1} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(t-x)^2 + y^2} dt + c' y,$$

where

$$B(z) = \prod \frac{1 - z/z_n}{1 - z/\bar{z}_n}$$

is the Blaschke product formed with the zeros z_n of $f(z)$ in $y>0$. ((ii) is actually established under weaker conditions.) The author next proves the theorem of Ahlfors and Heins (Ann. of Math. vol. 50) which gives a quite accurate description of the asymptotic behavior of $\log|B(z)|$. (iii) It follows that under the assumptions of (ii) the quantity $r^{-1}\log|f(re^{i\theta})|$, in the terminology of the author, tends effectively to $c'\sin\theta$ in $0<\theta<\pi$ when $r\to\infty$. Thus in particular $c'=c$. A large variety of applications of (iii) are made.

Chapter 8 (19 pages) connects growth and distribution of zeros in the case of entire functions of exponential type whose zeros z_n are close to the real axis in the sense that $\sum|\operatorname{Im}(1/z_n)|$ converges. The principal result of the chapter grew out of work by Paley and Wiener, with Pfluger its main architect (Comment. Math. Helv. vol. 16), and Noble and the author adding a finishing touch. Let $f(z)$ satisfy the above hypotheses, and let $f(0)=1$. Then the following three conditions are equivalent. (i) $\lim n(r)/r$ exists; (ii) $\lim\int_{-r}^{r}x^{-2}\log|f(x)|\,dx$ exists; (iii) $\lim r^{-1}\log|f(ir)|$ exists if a set of finite logarithmic length is excluded. Various related results of Pfluger, Levinson and Cartwright are also proved.

Chapter 9. Uniqueness theorems (26 pages). Most theorems in this chapter state that a function with certain growth properties must vanish identically if it or a certain transform have too many zeros. (i) A function regular for $x\geqq 0$ and of exponential type less than that of $\sin\pi z$ which vanishes on the positive integers is identically zero (Carlson). Many generalizations are given in which the growth conditions are weakened and/or the sequence $\{n\}$ is replaced by a more general sequence $\{\lambda_n\}$. Of these, Fuchs's theorem is especially interesting because it is in a certain sense a best possible result. (ii) There is a similar class of results for entire functions vanishing on or near the integers, with somewhat weaker growth conditions (Pólya, Valiron and others). (iii) When $f(z)$ is entire and of exponential type less than 1, and $f^{(n)}(n)=0$, $n=0, 1, \cdots$, then $f(z)\equiv 0$. Results of the kind (i) and (ii) are derived from the growth properties of simple specific functions vanishing at the points n or λ_n, while (iii) is obtained as a special case of a general expansion theorem due to Buck.

Chapter 10. Growth theorems (28 pages). In this chapter the typical theorem asserts that a function of exponential type which is of certain growth on a sequence of points on or near some line has certain growth all along the line. The chief tool is a suitable interpolation series on the given sequence of points which should not differ too much from the function along the given line. Let $f(z)$ be regular and of exponential type in $|\arg z|\leqq\alpha\leqq\pi/2$, and let $h(\theta)\leqq a|\cos\theta|$

$+b \sin \theta$, $b<\pi$. Then the main theorems are as follows. (i) If $\{\lambda_n\}$ is an increasing sequence of positive numbers such that $\lambda_{n+1}-\lambda_n \geqq \delta > 0$, $|\lambda_n - n| \leqq L$, and $\{f(\lambda_n)\}$ is bounded, then $f(x)$ is bounded for $x>0$ ($\lambda_n = n$: Cartwright; general case: Duffin and Schaeffer). (ii) If $\{\lambda_n\}$ is a (complex) sequence such that $n/\lambda_n \to 1$ and $|\lambda_m - \lambda_n| \geqq \delta|m-n|$, $\delta>0$, then lim sup $|\lambda_n|^{-1} \log |f(\lambda_n)| = h(0)$ ($\lambda_n = n$: Pólya; general case: V. Bernstein).

Chapter 11. Operators and their extremal properties (27 pages). One must be grateful indeed that the author has included this very interesting material. Until now a considerable portion of it could be found only in the Russian literature. Let L be an operator which carries every entire function of exponential type into another such function. Question: does the rate of growth of $L[f(z)]$ along a line follow from that of $f(z)$ on a certain set of points on the line? The main interest is in obtaining exact bounds. We cite two theorems due to S. Bernstein; $f(z)$ is supposed to be of exponential type τ and bounded on the real axis. (i) If $|f(x)| \leqq M$ for all real x, then $|f'(x)| \leqq \tau M$. (ii) $|f(x+\pi/2\tau)+f(x-\pi/2\tau)| \leqq (8/\pi)$ sup $|f(n\pi/\tau)|$. Among the many refinements and related results given in the chapter are inequalities due to Szegö, Schaeffer, and the author. Two methods of proof are exhibited. The first, developed by Civin and also by Krein, applies to operators L which carry functions $f(z) = \int_{-\tau}^{\tau} e^{izt} d\alpha(t)$ into $L[f(z)] = \int_{-\tau}^{\tau} \lambda(t) e^{izt} d\alpha(t)$ with a fixed $\lambda(t)$. The second method of proof depends on the use of an interpolation series for $f(z)$ due to Valiron. The chapter closes with a discussion of Levin's B-operators, which enable one to prove results of the form "$|f(x)| \leqq |\omega(x)|$ implies $|L[f(x)]| \leqq |L[\omega(x)]|$."

Chapter 12. Applications (19 pages). These include results on the asymptotic behavior of moments and the completeness of sets $\{t^{\lambda_n}\}$ and $\{\exp(i\lambda_n x)\}$. Uniqueness theorems for Fourier series are proved, and theorems on the behavior of a function represented by a power series on or near the circle of convergence. Finally there are theorems on expansions of analytic functions in series of polynomials, on differential equations of infinite order and on approximation on the real axis by entire functions of exponential type.

A bibliography of 17 pages mentions the papers which are referred to in the book. Only one or two of these papers are prior to 1920, many are as recent as 1953. The book closes with an unusually complete index of 8 pages. (Surprisingly, the index even shows the extent of Bourbaki's contributions to the subject matter of the book.)

Several years before he wrote this book the author confided to the reviewer that on the average every page of mathematics he had seen contained a minor error, and that he had never seen 50 pages of mathematical work without a serious mistake. The author now informs the reviewer that a mimeographed list of errata for this book can be obtained by writing to him.

J. KOREVAAR

Probability theory. Foundations. Random sequences. By M. Loève. Toronto, Van Nostrand, 1955. 15+515 pp. $12.00.

The mathematical theory of probability has been well established in its modern form for about twenty years, but Loève's is the first (non-elementary) textbook on the subject. That is to say, there has been no textbook which, like a textbook covering any other advanced mathematical subject, gives the fundamental definitions, basic theorems, and enough further development to lead the reader into the really advanced literature. Feller's *Probability theory and its applications* (1950) is superlative as far as it goes (only through discrete probabilities), but the promised further volumes are still keeping company with the other unborn descendants of first volumes, an illustrious group. The reviewer's *Stochastic processes* (1953) has been the closest approach to a probability textbook, but, as its title indicates, this book was neither written as nor intended to serve as a general textbook, and its choice of topics and emphasis were dictated by its title. Authors have been willing to write specialized probability books, of which there have been many, besides the two just mentioned, by Bartlett, Blanc-Lapierre and Fortet, Fortet, Gnedenko and Kolmogorov, Ito, Lévy, but the drudgery involved in writing a systematic and complete text has not been attractive. Thus, even if Loève's book were not as successful as it is, he would still deserve the thanks and respect of the mathematical community for writing it.

The mathematical theory of probability is now a branch of measure theory, with certain specializations and emphasis derived from the applications and the historical background. As the historical conditioning loses its significance for newer generations of mathematicians, the place of probability theory in measure theory becomes more and more difficult to describe. One slightly frivolous description, which, however, is about as accurate a description as can be given, is that probability is the one branch of measure theory, and in fact the one mathematical discipline, in which measurable functions as such are considered in detail, and their integrals evaluated. (The fact that the integration of smooth functions on intervals can be considered as that of measurable functions is of course discounted here.) In fact, in most applications of measure theory, say to the study of L_1 spaces, or to potential theory, the abstract measure theory serves to make the final results elegant—to make certain spaces complete, and so on, but any integrals actually evaluated are integrals that could have been evaluated by Euler. The point is that probability theory is more than just an application of measure theory. It is an intrinsic part of the theory, which uses the general ideas so intensively that the probabilist must have at his disposal concepts which students are unlikely to study in a first course in measure theory. The writer of a probability text must therefore either devote part of the book to measure and integration, covering more than is usually covered in a first course, or assume as known the content of such a course and develop the subject from there, or, under the same assumption, simply list a long collection of assorted results developed in different notations in different places. The second and third choices would be very difficult to carry through in a satisfactory way, and Loève has wisely chosen the first. He has thus written a book which, assuming the usual prerequisites to a first course in measure theory, as well as a considerable amount of mathematical maturity, conducts the reader through a remarkably thorough treatment of abstract measure and integration theory, and an extraordinary coverage of the basic concepts and results of probability theory. (The statement, in an otherwise humorless book, that “the prerequisite is calculus” should be taken in the spirit in which it was presumably intended.) The book has as subtitle *Foundations. Random sequences* and there is correspondingly no discussion of continuous parameter stochastic processes (c.p. “random functions” in Loève's terminology) except for the material on second order random functions.

Loève could not include so much material in his book without some sacrifice, and in fact he has, to some extent, sacrificed motivation, specialization and application of general results, and readability. Only the individual readers can decide whether the coverage is worth this sacrifice, but the extremely terse style is frequently not easy to follow. Loève's definitions are exact and receive no commentary even when they contain easily overlooked details which may be critically important. His theorems are tightly linked together, so that every statement can be and is justified simply by a reference to a previous theorem. He frequently does not make the specializations that would illuminate the subject. The classical discrete problems and applications, treated so beautifully in Feller's book, are absent, and in fact there are essentially no applications, except to other parts of mathematics. In other words, and this is or is not a criticism according to the preferences of the reader, the book is purely a mathematical text. (There is, however, a reference to a second volume, and the remarks on omissions are subject to correction if this volume breaks a tradition by appearing.)

The book begins with an introductory section, intended to provide intuitive meaning to the concepts and problems of probability theory, although this section is rather abstractly written. It is an independent section of the book, and in fact is so independent of the remainder that when, in a later chapter, Markov chains with enumerably many states are discussed, and a result is proved which is more special than one proved in the introductory section, no reference is made to the earlier result, or even to the earlier treatment! The introductory section contains a discussion of the basic probability definitions in the discrete case, and the work is carried through the central limit theorem and strong law of large numbers for Bernoulli trials, and the Kolmogorov treatment of Markov chains with enumerably many states and stationary transition probabilities. The first two chapters, constituting *Part One*, cover measure and integration. Chapters III and IV, constituting *Part Two*, cover the standard fundamental concepts of probability theory: random variables, distributions, characteristic functions, convergence. Chapters V and VI, constituting *Part Three*, concentrate on problems connected with independent random variables: convergence of series, the laws of large numbers, the central limit problem, including a detailed discussion of infinitely divisible and stable distributions. Chapters VII–X, constituting *Part Four*, involve dependent random variables. Chapter VII takes up conditional probabilities and expectations, giving among other things a careful discussion of when conditional probabilities can be assumed to define completely additive set functions. There is a very elegant treatment of the Markov dependence property. Chapter VIII shows how the theorems of Chapters V and VI can be extended to dependent random variables, and takes up martingales, proving the essential convergence theorems. Chapter IX covers ergodic theorems, including sufficient material on ergodic theorems for Banach space operators to obtain the Doeblin theorems on convergence of iterated conditional probabilities, following Yosida and Kakutani. Chapter X studies second order properties, including considerable material on second order random functions. The stationary case is studied as a special case.

A bibliography is appended to the introduction, to each of the four parts, and to each chapter. These separate bibliographies represent the sources of the author's material, but are by no means slavishly followed. The arrangement of bibliographies and the references make it difficult in some cases to find out exactly which work is the source for a given result, and therefore where further material may be found.

The above short outline does not suggest the remarkable amount of material in this book. The author's compact style enables him to include an extraordinary mass of ideas and theorems in a systematic and coherent fashion. At each stage he usually succeeds in achieving the utmost in generality, both in the large and in minor details—for example in allowing infinite-valued expectations wherever possible. This book will be a standard reference text for years, and students will find it indispensable, although difficult.

J. L. DOOB

General topology. By J. L. Kelley. New York, Van Nostrand, 1955. 14+298 pp. $8.75.

The appearance of a comprehensive treatise in English on present day set-theoretic topology is an important event. The rapid progress in set-theoretic topology during the past 20 years, and the ever increasing applications of this discipline to analysis, make the appearance of the volume under review particularly appropriate at the present time. Professor Kelley has set himself the task of producing a book useful for both students and specialists, and he has succeeded to a remarkable extent in reaching both of these somewhat inconsistent goals. Like many other texts of this genre, the present volume could be understood, at least in theory, by any intelligent person who can read English. All of the machinery is supplied; but a knowledge of the real numbers and elementary abstract algebra as set forth for example in Birkhoff-MacLane (*A survey of modern algebra*, rev. ed., New York, Macmillan, 1953) and a thorough knowledge of elementary analysis are certainly minimal prerequisites for appreciating this book.

The author's style is spirited, to say the least. The atmosphere of an informal and humorous lecture pervades the book, especially in the first part. An essential difference between oral and written communication is well illustrated here, for some sallies that would clearly enliven a lecture are less felicitous in print. (For example, a very special case of Fubini's theorem is designated without further comment as Fubinito (p. 78), and a *topologist* is defined as a man who doesn't know the difference between a doughnut and a coffee cup (p. 88).) An occasional solecism was noted (e.g., p. 135, line 7), and one might wish that the author adhered to the "which" and "that" precepts of Fowler. However, these are but trifling criticisms of a large and serious mathematical enterprise. We proceed to a discussion of the subject matter.

Chapter 0 and Appendix. The necessary facts about the algebra of sets, relations, functions (i.e., single-valued functions), orderings, ordinary algebra, the real numbers, cardinal and ordinal numbers, and the axiom of choice and certain useful equivalents such as Zorn's lemma, are collected here. Many of the proofs are postponed to the Appendix, where an excellent account of elementary set theory is given. The real numbers are defined as an ordered field in which every set bounded above has a supremum, and brief but unexceptionable proofs are given for a few important facts about real numbers. The treatment of the various equivalents of the axiom of choice (maximum principles and the like) is extremely clear.

Chapter 1. Topological spaces. The material in this chapter is quite standard. A topology on a set X is defined as the family $\mathfrak{T}$ of open sets. Closed sets, closure operators, and (in an exercise) interior operators are discussed. Hausdorff's neighborhood axioms appear, in slightly altered form, in an exercise. Since definition of bases at each point is one of the commonest methods of defining a topology, a more detailed discussion of this matter seems called for. Throughout the book, the exercises are called upon to carry a large part of the load of exposition. Examples, counter-examples, minor theorems, and occasionally very major theorems appear with hints of proof as exercises.

Chapter 2. Moore-Smith convergence. This chapter contains an excellent discussion of convergence of functions on directed sets, embodying among other things the author's own contributions. The axioms for convergence that generate a topology are given. Filters, which are of course in their applications equivalent to functions on directed sets, are touched on only briefly in an exercise. The exercises also contain a discussion of maximal ideals in lattices, a miniature integration theory, a treatment of Riemann integration on an interval $[a, b]$, and a proof of the semi-simplicity of Boolean rings (Boolean rings are unnecessarily postulated to satisfy the identity $r+r=0$).

Chapter 3. Product and quotient spaces. Continuous functions and topological equivalence are introduced here for the first time. The treatment of products and quotients is lucid and standard.

Chapter 4. Embedding and metrization. The following are proved: Tychonoff's theorem on embedding completely regular spaces (here called Tychonoff spaces) in products of closed unit intervals; Urysohn's lemma; Urysohn's metrization theorem; the general metrization theorem of Nagata, Smirnov, and Bing.

Chapter 5. Compact spaces. A lucid treatment of the basic facts about compact and locally compact spaces is given. The Alexandroff and Stone-Čech embedding theorems are proved. The main facts about paracompact spaces (including the theorem of A. H. Stone that every metric space is paracompact) are established in a brief and very readable section. Wallman's compactification appears in an exercise.

Chapter 6. Uniform spaces. This chapter contains the best account the reviewer has seen of the theory of uniform spaces. Every essential fact is found here. The closed graph theorem and the Banach-Steinhaus theorem are exercises.

Chapter 7. Function spaces. This chapter is couched in such generality as to lose considerable interest. The function spaces considered are mostly sets of mappings of a topological space into a uniform space. The important fact that the uniform limit of a sequence of continuous real-valued functions is continuous appears in concealed form and is explicitly mentioned only in a footnote to an exercise. The section dealing with a concept called even continuity seems out of place in a general text. The Stone-Weierstrass theorem appears only in an exercise. In view of the huge importance of this theorem, more extended treatment is surely called for. The complex form of this theorem is not mentioned at all.

The clarity of the author's thought and the carefulness of his exposition make reading this book a pleasure. He makes a large concession to human weakness in frequent repetition of definitions and even theorems. A few remarks on the subject matter may be in order. The author avoids the use of separation axioms whenever possible. For example, many assertions ordinarily proved only for metric spaces or Hausdorff spaces with a uniformity are established here for spaces that need not satisfy even the T_0 axiom. It would be useful, however, to have a more thorough discussion of the various separation axioms and a clear statement of their importance in applications. In particular, the reduction of arbitrary topological spaces to T_0-spaces should be given. The reviewer would have been glad to see some treatment of spaces of dimension zero (perhaps in exercises) and a discussion of Borel and Baire sets, including the theory of functions of the various Baire classes. No reasonable person would alter established terminology without grave reasons. While recognizing that the author undoubtedly has such reasons, the reviewer takes exception to the new definitions given for complete regularity (p. 117) and compacta (implicitly on p. 229) and to the introduction of the new term "Tychonoff space."

The reviewer noted a few inaccuracies and other defects of exposition, as follows. The term "nowhere dense" is defined in two quite different ways on pages 145 and 201 (the latter being the customary one). The attribution to Tong [1] on p. 167 should be to Hewitt [2]. The reference to Loomis [2] on p. 248 is inexact: Loomis deals with complex function spaces and homomorphisms onto the complex numbers. The terms "real linear function" and "linear functional" are used interchangeably, to the probable confusion of unsophisticated readers (pp. 108 and 241). The author cites Banach spaces (p. 110) and Haar measure (pp. 166 and 210) without explanation. Such references are bound to be unintelligible to many student readers. Only a few misprints, all of them inessential, were noted. The convenience of the book would be enhanced by numbering the chapter at the top of each page, or by designating theorems, etc., as they appear, with the number of the chapter containing them. As it is, locating back references in the text is troublesome.

Comparison of Professor Kelley's treatise with those of Kuratowski and Bourbaki is of some interest. Kuratowski's work (*Topologie* I and II, Monografie Matematyczne, Warszawa, 1948–1950) is an encyclopedic affair, strictly for professionals, and overlapping Kelley's book very little in subject matter. Bourbaki's work (*Topologie générale*, Actualités Sci. et Ind., Nos. 858–1142, 916, 1029, 1045, 1084, Paris, Hermann, 1940–1949) covers a good deal of the same ground as Kelley's, but in an austere manner very far from the sprightly style used by Professor Kelley. The treatments are in some ways complementary, and an abstract analyst could profitably be acquainted with both. In the United States, at least, Kelley's book will undoubtedly be more widely read than Bourbaki's, and it may be expected to exert a decisive and beneficent effect upon the future development of set-theoretic topology.

EDWIN HEWITT

Theory of functions of a real variable. By I. P. Natanson. Trans. by L. F. Boron, with the editorial collaboration of and with annotations by E. Hewitt. New York, Ungar, 1955. 277 pp. $6.50.

This volume differs from the original Russian book on which it is based (*Teorii funkcii veščestvennoi peremennoi*, Moscow-Leningrad, 1950) and from the more or less literal German translation (*Theorie der Funktionen einer reellen Veränderlichen*, Berlin, 1954 (reviewed in this Bulletin, vol. 61 (1955) p. 346) in two major respects. These changes appear to have been made by the editor to fit the book more closely to a course in Lebesgue measure and integration on the real line, rather than to a course covering a somewhat less concentrated list of topics in real function theory. The first, unarguably positive, change is the addition by the editor of appendices to several chapters; these show how results stated in the original text only for bounded measurable sets can be extended or adapted for unbounded sets. The second, or negative, change is the omission of the last eight chapters of the original work. This brings the scope of the text down to approximately the material needed for a one semester course of measure and integration on the real line, if it is assumed that the student already knows his ϵ from his δ, lim sup and lim inf, and uniform convergence; all these matters are used freely through the text. The present book ends with the chapter on absolutely continuous functions of one variable and their relation to differentiation and integration. It is easy to agree with the editor that, in an introductory course in measure theory given in a limited time, such topics as singular integrals, transfinite induction, set functions, Baire classification of functions, a little functional analysis, and a chapter of the contributions of Russian mathematicians to the theory of real functions, can be omitted without unfairly distorting the subject. Unfortunately Fubini's theorem has also been lost in this cut; as this is the last of the basic theorems of the Lebesgue theory, it would have been good for the student if it could have been kept available. The author and editor have both displayed considerable knowledge of the subject under discussion. All the kindly reviews of the other versions of this book, as far as they refer to the early chapters, apply as well to this edition. The style of the present work is very smooth; the clarity of the author's discussion is carried into English with none of the stiffness of expression which plagues translators.

M. M. DAY

The construction and study of certain important algebras. By C. Chevalley. (Publications of the Mathematical Society of Japan, no. 1.) The Mathematical Society of Japan, 1955. 6+64 pp.

This booklet reproduces lectures given by the author in Japan in 1954 on the main topics in multilinear algebra. But the subject is approached from an unusual point of view, the importance of which has only been realized during the last few years, mainly in the recent work on homology and Lie theory; and it is chiefly for the introduction it presents to these new methods that the book is especially valuable, being probably the first of its kind to appear in print.

The main themes of the book are the notion of "*universal*" algebra with respect to a given property, and the notion of *graded* algebra.

The first example of universal algebra is given by the free associative algebra, which forms the topic of the first chapter; of course, it provides at the same time an excellent illustration of the concept of graded algebra (with gradation either in the group of integers or in the group of integers mod 2); the author also develops here the general notion of derivation in a graded algebra, and shows how a derivation is determined in the free algebra by its values at the generators.

Chapter II takes up the theory of tensor algebras and tensor product of modules or algebras. The tensor algebra T of a module E is defined by its universality property (extension of a linear mapping of E into an algebra A, to an homomorphism of T into A) and its existence is proved by considering it as a quotient algebra of the free algebra generated by the set E. The tensor product of two modules is then defined a little artificially by embedding it in the tensor algebra of their direct sum, and its characteristic "universal" property with respect to bilinear mappings is only mentioned later on. It is shown also how derivations can be defined in a tensor algebra, and finally the "skew" tensor product of two "semi-graded" algebras (i.e. with gradations mod 2) is defined in view of applications in chapter III.

The next chapter treats at the same time Clifford and exterior algebras of a module: the first is introduced by its universality property (extension to an algebra homomorphism of a linear mapping u of E into an algebra A, such that $(u(x))^2 = Q(x)\cdot 1$, where Q is a quadratic form on E), and the exterior algebra of E is just the special case corresponding to $Q=0$. The (usually cumbersome) determination of a basis of the Clifford algebra is here achieved very neatly via a general theorem which proves that the Clifford algebra of a direct sum of two orthogonal submodules is the skew tensor product of the Clifford algebras of the submodules. Another novelty is the definition of the trace of an endomorphism by means of a derivation of the exterior algebra extending the given endomorphism. The relations between Clifford algebras, orthogonal groups, and spinors are briefly touched upon (without proofs).

The last chapter is entitled *Some applications of exterior algebras.* After a few words on Plücker coordinates, the main developments are devoted to the definition of the author's "exponential mapping" from the algebra F of even elements of the exterior algebra into itself (introduced already in the author's book *The algebraic theory of spinors*, and which is a nice way of presenting the Papy "reduced powers" with a minimum of computations); this is achieved by means of some clever lemmas on derivations. As an application, the pfaffian of a 2-vector is defined, and an interesting proof is given of the equality of the determinants of an endomorphism and of its transposed endomorphism, both endomorphisms being considered (following another idea of Papy) as restrictions of a single endomorphism of the direct sum of the module E and its dual.

Due to the usual time lag between research and teaching, multilinear algebra, although fundamental in modern mathematics, is still hardly taught (if taught at all) in most universities. It is to be hoped that many professors will avail themselves of the opportunity created by the publication of this little book, and let their students share the experience of the young Japanese mathematicians who first listened to these challenging lectures.

J. DIEUDONNÉ

Konstruktive Funktionentheorie. By I. P. Natanson. Trans. by K. Bögel. Berlin, Akademie-Verlag, 1955. 14+515 pp. 36 DM.

The phrase "constructive theory of functions" was coined by S. Bernstein to describe the part of analysis that deals with the approximate representation of functions of a real variable by means of combinations of other functions. Thus it includes the theory of approximation, in various metrics, by polynomials and trigonometric polynomials; the theory of interpolation and approximate integration (formerly known inappropriately as "mechanical quadratures"); large portions of Fourier analysis and the theory of orthogonal functions; and related subjects like moment problems. All this may fairly accurately be thought of as the classical part of the subject. Although its fine structure (to borrow a term from atomic physics) is still undergoing investigation, the main results are at least 25 years old, often much older. There are also a number of topics that clearly belong to the subject but are of more recent development: approximation by translations of a function (and hence Wiener's Tauberian theorems); approximation by entire functions (developed by Kober and Bernstein within the last ten years); weighted polynomial approximation on infinite intervals (here the fundamental problem was finally solved by Pollard, and independently by Ahiezer and Bernstein, in 1953); closure and completeness theorems; extremal problems for polynomials, trigonometric polynomials, and more general classes of functions (currently enjoying a renaissance at the hands of Rogosinski and others); and the theory of special classes of functions that admit simple representations, such as absolutely monotonic functions and completely convex functions (which enjoyed a flurry of intensive activity a few years ago). The subject has been a favorite of Russian mathematicians from Chebyshev onwards, and especially of S. Bernstein and his students.

Ahiezer's *Vorlesungen über Approximationstheorie* (translated into German in 1953 and reviewed in this Bulletin vol. 61, p. 369) dealt with parts of the subject at a rather advanced level and in monographic style, with emphasis on the modern theory. The present volume, translated from the Russian edition of 1951, is written as a textbook and covers the more elementary and classical portions in a more detailed style, especially in the first part. The author suggests that his book can be used as an introduction to the current literature or to such monographs as Ahiezer's. It is divided into three parts, dealing respectively with uniform approximation, L^2 approximation, and interpolation and approximate integration. The selection of modern topics seems to have been governed by their ability to fit into the classical framework. Thus Zygmund's smooth functions are included for their connection with best approximation, and there are some theorems of Marcinkiewicz and of Grünwald and Turán on interpolation. On the other hand, although Müntz's theorem on the completeness of $\{t^{\lambda_n}\}$ is included, there is no mention of the results of Clarkson and Erdös and of Schwartz on the span of such a set when it is not complete. The translator has added a rather spotty supplementary bibliography: it contains many papers published after the Russian edition of the book, but does not, for example, include anything by Mandelbrojt.

The contents are, in more detail, as follows. Part I: Approximation to continuous functions by polynomials and trigonometric polynomials. Best approximation of functions satisfying Lipschitz conditions, and converse theorems deducing properties of functions from the degree of approximation (the basic results are associated with the names of D. Jackson and S. Bernstein). Approximation by partial sums of Fourier series and by various transforms of these sums. The problem of best approximation to an analytic function of a real variable receives an original and completely real-variable treatment. Part II: Weighted L^2 approximation. Orthogonal polynomials and in particular the Legendre, Jacobi, Laguerre and Hermite systems. The Hausdorff and Hamburger moment problems. Part III: Interpolation polynomials of various kinds, general convergence and divergence theorems, modified interpolation processes; approximate integration formulas of various types. In part II the author mentions Nikolaev's theorem that no system of orthogonal polynomials can provide a convergent expansion for every continuous function. In an appendix he proves the following generalization by Lozinskii and Harshiladze: there exists no sequence $\{U_n\}$ of linear operators on C such that U_n takes every element of C into a polynomial of degree at most n, and leaves such polynomials invariant, while $U_n(f) \to f$ for every f in C.

The translation reads smoothly, with no relics of Russian mathematical style; it appears to be quite idiomatic, even to the extent of rendering "Buniakovskii's inequality" by "Cauchy's inequality" or "Schwarz's inequality" according to circumstances.

R. P. BOAS, JR.

BRIEF MENTION

Allgemeine Theorie der algebraischen Zahlen. By Ph. Furtwängler. Reworked by H. Hasse and W. Jehne. Enzyklopädie der mathematischen Wissenschaften, vol. I_2, no. 8, part II. 2d ed. Leipzig, Teubner, 1954. 50 pp. 6 DM.

This subject being surrounded in the Enzyklopädie by articles on Allgemeine Modul-Ring- und Idealtheorie, Bewertungstheorie, Theorie der abelschen Zahlkörper, and a section on analytic number theory, the authors faced a considerable problem in choice of material. In the reviewer's opinion they solved it admirably, allowing enough overlapping to make the various approaches to the subject clear. The first 40 pages are devoted to the arithmetic in integral domains of algebraic number fields as worked out by Kummer, Dedekind, and Kronecker, with an indication of the approach by valuation theory. The description of Kummer's approach, which is seldom mentioned but turns out to be surprisingly modern, is of interest. Main topics: ideals, ideal classes, different and discriminant, units. Except for a brief hint under "Axiomatische Begründung der Idealtheorie," there is no indication that practically all of this applies equally well to function fields of transcendence degree 1. The last 10 pages are devoted to Artin's theory of the conductor and L-series, and analytic formulas for the class number.

G. WHAPLES

An introduction to stochastic processes with special reference to methods and applications. By M. S. Bartlett. Cambridge University Press, 1955. 312+14 pp. $6.50.

This book deals primarily with elementary heuristic applications to genetics, population growth, insurance risk, statistics, queuing and waiting time, epidemiology, particle physics, turbulence, prediction, information theory, time series, etc. There are a number of illustrative numerical examples.

DONALD A. DARLING

Numerische Behandlung von Differentialgleichungen. By L. Collatz. Berlin, Springer, 1955. 15+526 pp., with 118 illustrations and a portrait. Paperbound, 56 DM.; clothbound, 59.60 DM.

The first edition of Collatz's Numerische Behandlung (reviewed in this Bulletin, vol. 59, pp. 94–96) was noteworthy as the most extensive and most complete treatment of the numerical solution of differential equations that had yet appeared. The second edition now at hand continues to maintain this leadership. It is still larger (526 pages) and has undergone considerable reorganization. A major part of the reorganization consists in the insertion of a new chapter at the beginning devoted to basic material needed later, such as finite differences, interpolation, formulas for numerical differentiation and integration, Green's theorem and related topics, least squares, orthogonality, and concepts from functional analysis. The remaining five chapters cover substantially the same material as in the first edition except for the topics now collected in Chapter I and the expansion of the remaining topics by more detailed treatment and the addition of new items. The high character of the first edition has been well preserved.

W. E. MILNE

Curso de análisis matemático. Vol. 3. By C. de Losada y Puga. Lima, Universidad Católica del Perú, 1954. 22+814 pp.

This final volume of Professor de Losada y Puga's treatise covers trigonometric series, divergent series, functions of a complex variable, differential equations, calculus of variations (very briefly) and probability. The exposition is for the most part at the advanced calculus level, and in a leisurely and readable style. The section on differential equations (350 pages) is more up-to-date and detailed than many textbooks on the subject in English.

R. P. BOAS, JR.

Opere. Vol. 3, *Sistemi tripli ortogonali.* By L. Bianchi. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Rome, Cremonese, 1954. 6+851 pp. 7500 lire.

For the first two volumes cf. this Bulletin, vol. 60, p. 288.

Collected works. By G. A. Miller. Vol. 4. Urbana, University of Illinois, 1955. 12+458 pp. $7.50.

This volume covers publications from 1916 to 1929.

Tables of the cumulative binomial probability distribution. By the Staff of the Computation Laboratory. Cambridge, Harvard University Press, 1955. 61+503 pp. $8.00.

The function $\sum_{s=r}^{n} C_{n,s}p^{s}(1-p)^{n-s}$ is tabulated to 6 decimals for n at various intervals up to 1000 and for 60 values of p. A long introduction explains the applications of the function.

The real projective plane. By H. S. M. Coxeter. 2d ed. Cambridge University Press, 1955. 12+226 pp. $4.75.

This is a revision of the first edition, which was reviewed in this Bulletin, vol. 56, p. 376.

Science awakening. By B. L. van der Waerden. Trans. by A. Dresden with additions of the author. Groningen, Noordhoff, 1954. 4+306 pp. $5.00 or 19 florins.

As the title does not suggest, this is a book about ancient Egyptian, Babylonian and Greek mathematics. For a detailed review of the Dutch edition of 1950, see Mathematical Reviews, vol. 12, p. 381. The present edition is sumptuously illustrated.

Lectures on functions of a complex variable. Ed. by W. Kaplan with the assistance of M. O. Reade and G. S. Young. Ann Arbor, University of Michigan Press, 1955. 10+435 pp. $10.00.

This volume contains, in 31 articles, the proceedings of the Conference on Functions of a Complex Variable which was held at the University of Michigan in 1953.

Second colloque sur les équations aux dérivées partielles. Tenu à Bruxelles du 24 au 26 mai 1954. Liège, Thone; Paris, Masson, 1955. 132 pp. 200 Belgian fr. or 1500 French fr.

The colloquium was sponsored by the Centre Belge de Recherches Mathématiques. The volume contains articles by Picone, Schwartz, Lions, Leray, Brelot and Choquet, de Rham, Garnir, and Fantappiè.

The Computer Director, June 1955. New York, Berkeley Enterprises, Inc. 162 pp.

This is vol. 4, no. 6, of the journal *Computers and Automation*; it contains a directory of people and organizations active in the field.

Cinquant' anni di relatività, 1905–1955. Ed. by M. Pantaleo. Firenze, Editrice Universitaria, 1955. 50+634 pp.

This volume contains, besides a preface by Einstein and a general introduction by the editor, articles by A. Aliotta, G. Armellini, P. Caldirola, B. Finzi, G. Polvani, F. Severi, and P. Straneo, and translations into Italian of seven of Einstein's papers.

RESEARCH PROBLEMS

1. John Nash: *Generalized Brouwer Theorem.*

Define a "connectivity map" from a space A into a space B as one such that the induced map $A \to A \times B$ preserves the connectedness of any connected set in A. Must every connectivity map of a cell into itself have a fixed point? (Received August 24, 1955.)

2. C. S. Herz: *The Bohr spectrum of bounded functions.*

Let ϕ be a bounded, uniformly continuous function on the real line. Is it true that for almost all t, $\lim_{N\to\infty} (2N)^{-1}\int_{-N}^{N} \exp(-itx)\phi(x)dx = 0$? (Received October 10, 1955.)

3. J. L. Brenner: *Group Theory.*

Find the (algebraic) real values of m (between 0 and 2) for which the matrices $\begin{pmatrix}1,0\\ m,1\end{pmatrix}$, $\begin{pmatrix}1,m\\ 0,1\end{pmatrix}$ do not generate a free group. 1. When m is transcendental the group is known to be a free group. 2. When m is real and greater than or equal to 2 the group can be shown to be a free group.) (Received October 20, 1955.)

4. H. L. Alder: *Number theory.*

Let $q_d(n)$ = the number of partitions of n into parts differing by at least d; let $Q_d(n)$ = the number of partitions of n into parts congruent to 1 or $d+2$ (mod $d+3$); let $\Delta_d(n) = q_d(n) - Q_d(n)$. It is known that $\Delta_1(n) = 0$ for all positive n (Euler's identity), $\Delta_2(n) = 0$ for all positive n (one of the Rogers-Ramanujan identities), $\Delta_3(n) \geq 0$ for all positive n (from Schur's theorem which states $\Delta_3(n)$ = the number of those partitions of n into parts differing by at least 3 which contain at least one pair of consecutive multiples of 3). a. Is $\Delta_d(n) \geq 0$ for all positive d and n? b. If (a) is true, can $\Delta_d(n)$ be characterized as the number of a certain type of restricted partitions of n as is the case for $d = 3$?

References

1. D. H. Lehmer, *Two nonexistence theorems on partitions*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 538–544.

2. H. L. Alder, *The nonexistence of certain identities in the theory of partitions and compositions*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 712–722.

(Received October 24, 1955.)

5. V. L. Klee: *Topology.*

A topological space S is called *homogeneous* provided for each two points x and y of S there is a homeomorphism h of S onto S such that $hx = y$. Clearly each product of homogeneous spaces is homogeneous. O. H. Keller [Math. Ann. vol. 105 (1931) pp. 748–758] has proved that the Hilbert parallelotope $[0, 1]^{\aleph_0}$ is homogeneous. Thus $[0, 1]^{\aleph_0}$ and $[0, 1]$ are spaces whose product is homogeneous, even though $[0, 1]$ is not homogeneous. Questions: Is there a manifold M (with or without boundary) such that $M^{\aleph_0}$ is non-homogeneous? Are there compact metric spaces A and B such that $A \times B$ is homogeneous even though *neither* A nor B is homogeneous? (Received November 9, 1955.)

6. V. L. Klee: *Topology.*

It is known [Trans. Amer. Math. Soc. vol. 74 (1953) p. 36] that Hilbert space H has the following property: If Y is a metric space and f and g are homeomorphisms of Y into H, then f and g are isotopic in H. Is this property possessed by any nondegenerate finite-dimensional metric space? (Received November 9, 1955.)

1955

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SCIENTIFIC REPORT ON THE SECOND SUMMER INSTITUTE

SEVERAL COMPLEX VARIABLES

The Second Summer Institute, devoted to several complex variables, was held from June 21 to July 31, 1954 at the University of Colorado. It was supported by a grant from the National Science Foundation to the American Mathematical Society. This report outlines much of the work presented at the seminars at the Summer Institute. The report has been written under the supervision of an Editorial Committee consisting of S. S. Chern, W. T. Martin, and Oscar Zariski. The report is presented herewith in three parts, each of which has its own bibliography and may be read independently of the other two parts. Part I, which is based on the Analysis seminar, was prepared under the editorial supervision of W. T. Martin; Part II, Complex Manifolds, was written by S. S. Chern; and Part III, Algebraic Sheaf Theory, by Oscar Zariski. Detailed acknowledgments for each of the three parts of the report are given in the separate parts.

The report does not cover the work done on a seminar on Algebraic Geometry since the topic of that seminar—the problem of existence of minimal models for algebraic surfaces—will be the subject of a separate memoir by Oscar Zariski. Several lectures on the Albanese variety were given in that seminar by W. L. Chow.

PART I. REPORT ON THE ANALYSIS SEMINAR

Introduction. In the analysis seminar talks were given by several members of the Institute on topics ranging from the kernel function and automorphic functions to functions on abstract algebras and applications of function theory to linear partial differential equations. The talks included not only reports of work previously done in the field but also recent work done by the participants.

In this part we present a brief report of the seminar. This part is divided into sections corresponding to the lectures on a given topic. Some unsolved problems of interest have been included in the sections to which they are related. The material in each section is based upon a written summary prepared by the speaker or speakers as follows:

§§1A and 4: H. J. Bremermann;

§1B: M. Maschler;

§§2A and 6: S. Bergman;
§2B: R. C. Gunning;
§3A: W. L. Baily;
§3B: J. H. Sampson;
§5A: H. G. Haefeli;
§5B: N. S. Hawley.

In addition the Editorial Committee wishes to thank Doctors W. L. Baily and R. C. Gunning for their assistance in the planning and coordinating of the material of this part.

1. Applications of kernel functions to some geometrical problems. In this section we discuss some applications of the theory of the kernel function. A general treatment of the kernel function with further applications may be found in Bergman [3; 5; 6] and other papers cited there.

A. *Domains of holomorphy.* The Bergman kernel function $K_D(z, \bar{t})$ of a domain $D \subset C^n$ is a holomorphic function of (the $2n$ variables) z, $\bar{t}$ in the product domain $D_z \times D_{\bar{t}}$. The envelope of holomorphy E of a product domain is equal to the product of the envelopes of holomorphy of the domains: $E(D_z \times D_{\bar{t}}) = E(D)_z \times E(D)_{\bar{t}}$. Therefore for any domain D, $K_D(z, \bar{t})$ is holomorphic in $E(D)_z \times E(D)_{\bar{t}}$. In particular (letting $\bar{t} = \bar{z}$), $K_D(z, \bar{z})$ has a (plurisubharmonic) continuation into $E(D)$. Some of the properties of the kernel function continue to hold in the larger domain $E(D)$. For instance the reproducing property holds in the following sense: *Let $f(z)$ be holomorphic and let $f \in \mathcal{L}^2(D)$. Then $f(z) = \int_D K_D(z, \bar{t})\overline{f(t)}d\omega_t$ not only for $z \in D$ but also for $z \in E(D)$.* (However the property $f \in \mathcal{L}^2(D)$ does not in general imply $f \in \mathcal{L}^2(E(D))$.) Also the Hermitian form

$$ds^2 = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 K_D(z, \bar{z})}{\partial z_\mu \partial \bar{z}_\nu} dz_\mu d\bar{z}_\nu$$

is positive definite and invariant with respect to holomorphic transformations not only in D but also in $E(D)$.

We say that "$K_D(z, \bar{t})$ becomes infinite everywhere at the boundary of D" if for arbitrarily large real M the closure of $\{z \mid K_D(z, \bar{z}) < M\}$ is contained in D. Bremermann [2; 5] has shown that a necessary condition for K to become infinite everywhere at the boundary of D is that D be a domain of holomorphy. He has given counter-examples to show that this is not a sufficient condition. However he has shown that any domain of holomorphy can be approximated by domains D_ν for which K does become infinite everywhere at the boundary. Each of the D_ν can be approximated by $\{z \mid K_{D_\nu}(z, \bar{z}) < M\}$, which turn out to be regions of holomorphy. Both approximations together give Bremermann's theorem [2]:

Any domain of holomorphy can be approximated from the interior by regions of holomorphy each having an infinitely differentiable boundary surface.

Bremermann has proposed the following two problems:

1. Express the kernel function for simple domains explicitly in terms of known transcendental functions.
2. Develop this theory for complex manifolds.

B. ***Minimal and representative domains.*** As there is no known analogue to the Riemann mapping theorem in the space of two complex variables, it is interesting to know that it is at least possible to map an arbitrary domain onto a domain which has certain properties. (In order to save space, the results of this portion are stated for the case of two complex variables. Similar theorems hold for the space of n complex variables, $n \geqq 1$.) In Bergman [5] (and other papers mentioned there) two kinds of such domains were introduced: ***minimal domains*** and ***representative domains.*** A minimal domain D in z_1, z_2-space with respect to a point $t=(t_1, t_2) \in D$ as center is a domain which has the property that under any pseudo-conformal transformation $w_k = w_k(z_1, z_2)$, $(k=1, 2)$, $\partial(w_1, w_2)/\partial(z_1, z_2) = 1$ at $z=t$, its (four-dimensional) volume does not decrease. We allow also mappings in which w_k are not single-valued functions, provided that the Jacobian of the transformation is a single-valued function in D. In this case we identify the image points which correspond to the same point in D. A representative domain B with $r \in B$ as center is a domain which satisfies

$$z_j = M^{(j)}(z_1, z_2)/M(z_1, z_2) + r_j, \qquad j = 1, 2,$$

where M is the function which minimizes the integral $\int_B |f(z)|^2 d\omega$ under the conditions $f \in \mathcal{L}^2(B)$, $f(r)=1$ ($d\omega$ is the volume element), and where $M^{(j)}$ is the function which minimizes the same integral under the conditions $f \in \mathcal{L}^2(B)$, $f(r)=0$, $\partial f/\partial z_j = 1$, $\partial f/\partial z_i = 0$ $(i \neq j)$ at $z=r$. (We do not explicitly denote the dependence of the functions M, M' and M'' upon r and upon the domain B.)

A minimal or a representative domain does not have to be *schlicht* provided the point t does not lie on a branch manifold. Schiffer [1] has given sufficient conditions for a minimal domain to be schlicht. Very little is known about the geometric shape of minimal or representative domains; yet, since *any* domain which has a Bergman kernel function can be mapped pseudo-conformally onto them, knowledge of properties of these domains enables us to deduce various results which are of interest in the theory of pseudo-conformal transformations, Bergman [5], Maschler [1].

Maschler [1] has proved the following theorem.

A necessary and sufficient condition for a domain D to be a minimal domain with respect to a point t as center is that its Bergman kernel function satisfy a relation $K_D(z, \bar{t}) = \text{const.}$ *for* $z \in D$. *The value of this constant is the reciprocal of the volume of* D.

From this result Maschler has deduced other properties of minimal domains which give some information about their shape and also lead to distortion theorems. It can be shown that Reinhardt circular domains are both minimal and representative domains with their center at their center of gravity. Maschler has pointed out, however, that this phenomenon is not true in general. In fact he has obtained the following result, Maschler [2]: *A necessary and sufficient condition for a minimal domain D to be also a representative domain with the same center t is that* $\partial K_D(z, \bar{t})/\partial \bar{t}_j = A_j(z_1 - t_1) + B_j(z_2 - t_2)$, $(j = 1, 2)$, *for* $z \in D$, *where A_j and B_j are constants* (A_1, B_2 *are real*; $A_2 = \bar{B}_1$ *and* $A_1B_2 - A_2B_1 \neq 0$). *The kernel function of any domain Δ which can be mapped pseudo-conformally onto a domain D of this type such that the Jacobian of the transformation is regular and different from zero at τ (the inverse image of t in Δ) has the property that its kernel function satisfies the following differential equation*:

$$S(w; \tau) = \frac{1}{K^4} \begin{vmatrix} K & K_{\bar{1}} & K_{\bar{2}} \\ K_1 & K_{1\bar{1}} & K_{1\bar{2}} \\ K_2 & K_{2\bar{1}} & K_{2\bar{2}} \end{vmatrix} = \text{const.}$$

for $w \in \Delta$, *where the elements of the determinant are defined as follows*:

$$K = K_\Delta(w; \bar{\tau}), \qquad K_j = \partial K_\Delta(w; \bar{\tau})/\partial w_j,$$

$$K_{\bar{j}} = \partial K_\Delta(w; \bar{\omega})\partial \bar{\omega}_j \text{ at } \omega = \tau$$

and

$$K_{j\bar{k}} = \partial^2 K_\Delta(w; \bar{\omega})/\partial w_j \partial \bar{\omega}_k) \text{ at } \omega = \tau,$$

$$w = (w_1, w_2), \qquad \omega = (\omega_1, \omega_2).$$

This identity yields information about the kernel function for the case $w \neq \tau$. Since Bergman [3] has shown that the expression $S(w; w)$ is invariant under pseudo-conformal transformations, the value of the constant can be computed.

2. Distinguished boundary surfaces.

A. *Value distribution problems*; *examples*. When we attempt to generalize the methods of the theory of functions of one complex variable to the case of two variables the following class of domains plays an important role. On the (three-dimensional) boundary of a domain of this class lies a (not necessarily connected) two-dimensional surface $\mathfrak{F}^2$ such that every function regular in the closed domain assumes the maximum of its absolute value *not only on the three-dimensional boundary but even at a point of* $\mathfrak{F}^2$. (Superscripts denote the dimension of manifolds.) The surface $\mathfrak{F}^2$ is denoted as the distinguished boundary surface of the domain under consideration. For instance, if the domain is bounded by finitely many segments of analytic hypersurfaces, then the sum of intersections of these hypersurfaces forms the distinguished boundary surface $\mathfrak{F}^2$.

The Cauchy and Poisson formulas are important tools in investigating functions of one complex variable. As has been shown, Bergman [1; 2; 12]; Weil [1; 2] there exists, for a large class of domains with distinguished boundary surfaces in several complex variables, an integral formula expressing the value of the function inside the domain in terms of its value on the distinguished boundary surface. This formula can be considered as a *generalization of the Cauchy formula* in one variable. However *it has the disadvantage that in the case of more than one variable its kernel depends on the domain.* Functions which are orthogonal when integrated over the distinguished boundary surface are introduced in Bergman [3; 13; 16]; these generalize the functions introduced by Szegö [1], which are orthogonal when integrated over the boundary curve of a domain in one variable. By using various positive weighting functions we then obtain various formulas expressing a function inside the domain in terms of its values on the distinguished boundary surface.

We proceed now to the question of analogs of the Poisson formula and of their applications, in particular, applications to the question of value distributions of functions of two variables. Let us note that the situation in this case differs in many respects from that in one variable. A function of two variables vanishes on segments of analytic surfaces. One of the problems of the theory of functions of two complex variables is to associate with these segments certain functionals which characterize some properties of these segments and which on the other hand are connected with the growth of the function. We are, in particular, interested in determining functionals which can be considered as generalizations of such notions as

$$D_\lambda(a, r) = \sum \left| z^{(\nu)}(a) \right|^{-\lambda},$$

where $z^{(\nu)}(a)$ are roots of $f(z) = a$ in the circle $|z| < r$. In the case of one variable derivations of many results in this direction are based on the possibility of solving boundary value problems for harmonic functions. In the case of a domain with a distinguished boundary surface, the boundary value problem for B-harmonic functions, (i.e., for the real parts of analytic functions of two complex variables) with arbitrary values prescribed on this surface does not always have a solution. In Bergman [1; 6; 16], Bergman and Schiffer [1] and Bremermann [5] various types of *functions of extended class* are introduced. Using these we can always solve the boundary value problem mentioned above and generalize potential-theoretic methods (in particular those of Nevanlinna and Ahlfors) to the case of two variables.

EXAMPLE 1. Consider the 2-parameter family of closed surfaces $\mathfrak{F}^2(r, s)$ in (z_1, z_2)-space defined by

$$\mathfrak{F}^2(r, s) = \{(z_1, z_2) \mid |z_2| = r,\ z_1 = h(z_2, t, \bar{t}),\ |t| = s\},$$

where $h(z_2, t, \bar{t})$ is a single-valued analytic function of $z_2, t, \bar{t}$. For any conveniently chosen real function $s = s(\rho)$, the set $\mathfrak{S}_r^3 = \bigcup_{r_0 \leq \rho \leq r} \mathfrak{F}^2(\rho, s(\rho))$ is a segment of the hypersurface $\mathfrak{S}^3 = \lim_{r \to \infty} \mathfrak{S}_r^3$. Then for an entire or meromorphic function $f(z_1, z_2)$ the set

$$\mathfrak{k}^1(a, r) = \mathfrak{S}_r^3 \cap \{(z_1, z_2) \mid f(z_1, z_2) = a\},$$

will be one-dimensional and will have a parametric representation

$$z_1 = z_1(\Psi, a), \qquad z_2 = z_2(\Psi, a), \qquad |z_2| < r$$

(with the parameter Ψ). The integral

$$B_\lambda(a, r) = \frac{1}{2\pi} \int_{\mathfrak{k}^1(a,r)} |z_1(\Psi, a)|^{-\lambda} d\Psi \tag{3}$$

can be considered as a generalization of the quantity $D_\lambda(a, r)$ in one variable. Generalizing the theorems of Hadamard and Borel it is possible to show that

$$\lim_{r \to \infty} B_\lambda(a, r)$$

exists if and only if λ is larger than a quantity connected with the growth of $f(z_1, z_2)$ on $\mathfrak{S}^3$. (See Bergman [14; 16], Bergman and Schiffer [1] and references to earlier papers of Bergman cited there.)

EXAMPLE 2. Since in pseudo-conformal transformations a pair of functions of two variables plays the same role as one function of one variable in conformal transformations, it is of interest to consider functionals connected with a pair of functions.

Let $\mathfrak{M}^4$ be a domain in (z_1, z_2)-space bounded by $\mathfrak{a}^3 \cup \mathfrak{b}^3$, where

$$\mathfrak{a}^3 = \{(z_1, z_2) \mid z_2 = \exp(i\lambda),\ 0 \leq \lambda \leq 2\pi\},$$
$$\mathfrak{b}^3 = \{(z_1, z_2) \mid z_1 = h(\lambda, z_2),\ 0 \leq \lambda \leq 2\pi,\ |z_2| \leq 1\},$$

and having distinguished boundary surface $\mathfrak{F}^2 = \mathfrak{a}^3 \cap \mathfrak{b}^3$. For each $|z_2^0| < 1$ we form the function $w(z_1, z_2^0)$ mapping the set

$$\mathfrak{B}^2(\overset{0}{z_2}) = \mathfrak{M}^4 \cap \{(z_1, z_2) \mid z_2 = \overset{0}{z_2}\},$$

which is assumed to be simply connected, onto the unit disk. For a pair of analytic functions (f_1, f_2), let $P_1(z_2^0)$ and $P_2(z_2^0)$ be the products of the $C(\mathfrak{B}^2)$ distances between every zero of f_1 and every zero and pole respectively of f_2 on $B^2(z_2^0)$; here by the $C(\mathfrak{B}^2)$ distance between two points (z_1, z_2) and (z_1', z_2^0) is meant the quantity

$$\frac{w(z_1, \overset{0}{z_2}) - w(z_1', \overset{0}{z_2})}{1 - w(z_1, z_2^0)\overline{w(z_1', z_2^0)}}\,.$$

Similarly $P_3(z_2^0)$ and $P_4(z_2^0)$ are defined by replacing the zeros of f_1 by poles of f_1.

The first functional to be considered is $\mathfrak{M}_1(\mathrm{P})$, the average of the *generalized Blaschke products*

$$\frac{P_1(z_2)P_4(z_2)}{P_2(z_2)P_3(z_2)}$$

over the circle $|z_2| < 1$, from which have been removed circles of radius P around certain exceptional points (such as the projections of the points of intersection of pole and zero surfaces of f_1 with pole and zero surfaces of f_2). A second functional $\mathfrak{M}_2(\mathrm{P})$ is introduced similarly.

In Bergman [11] an upper bound is derived for $\mathfrak{M}_1(\mathrm{P}) + \mathfrak{M}_2(\mathrm{P})$ in the following manner: we draw in M^4 tubes $\mathfrak{t}^4_{k\nu}$ of radius ρ around segments of zero-surfaces $\mathfrak{A}^2_{k1} = [z_1 = \alpha_{k1}(z_2)]$ and pole-surfaces $\mathfrak{A}^2_{k2} = [z_1 = \alpha_{k2}(z_2)]$ of f_k, $k = 1, 2$. Upper bounds for $|(z_1 - \alpha_{k\nu}(z_2))^{\tau} f_k|^{-1}$ and $|\partial[(z_1 - \alpha_{k\nu}(z_2))^{\tau} f_k]/\partial z_1)|$ in $\mathfrak{t}^4_{k\nu}$ are denoted by B_k and D_k respectively, while A_k and C_k are the respective bounds for $|f_k|$ and $|\partial f_k/\partial z_1|$ in the complementary part $M^4 - \sum \mathfrak{t}^4_{k\nu}$ of M^4. ($\tau = 1$ if f_k has a pole, $\tau = -1$ if f_k has a zero.) The upper bound for $\mathfrak{M}_1(\mathrm{P}) + \mathfrak{M}_2(\mathrm{P})$ is given in the form $S_1 + S_2$ where S_1 depends essentially upon P, ρ, A_k, B_k, C_k, D_k, $k = 1, 2$, the area of the above mentioned segments $\mathfrak{A}^2_{k\nu}$ and the volume of M^4. S_2 depends essentially, in addition to the previously mentioned constants, also on the upper bounds L_k of $|(z_2 - \gamma_{k2}(\lambda))\partial f_k/\partial z_2|$ in tubes of radius ρ around the intersection lines $[z_2 = \gamma_{k2}(\lambda)]$ of b^3 with the surfaces $(\partial f_k/\partial z_2) = \infty$ and on the upper bounds M_k of $|\partial f_k/\partial z_2|$ in the complementary part of $\mathfrak{b}^3$. Further, S_2 depends also upon the length of the lines $\mathfrak{b}^3 \cap [f_k = 0]$, $\mathfrak{b}^3 \cap [f_k = \infty]$.

EXAMPLE 3. See Bergman [7, pp. 170 ff.]. We note that in this case topological methods can be used, in particular Morse's theory of critical points.

B. *Poisson formulas in several variables.* The boundary value problem for the real parts of analytic functions of several complex variables (polyharmonic functions) in a domain D with a distinguished boundary surface B is in general insolvable. We may ask then for a polyharmonic function in D whose boundary values on B give a best approximation in L_2-norm to a prescribed set of values on B. A natural approach is to introduce a Bergman kernel function with respect to the distinguished boundary surface, Bergman [3]. However an approach based on Fourier series and integrals, Bochner [2], Bochner and Martin [1], will give an effective method of calculation in some special but interesting cases; only the results of this approach will be mentioned here.

Any polycylindrical domain may be transformed by a pseudo-conformal map onto a circular polycylinder

$$D: |z_j| < 1 \qquad (1 \leqq j \leqq n),$$

with distinguished boundary surface

$$B: |\zeta_j| = 1 \qquad (1 \leqq j \leqq n).$$

For any continuous distribution $u(\zeta_1, \cdots, \zeta_n)$ on B, Gunning has shown that the solution to this weak boundary value problem is given by:

$$u(z_1, \cdots, z_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} u(\zeta_1, \cdots, \zeta_n) \cdot \left\{ 2\Re \left[\frac{\zeta_1 \cdots \zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \right] - 1 \right\} d\phi_1 \cdots \phi_n,$$

where $\zeta_j = e^{i\phi_j}$ $(1 \leqq j \leqq n)$.

The space of n^2 complex variables may be represented as the space of $n \times n$ complex matrices $z = (z_{ij})$; this representation imposes a ring structure on the argument z, in terms of which some of the approaches used in the case of functions of one variable may be extended to yield similar formulas for matrices. Replace the unit disk by the generalized unit sphere

$$D = \{ z \mid z \cdot \bar{z}' < I \}$$

where I is the unit matrix and the ordering $u < v$ means $v - u$ is positive definite. The distinguished boundary surface of D is the set

$$B = \{\zeta \mid \zeta\cdot{}'\bar{\zeta} = I\},$$

the unitary group of $n\times n$ matrices. The equivalent of the angular measure $d\phi$ on the circle is the group-invariant measure $d\mu$ on the unitary group. Then for a continuous function $u(\zeta)$ on B vanishing at $-I$, Gunning has shown that the weak boundary value problem has the solution

$$u(z) = 2\int_B u(\zeta)\cdot\Re\left[\frac{\det(I+z)\cdot\det(\zeta)}{\det(I+\zeta)\cdot\det(\zeta - z)}\right]^n d\mu.$$

3. **Projective models for fields of automorphic functions.**

A. *Mappings and imbeddings.* We would like to state, in a special case, some results on analytic mappings in several complex variables recently worked out in some degree of completeness by Cartan [1; 2] (but which in principle go back at least as far as Osgood [1]) and to indicate their applications to a global imbedding theorem recently proved by Baily [1]. The problem in general with which these results deal is as follows:

Given analytic functions $f_1, \cdots, f_m$ at the origin a of the space of n complex variables C^n such that $f_1(a) = \cdots = f_m(a) = 0$ and a small neighborhood V of a, what does the image $\Phi(V)$ look like at the origin c of C^m, where $\Phi(z) = (f_1(z), \cdots, f_m(z))$? Osgood [1] proved a theorem which in part can be stated thus: *If $m=n$ and a is an isolated point of the set of common zeros of $f_1, \cdots, f_m$, then Φ is an open mapping in some neighborhood V of a, and with an appropriate choice of V, Φ represents V as a ramified N-fold covering of $\Phi(V)$, where N is some integer*. More recently Cartan [1; 2] has obtained somewhat more general results which are formulated in terms of the notion of a germ of a general analytic space. Though space is lacking to develop the latter concept, part of Cartan's results can be stated as they apply to our problem thus:

If a is an isolated point of the common zeros of $f_1, \cdots, f_m$, then every holomorphic function g at a satisfies a distinguished polynomial relation of the form

$$g^N + a_1 g^{N-1} + \cdots + a_N = 0$$

in some small neighborhood of a, where $a_1, \cdots, a_N$ belong to the ring of holomorphic functions analytically generated by $f_1, \cdots, f_m$; moreover, $\Phi(V)$ is an irreducible analytic variety at a (in the sense of being the set of common zeros of analytic functions).

These results have an application to the problem: *If H is a finite subgroup of the general complex linear group in n variables, and V is a* *small neighborhood of the origin a of C^n stable under H, how may V/H be regarded as an analytic variety?* Cartan has shown, on the basis of results related to those stated previously, that if I is the ring of holomorphic functions at a invariant under H and $f_1, \cdots, f_m$ generate I analytically (so that a is automatically an isolated point of the set of their common zeros), then, for a sufficiently small neighborhood V of a, $\Phi(V)$ is an irreducible variety at $\Phi(a)$ and its local ring there is naturally isomorphic to I; since it is easily seen that I is integrally closed in its quotient field, this means that $\Phi(V)$ is analytically normal at $\Phi(a)$. This result has been used by Baily in examining the nature of the quotient space of an analytic manifold D by a discrete group G of analytic self-homeomorphisms in case D/G is compact and there exists a positive G-complex line bundle over D, Baily [1]. In this case he has shown that D/G can be regarded as an algebraic variety, which is locally analytically normal. A similar result has also been obtained by Cartan and Serre, see Cartan [1; 2], in the special case when D is a bounded domain in C^n.

B. *Automorphic varieties.* Consider a bounded domain D in the space of n complex variables $z_1, \cdots, z_n$, and let Γ be a group of analytic homeomorphisms $\gamma: D \to D$. The elementary properties of Γ are established by means of Cauchy's integral theorem. (a) If we assume Γ is discontinuous at some point of D, then it is countable and is totally discontinuous in D. (b) Let the elements of Γ be γ_0, γ_1, γ_2, etc., where γ_0 is the identity, and let $J_\nu(z)$ denote the Jacobian

$$J_\nu(z) = \frac{\partial(\gamma_\nu z_1, \cdots, \gamma_\nu z_n)}{\partial(z_1, \cdots, z_n)} \qquad (\nu = 0, 1, 2, \cdots).$$

Then, under our assumption, the series

$$\sum_\nu |J_\nu(z)|^2$$

is majorized on every closed subset of D by a convergent series of constants.

Therefore, if $H(z)$ is bounded and holomorphic in D, the Poincaré θ-series

$$\vartheta(z) = \sum_0^\infty H(\gamma_\nu z) \cdot [J_\nu(z)]^k$$

represents a holomorphic function in D for any integral $k \geq 2$. $\vartheta(z)$ satisfies the relation

$$\vartheta(\gamma_\mu z) = \vartheta(z) \cdot [J_\mu(z)]^{-k}.$$

A key result, Giraud [1] concerning θ-series is the following: If points $a_1, \cdots, a_r$ are selected in D, none of which is a fixed point of any γ in Γ other than γ_0, and if values for $\vartheta(z)$ and its partial derivatives of order $\leqq m$ (m fixed) are specified at the points $a_1, \cdots, a_r$, then for all suitably large weights k there exist θ-series $\vartheta(z)$ which, with their partial derivatives of orders $\leqq m$, assume at $a_1, \cdots, a_r$ the prescribed values within any prescribed error $\epsilon > 0$.

For simplicity we suppose now that Γ has no maps other than γ_0 which have fixed points in D. The space $M = D \pmod{\Gamma}$ can always be regarded as a complex manifold, and we further assume that Γ is such that M is compact.

Quotients of θ-series of the same weight k define meromorphic functions on M, and it is easily shown by the above result that the field $\mathcal{J}$ of meromorphic functions on M has transcendance degree n over the complex numbers. Moreover, the functions of $\mathcal{J}$ separate points of M. From the lemma, Sampson [1] has established that the dimension ρ_k of the linear space of holomorphic densities of weight k on M tends to infinity with k, and from this that M admits only finitely many analytic self-homeomorphisms. Hawley [1] obtains this result from his theory of Picard domains.

In an oral communication to Sampson, G. Washnitzer has indicated how the holomorphic densities on M can be used to obtain a nonsingular imbedding of M in complex projective space, whence, by Chow's theorem, Chow [1], M is algebraic.

4. Domains of holomorphy; plurisubharmonic functions and Oka's lemma.

1. A real valued function V is called *plurisubharmonic* in a domain $D \subset M^n$ (M^n a complex manifold) if and only if the following conditions hold: (a) $-\infty \leqq V < \infty$; (b) V is upper semi-continuous; and (c) $\partial^2 V(z+\lambda a)/\partial\lambda\partial\bar{\lambda} \geqq 0$ (defined as a distribution) holds for all $z+\lambda a \in U_i$ and all U_i, where $\{U_i\}$ is a covering of D, the z's are local coordinates in U_i, λ is a complex number, and $a = (a_1, \cdots, a_n) \in C^n$ (the space of n complex variables).

There are equivalent definitions (see Lelong [1; 4] and Bremermann [1]). In a *schlicht* space condition (c) means that the function V is subharmonic in the intersection of any one-dimensional analytic plane with D. This definition can be used for Banach spaces, see Bremermann [8]. The plurisubharmonic functions are a proper subclass of the subharmonic functions of $2n$ real variables for $n > 1$; for $n = 1$ they coincide.

2. The functions built up from $\log |f|$, f holomorphic in D, by addition, multiplication with positive (real) numbers, taking upper

envelopes, and a closure operation are called *Hartogs functions*, Bochner-Martin [1]. Every (upper semi-continuous) Hartogs function is a plurisubharmonic function. If D is a domain of holomorphy the converse is true. This is a consequence of Oka's lemma (see Bremermann [7]). Bremermann has shown that the converse is not true if D is not a domain of holomorphy. In this case every Hartogs function has a "Hartogs continuation" into the envelope of holomorphy of D, but by the use of tube domains, Bochner [1], Bremermann has shown that plurisubharmonic functions can be constructed which do not possess a plurisubharmonic continuation, Bremermann [6]. This disproves the conjecture by Bochner-Martin [1] that the plurisubharmonic functions and the Hartogs functions coincide. This conjecture has been investigated by Lelong [2] and Hitotumatu [1].

3. Let $P = \{z \mid z \in D \wedge \log|f_1(z)| < 0 \wedge \cdots \wedge \log|f_k(z)| < 0\}$, where $f_1, \cdots, f_k$ are holomorphic in D. Let the closure $\bar{P}$ of P be contained in D, $\bar{P} \subset D$. Then P is called an analytic polyhedron, and $B = \{z \mid z \in \bar{P} \wedge \log|f_1(z)| = 0 \wedge \cdots \wedge \log|f_k(z)| = 0\}$ its distinguished boundary surface. Given continuous boundary values on B, take the class of all functions plurisubharmonic in P and less than or equal to the given boundary values on B. Bremermann has shown that the envelope of this class exists, is plurisubharmonic, and assumes the boundary values. The upper envelope function is different from Bergman's "function of the extended class" for the same domain P but serves the same purpose, Bergman [6] (and references cited there); Bremermann [7].

4. Plurisubharmonic functions generate metric forms since $\sum_{\mu,\nu=1}^{n} (\partial^2 V/\partial \bar{z}_\mu \partial z_\nu) dz_\mu d\bar{z}_\nu$ (defined as current) is positive semidefinite. Also several exterior differential forms are connected with plurisubharmonic functions, Lelong [4]. In particular: Let W^p be an analytic set in D of pure complex dimension p. Let $\{U_k\}$ be a covering of D. In each U_k, W^p can be represented as the intersection of the zero manifolds of $n-p$ functions f_s^k holomorphic in U_k. Let $V_s = \log|f_s^k|$ and $dzd\bar{z} V_s = \sum_{\mu,\nu=1}^{n} (\partial^2 V/\partial z_\mu \partial \bar{z}_\nu) dz_\mu \wedge d\bar{z}_\nu$. Then the current $\Theta_k = (i/\pi)^{n-p} dzd\bar{z} V_1 \wedge \cdots \wedge dzd\bar{z} V_{n-p}$ is defined in U_k. Then $\Theta = \Theta_k$ defines a closed current in D. For the integral over W^p of a p-form with compact support in D we have $\Theta(\phi) = \int_{W^p} \phi$, Lelong [4].

5. Given a domain D in a metric space, then we define:

$$\delta_D(z) = \sup r \ni \{z' \mid |z' - z| < r\} \subset D;$$

$\delta_D(z)$ is the distance of the point z from the boundary of D. $\delta_D(z)$ is continuous for any domain. A domain D in a complex Banach space is called "pseudo-convex" if and only if $-\log \delta_D(z)$ is plurisubharmonic in D. The definition of convex functions can be made completely analogous to the definition of the plurisubharmonic functions. (We could call convex functions for one real variable "sublinear" and for several real variables "plurisublinear.") Bremermann [5] has proved that a domain is convex if and only if $-\log \delta_D(x)$ is a convex function in D, and he [8] has proved the same result for real Banach spaces. "Complex convexity" (as we denote pseudo-convex domains and plurisubharmonic functions together) is the extension of the notion of convexity from real to complex spaces. The theories can be developed in parallel. One example: Let S_ν, S_0 be domains on one-dimensional analytic surfaces, T_ν, T_0 the boundaries, and $\lim_{\nu\to\infty} T_\nu = T_0$ and $\lim_{\nu\to\infty} S_\nu = S_0$. If for any such sequence for which S_ν, T_ν and $T_0 \subset D$ also $S_0 \subset D$, then we say "the theorem of continuity holds for D." D is pseudo-convex if and only if the theorem of continuity holds for D. (Bremermann [1], Lelong [3].) From this follows: D is pseudo-convex if and only if a function V, plurisubharmonic in D, exists such that for all real M the closure of $\{z \mid V(z) < M\}$ is contained in D, in other words if and only if a plurisubharmonic function exists that becomes infinite everywhere at the boundary of D. The same theorem holds for convex domains, if we replace "one-dimensional analytic surfaces" by straight lines, and the plurisubharmonic functions by convex functions.

6. A domain is called a "tube" domain if and only if it is of the form $\{z \mid x \in B, y \text{ arbitrary}\}$ (x real part, y imaginary part), Bochner [1]; Bochner-Martin [1]. A tube domain is pseudo-convex if and only if B is convex. A function defined in a tube domain and not depending on the imaginary parts is plurisubharmonic if and only if its restriction to B is convex in B. Therefore one obtains for every theorem on plurisubharmonic functions and pseudo-convex domains (that does not involve "existence") by specialization to tube domains a corresponding theorem on convex functions and domains. The converse is not true, of course. However, to every convexity theorem we have the problem: does the corresponding theorem on complex convexity hold? Bremermann [5].

7. Oka's lemma states: A schlicht and finite domain is a domain of holomorphy if and only if it is a pseudo-convex domain. The lemma solves a problem established 1910 by E. E. Levi [1] (compare Behnke-Thullen [1]; and Behnke-Stein [2]). The lemma was proved for two variables by Oka [2] and for n variables by Norguet [1] and Bremermann [4]. Recently a new proof for arbitrary n and for a certain class of complex manifolds was given by Oka [5]. The proof of the main part "if pseudo-convex, then domain of holomorphy" is

too involved to be sketched here. The other part can be proved in an easy way from the theorem: Let D be a domain of holomorphy and S, T sets such that $\overline{S \cup T} \subset D$, S, open and for any function $f(z)$ holomorphic in D assume that $\sup_{z \in S \cup T} |f(z)| = \sup_{z \in T} |f(z)|$. Let $g(z)$ be holomorphic in D. Then $\inf_{z \in S \cup T} \delta_D(z) |e^{g(z)}| = \inf_{z \in T} \delta_D(z) |e^{g(z)}|$. Also the holomorph-convexity (in the sense of Cartan-Thullen [1]) follows from this theorem immediately with $g(z) = 0$, Bremermann [5].

PROBLEMS: 1. To find a substitute for $\delta_D(z)$ on complex manifolds.

2. To study pseudo-convex domains on complex manifolds. As a definition the following could be used: A domain D is called pseudo-convex if and only if there exists a function V, plurisubharmonic in D, such that the closure of $\{z | V(z) < M\}$ is contained in D for all real M.

3. Prove Oka's lemma for arbitrary complex manifolds.

4. Study the sheaf of germs of plurisubharmonic functions.

5. Functions on algebras.

A. *Functions on a Clifford algebra.* This is a report on functions of a hypercomplex variable, extensive work on which has been carried out by R. Fueter [1] at the University of Zurich, and his school including H. G. Haefeli [1] in this country.

Let C be a Clifford Algebra of order 2^{n-1} with the basis

$$e_1, \cdots e_{n-1}$$

and the relations

$$e_k^2 = -1 \text{ (elliptic case)} \tag{1}$$

and

$$e_h e_k = -e_k e_h \text{ for } h \neq k, \tag{2}$$

and let $\mathcal{L}$ be a module in C with the basis $e_0, e_1, \cdots, e_{n-1}$, where $e_0 = 1$ denotes the principal unit. A variable $x \in \mathcal{L}$ is defined $x = \sum_{k=0}^{n-1} x_k e_k$, where the x's are real variables, and a function $w \in \mathcal{L}$ as a mapping from $\mathcal{L}$ into itself $w = \sum_{k=0}^{n-1} u_k e_k$, where the u's are real functions of the n variables. One requires that all these functions are defined and possess continuous first partial derivatives in all variables in a region R of the n-dimensional euclidean space E^n. If D denotes

$$D = \sum_{k=0}^{n-1} \frac{\partial}{\partial x_k} e_k, \tag{3}$$

a function w is called right- or left-analytic in R, if

$$wD = 0 \text{ or } Dw = 0 \text{ holds in } R. \tag{4}$$

These are two systems of linear partial differential equations and they can be considered as extensions of the Cauchy-Riemann equations; they are equivalent to $(n^2-n+2)/2$ real conditions. This choice of definition of analyticity is not a formal one, but originates from the desire to generalize Cauchy's integral theorem. Indeed, if $R_1 \subset\subset R$, the integral

$$(5) \qquad \int_{BdR_1} w\,dX = 0 \quad \text{if and only if} \quad wD = 0.$$

Let $\bar{D} = \sum_{k=0}^{n-1} (\partial/\partial x_k)\bar{e}_k$ with $\bar{e}_0 = e_0$ and $\bar{e}_k = -e_k$ for $k \neq 0$; then $D\bar{D} = \Delta$ is the Laplacian in n variables and $wD = 0$ implies

$$(6) \qquad \Delta w = 0$$

and this implies $\Delta u = 0$. The components of w are again harmonic functions as in the classical case. Analogously to Cauchy's formula it is also possible to express the value of w at a point $x \in R$, by means of the values of w on the boundary of R_1:

$$(7) \qquad w(x) = \text{const.} \int_{BdR_1} w(\xi)\,d\Xi \frac{\xi - x}{n(\xi - x)^{n/2}}.$$

From this it is possible for even n to obtain a development of an analytic function around a regular point in a series of orthogonal, homogeneous polynomials.

Instead of the conditions (1) one can take

$$(1') \qquad \begin{aligned} e_k^2 &= 1 \quad \text{for } k = 1, \cdots, m \text{ and} \\ e_k^2 &= -1 \text{ for } k = m+1, \cdots, n-1. \end{aligned}$$

Analytic functions are defined as in (4), and (5) holds; but (6) is now a hyperbolic partial differential equation of second order and (7) becomes an integral equation, which has been solved for special cases with the *partie finie methode*.

One may take the variable x and the function w in two different modules or algebras, as long as the second is invariant by left multiplication with elements from the first. $x \in \mathcal{L}_1$ and $w \in \mathcal{L}_2$, $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2$ and $D \in \mathcal{L}_1$. For particular choice of $\mathcal{L}_1$ and $\mathcal{L}_2$, (4) gives the Dirac equations and (6) the wave equation; (7) reduces to an integration over the intersection of BdR_1 with the characteristic cone through x, and the resulting integral equation has been solved.

If one takes instead of $\mathcal{L}$ a product system $P = [1, e_n][1, e_1, \cdots, e_{n-1}]$ and identifies $e_n = i$ with the ordinary complex unit, one obtains a class of analytic functions, which contains as special subclass the complex analytic functions

$$w = f_0 + f_1 e_1 + \cdots f_{n-1} e_{n-1}, \tag{8}$$

where $f_k(z_1, z_2, \cdots, z_n)$ are analytic functions of n-complex variables. (5) holds again and gives for the subclass the first rigorous proof of Hartogs' theorem on analytic continuation of f_k. (7) holds only for the subclass, from which one concludes that f_k can be approximated with rational functions in P.

For $n=2$ one obtains the algebra of the quaternions. (5), (6) and (7) and Runge's approximation theorem hold without exceptions. Around an isolated singularity there exists a Laurent expansion and one defines nonessential and essential singularities. The first need not be poles and one has essential singularities along curves and surfaces, in the neighborhood of which the function can be expressed with the help of a Stieltjes integral. Analytic continuation and first attempts to define generalized Riemann surfaces have been made. Also quadruply periodic functions and applications to the theory of numbers have been studied.

PROBLEMS: 1. For what Clifford Algebras does (7) hold without restrictions? Conjecture of Fueter: If the absolute term in the characteristic equation has no zero divisor.

2. What are the different systems of linear partial differential equations (4) which imply the same partial differential equation of second order (6)? (elliptic and hyperbolic case).

3. When does there exist for a system of linear partial differential equations an algebra such that (4) implies (5)?

4. Can every *schlicht* region be the domain of definition for an analytic quaternion function?

5. Is a product representation of an analytic quaternion function possible?

6. What is the behavior of an analytic quaternion function around an isolated essential singularity?

B. *Functions on a complex algebra.* Let E denote the algebra defined on pairs (x_1, x_2) of complex numbers by the multiplication rule $(x_1, x_2)\cdot(y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)$ and the addition rule $(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$. This algebra is commutative, associative, and has no nilpotent elements. Hawley has developed a theory of E-valued functions on E which very closely parallels the classical theory of functions of one complex variable.

This E-function theory has Cauchy-Riemann equations, a Cauchy's theorem, and formula, calculus of residues, Taylor's and Laurent's expansions, etc. Let $X=(x_1, x_2)$ and $Y=(y_1, y_2)$, then the functions $Y=f(X)$ considered are those such that $y_1=u_1(x_1, x_2)$, $y_2=u_2(x_1, x_2)$ satisfy $\partial u_1/\partial x_1=\partial u_2/\partial x_2$ and $\partial u_2/\partial x_1=-\partial u_1/\partial x_2$, i.e. a very special class of mappings in two complex variables. The property of being an E-function, or E-analytic, is *not* invariant under arbitrary pseudo-conformal mappings. This last fact ultimately proves to be of great advantage, bringing within the scope of the E-theory a much larger class of analytic mappings than is apparent at first. A proper change of coordinates may reduce a quite general analytic mapping to an E-function. Partial differential equations (but not a unique set of them) for the new coordinates can be given. Of course, all of this is quite local.

Hawley has also developed a theory of conformal mapping in two complex variables on the basis of this E-function theory. Thus if C_1 and C_2 are two complex curves, they have a *complex* angle between them which is preserved by elements of E-function theory just as the real angle is preserved in ordinary conformal mappings.

Of course, the parallelism of the E-theory to the classical theory becomes less satisfactory the deeper we go. For the most part, analogies to existence theorems and boundary properties break down. Its strength lies in special cases, not generality.

Analogues to the theory of algebraic curves have been developed and one obtains a special class of algebraic surfaces for which one has an extended set of methods for investigating.

There are certain obvious extensions to more than two complex variables, but these will not be discussed here.

PROBLEM: "Characterise" a complex manifold which is complex analytically homeomorphic to a bounded domain in a space of several complex variables.

As a special case of this problem, consider the simply connected Picard domains. (For definition see Hawley [1, p. 638].) The Picard domains are a generalization from one to several variables of the domains possessing at least two boundary points. In the case of one variable it is trivial to prove that a simply connected Picard domain is analytically homeomorphic to a bounded domain. Is this proposition true for more variables?

6. **Operators transforming functions of complex variables into solutions of linear partial differential equations.** By the relation $\Psi=[g(z)+\bar{g}(\bar{z})]/2$ the linear space of harmonic functions Ψ is mapped onto the algebra of analytic functions of one complex variable. Here z and $\bar{z}$ are complex conjugates, $z=x+iy$, $\bar{z}=x-iy$, and x and y are real. If the function Ψ is continued to complex values of x, y the

variables z, $\bar z$ become two independent complex variables. If we consider the harmonic function Ψ in a neighborhood of a given point P with coordinates $z_0=x_0+iy_0$, then the associated analytic function g satisfies the relation $g(z)=2\Psi(z, \bar z_0)-\bar g(\bar z_0)$, and thus we note that the associate g changes by a constant when the point P varies.

Bergman has generalized this mapping to the case of linear partial differential equations

$$L(\Psi) = \Psi_{z\bar z} + a\Psi_z + \bar a\Psi_{\bar z} + c\Psi = 0 \tag{1}$$

whose coefficients are entire functions. In this case we have the relation

$$\Psi = [p(g) + \bar p(\bar g)]/2 \tag{2}$$

where $p(g)=p[g(z), P]$ is Bergman's integral operator of the first kind. As in the harmonic case, the function g which we associate with a given solution Ψ of (1) depends upon the point P in the neighborhood of which the relation (2) is defined. This relation is initially defined in a sufficiently small neighborhood of the point P. Bergman has shown however that it can be continued so that it holds in the large. For example, it holds in the domain of regularity of g if this domain is simply connected.

If P is the origin 0 the functional $p(g)$ of (2) can be expressed in the form

$$\begin{aligned} &p[g(z), 0] \\ &= \exp\left(-\int_0^{\bar z} a\,d\bar z\right)\left[g(z) + \sum_{n=1}^{\infty} T^{(n)}(z, \bar z)\int_0^{z} (z-\zeta)^{n-1}g(\zeta)\,d\zeta\right] \end{aligned} \tag{3}$$

where the $T^{(n)}$ are functions of z, $\bar z$ which *depend only* on equation (1) and possess the property that $T^{(n)}(z, 0)=0$. Bergman [4; 9]. It should be noted, however, that due to the fact that integrals appear in (3) certain complications arise when the solutions Ψ defined in the small are continued in the large.

In this theory it is of basic importance that *various relations between g and Ψ are independent of the coefficients a*, c occurring in the operator L, or that various other relations depend only upon certain properties of these coefficients. As an example for this fact one sees from (2) and (3) that the relation

$$\Psi(z, 0) = \frac{1}{2}\left[g(z) + \exp\left(-\int_0^{z} \bar a(z, 0)\,dz\right)\bar g(0)\right]$$

holds.

The use of the integral operator of the first kind shows that *most* *of the results of the theory of functions of one complex variable can be interpreted as theorems on real solutions of the differential equation* (1) (and not merely as theorems on harmonic functions). A survey of some of the results in this direction (with bibliographical data) can be found in Bergman [8, p. 38 ff.], and in Bergman [9] where multi-valued solutions Ψ are considered.

The theory of integral operators has been generalized in two directions: (1) the case when the coefficients a, c have singularities of certain types; and (2) the case of harmonic and more general equations in three variables as well as the case of certain systems of linear partial differential equations. See Bergman [9; 10]. As Bergman has pointed out, while investigations of flows of incompressible fluids lead to the theory of harmonic functions of two variables, the case of compressible fluids, when the hodograph method is used, introduces problems in the theory of a class of equations of the type described in (1). Further, as Bergman [14] and Kreysig [1] pointed out, special integral operators are useful for characterizing solutions of partial differential equations using the theory of ordinary differential equations.

In the remainder of this section we consider only the case (2). It seems natural that harmonic functions of three variables have to be mapped onto analytic functions of *two* complex variables. Bergman (see [4; 9; 10] and earlier references cited there) has introduced the operator

$$
\begin{aligned}
H(X) &= C_3(g, P) \\
&= \frac{1}{\pi i}\int_{|\zeta|=1}\int_0^1 u^{1/2}\,\frac{d[u^{1/2}g(u\zeta^{-1}T^2, u\zeta(1-T)^2)]}{du}\,dT\,\frac{d\zeta}{\zeta},
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
X &= (x, Z, Z^*) = (x, (z+iy)/2, -(z-iy)/2), \\
u &= x + Z\zeta + Z^*\zeta^{-1},
\end{aligned}
\tag{5}
$$

and he has shown that (4) *transforms analytic functions* g *of two complex variables into harmonic functions* $H(X)$ of three variables x, y, z. He has further considered, Bergman [10], integral operators

$$P_3(G, P) = G(X) + \int_0^1 M(r^2, \sigma^2)G(X\sigma^2)d\sigma$$

which transform harmonic functions $G(X)$ of three variables into solutions of the equation

$$\Delta(\Psi) = \Delta_3\Psi + C\Psi = 0$$

where C is an entire function of $r^2=x^2+y^2+z^2=x^2-4ZZ^*$. The inverse of $p_2[C_2(g, P)]$, $g(Z, Z^*)=\Psi[2(ZZ^*)^{1/2}, Z, Z^*]$, is *independent of the coefficient* C of the equation $\Lambda(\Psi)=0$. In Bergman [9] solutions of $\Lambda(\Psi)=0$ are investigated, whose associates $g=g_1+(ZZ^*)^{1/2}g_2$ have the property that g_1, g_2 are rational or algebraic functions of Z, Z^*. In his lectures Bergman showed that various theorems in the theory of functions of two complex variables can be interpreted as theorems on solutions of $\Lambda(\Psi)=0$, theorems on more general differential equations in three variables, theorems on harmonic vectors, Bergman [9], and theorems on certain differential equations in four variables.

In this section we have discussed only very special integral operators, mapping functions of complex variables into solutions of linear differential equations and systems of such equations. To every differential equation there exist infinitely many such operators, and the question of determining and classifying them is of great interest. In particular the investigation of how different operators permit us to use the theory of analytic functions for the study of solutions of differential equations (for real and complex values of the arguments) represents an interesting task.

REFERENCES

W. L. BAILY

1. *On the quotient of an analytic manifold by a group of analytic homeomorphisms*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 804–808.

H. BEHNKE and K. STEIN

1. *Konvergente Folgen nichtschlichter Regularitaetsbereiche*, Annali di Matematica Pura ed Applicata (4) vol. 28 (1949) pp. 317–326.
2. *Die Singularitaeten der analytischen Funktionen mehrerer Veraenderlichen*, Nieuw Archief voor Wiskunde, Amsterdam, 1951, pp. 227–242.

H. BEHNKE and P. THULLEN

1. *Theorie der Funktionen mehrerer komplexer Veraenderlichen*, Ergebnisse der Mathematik, vol. 3, 1934, no. 3.

S. BERGMAN

1. *Über ausgezeichnete Randflächen in der Theorie der Funktionen von zwei komplexen Veränderlichen*, Math. Ann. vol. 105 (1931) pp. 611–636.
2. *Über eine Integraldarstellung von Funktionen zweier komplexer Veranderlichen*, Rec. Math. (Mat. Sbornik) N.S. vol. 1 (43) (1936) pp. 851–862.
3. *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la theorie des fonctions analytiques*, Interscience, 1941, and Memorial des Sciences Mathématiques vol. 106, Paris, 1947.
4. *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155.
5. *Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes*, Mémorial des Sciences Mathématiques, vol. 108, Paris, 1948.
6. *The kernel function and conformal mapping*, Mathematical Surveys, no. 5, Amer. Math. Soc., 1950.

7. *Geometric and potential-theoretical methods in the theory of functions of several complex variables*, Proceedings of the International Congress of Mathematicians, vol. 2, 1950, pp. 165–173.
8. *Operatorenmethoden in der Gasdynamik*, Zeitschrift für angewandte Mathematik und Mechanik vol. 32 (1952) pp. 33–45.
9. *The coefficient problem in the theory of linear partial differential equations*, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 1–34.
10. *Essential singularities of solutions of a class of linear partial differential equations in three variables*, Journal of Rational Mechanics and Analysis vol. 3 (1954) pp. 539–560.
11. *On zero and pole surfaces of functions of two complex variables*, Trans. Amer. Math. Soc. (1955) vol. 77 (1954) pp. 413–453.
12. *Zur Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randflache*, Jber. Deutschen Math. Verein. vol. 42 (1932–1933) pp. 238–252.
13. *Bounds for analytic functions in domains with a distinguished boundary surface*, Math. Zeit. vol. 63 (1955) pp. 173–194.
14. *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen*, Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 1169–1198.
15. *Ueber meromorphe Funktionen von zwei komplexen Veranderlichen*, Compositio Math. vol. 6 (1939) pp. 305–335.
16. *Kernel function and extended classes in the theory of functions of complex variables*, Colloque sur les fonctions de plusieurs variables, Paris, Masson, 1953, pp. 135–157.

S. Bergman and M. Schiffer
1. *Potential-theoretic methods in the theory of functions of two complex variables*, Compositio Math. vol. 10 (1952) pp. 213–240.

S. Bochner
1. *A theorem on analytic continuation of functions in several variables*, Ann. of Math. vol. 39 (1938) pp. 14–19.
2. *Group invariance of Cauchy's formula in several variables*, Ann. of Math. vol. 45 (1944) pp. 686–706.

S. Bochner and W. T. Martin
1. *Several complex variables*, Princeton, 1948.

H. J. Bremermann
1. *Die Charakterisierung von Regularitaetsgebieten durch pseudokonvexe Funktionen*, Schriftenreihe des Math. Inst., Muenster, no. 5, 1951.
2. *Holomorphic continuation of the Bergman kernel function and the invariant metric in several complex variables*, Lectures on functions of a complex variable, . . . (Report of the Ann Arbor, Michigan 1953 conference.)
3. *Die Holomorphiehuellen der Tuben-und Halbtubengebiete*, Math. Ann. vol. 127 (1954) pp. 406–423.
4. *Ueber die Aequivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veraenderlichen*, Math. Ann. vol. 128 (1954) pp. 63–91.
5. *Complex convexity*, Technical Report 38 (ONR), Contract 106-5, Stanford, California, 1954.
6. *On a conjecture on Hartogs functions*, To appear.
7. *A plurisubharmonic extended class of functions*, To appear.
8. *Complex convexity in Banach spaces*, To appear.

H. Cartan
1. *Séminaire Ecole Normale Supérieure*, 1951–1952.

2. *Séminaire Ecole Normale Supérieure*, 1953–1954.

H. CARTAN and P. THULLEN

1. *Zur Theorie der Singularitäten der Funktionen mehrerer Veränderlichen: Regularitäts-und Konvergenzbereiche*, Math. Ann. vol. 106 (1932) pp. 617–647.

W. L. CHOW

1. *On compact complex analytic varieties*, Amer. J. Math. vol. 71 (1949) pp. 893–914.

R. FUETER

A list of the publications of Fueter and his students can be found at the end of the paper by H. G. Haefeli listed below.

G. GIRAUD

1. *Leçons sur les fonctions automorphes*, Paris, Gauthier-Villars, 1926, pp. 19–25.

H. G. HAEFELI

1. *Hyperkomplexe Differentiale*, Comment. Math. Helv. vol. 20 (1947) pp. 382–420.

NEWTON S. HAWLEY

1. *A theorem on compact complex manifolds*, Ann. of Math. vol. 52 (1950) pp. 637–641.

S. HITOTUMATU

1. *On some conjectures concerning pseudo-convex domains*, J. Math. Soc. Japan vol. 6 (1954) pp. 177–195.

E. KREYSZIG

1. *Integral operators in the theory of partial differential equations*, Journal of Rational Mechanics and Analysis vol. 4 (1955) pp. 907–923.

P. LELONG

1. *Les fonctions plurisousharmoniques*, Ann. École Norm. vol. 62 (1945) pp. 301–345.
2. *La convexité et les fonctions analytiques de plusieurs variables complexes*, J. Math. vol. 31 (1952) pp. 191–219.
3. *Domaines convexes par rapport aux fonctions plurisousharmoniques*, Journal d'Analyse Mathématique vol. 2 (1952–1953) pp. 178–208.
4. *Sur les dérivées d'une fonction plurisousharmonique*, C. R. Acad. Sci. Paris (1954) pp. 2276–2278.

E. E. LEVI

1. *Studii sui punti singolari essenziali delle funzioni analitiche di due o più variabili complesse*, Annali di Matematica Pura ed Applicata (3) vol. 17 (1910) pp. 61–87.

M. MASCHLER

1. *Properties of minimal domains*, Proceedings of the International Mathematical Congress, Amsterdam, 1954.
2. *Minimal domains and their Bergman kernel function*, to appear in Pacific Journal of Mathematics.

F. NORGUET

1. *Sur les domaines d'holomorphie des fonctions uniformes de plusieurs variables complexes (passage du local au global)*, Bull. Soc. Math. France vol. 82 (1954) pp. 137–159.

K. OKA

1. *Sur les fonctions analytiques de plusieurs variables*. II. *Domaines d'holomorphie*, Journal of Science of the Hiroshima University Ser. A vol. 7 (1937) pp. 115–130.
2. VI. *Domaines pseudoconvexes*, Tôhoku Math. J. vol. 49 (1942) pp. 15–52.

3. VII. *Sur quelques notions arithmetiques*, Bull. Soc. Math. France vol. 78 (1950) pp. 1–27.
4. VIII. *Lemme fondamental*, J. Math. Soc. Japan vol. 3 (1951) pp. 204–212, 259–279.
5. IX. *Domaines finis sans point critique interieur*, Jap. J. of Math. vol. 23 (1953) pp. 97–155.

W. F. OSGOOD
1. *Lehrbuch der Funktionentheorie*, 2d ed., vol. 2, part 1, Leipzig, Teubner, 1929.

J. H. SAMPSON
1. *A note on automorphic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) pp. 895–898.

M. SCHIFFER
1. *Sur les domaines minima dans la theorie des transformations pseudo-conformes*, C.R. Acad. Sci. Paris vol. 207 (1938) pp. 112–115.

G. SZEGÖ
1. *Über orthogonale Polynome, die zu einer gegebenen Kurve der Komplexen Ebene gehören*, Math. Zeit. vol. 2 (1921) pp. 218–270.

A. WEIL
1. *Sur les series de polynômes de deux variables complexes*, C. R. Acad. Sci. Paris vol. 194 (1932) p. 1304.
2. *L'integral de Cauchy et les fonctions de plusieurs variables*, Math. Ann. vol. 111 (1935) pp. 178–182.

W. T. MARTIN

PART II. COMPLEX MANIFOLDS[1]

The notion of a complex manifold is a natural outgrowth of that of a differentiable manifold. Its importance lies to a large extent in the fact that it includes as special cases the complex algebraic varieties and the Riemann surfaces and furnishes the geometrical basis for functions of several complex variables. Its development has led to clarifications of classical algebraic geometry and to new results and problems. Two notions from algebraic topology have so far played an essential rôle: sheaves (faisceaux) and fiber bundles. But the deeper problems on complex manifolds are not entirely topological.

1. **Topology of complex manifolds.** From the point of view of topology a fundamental problem would be to characterize the orientable manifolds of even dimension $2n$ which can be given a complex structure. But this is too difficult and, at least at the present moment, one

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[1] Acknowledgement. §4 on Stein manifolds is based on material prepared by Bally, Bremermann, and Gunning for Part I, later transferred to this part for the sake of harmony. N. Hawley prepared a summary of Atiyah's work on projective bundles. The whole manuscript has been critically read by A. Borel, K. Kodaira, D. C. Spencer, H. C. Wang, A. Weil, and many others. While it is a pleasant duty of the writer to thank these mathematicians for their help and criticisms, it should be remarked that the report is prepared by the writer as an organized article, for whose shortcomings and inaccuracies these mathematicians are not responsible.

should be satisfied with necessary conditions. An immediate necessary condition is obtained by the consideration of the tangent bundle.[1] For the existence of a complex structure implies that the tangent bundle whose structural group is the general linear group $GL(2n, R)$ in $2n$ real variables is equivalent to a bundle whose structural group is the general linear group $GL(n, C)$ in n complex variables, considered as a subgroup of $GL(2n, R)$. A manifold of even dimension $2n$ with the latter property is called almost complex. Various necessary conditions are known for a manifold to be almost complex.

Among such necessary conditions the most effective ones are expressed in terms of the characteristic classes of the manifold, if we make the further assumption that the latter is compact. These are the Stiefel-Whitney classes $W^i \in H^i(M, Z_2)$,[2] $1 \leq i \leq 2n$, and the Pontrjagin classes $p_k \in H^{4k}(M, Z)$, $1 \leq k \leq [n/2]$ [49]. If M is almost complex, its almost complex structure defines the Chern classes $c_k \in H^{2k}(M, Z)$, $1 \leq k \leq n$, [49]. The following relations between the characteristic classes give necessary conditions for a manifold to be almost complex [49]:

$$W^i = 0, \qquad i \text{ odd}, \tag{1}$$

$$\sum_{i=0}^{[n/2]} (-1)^i p_i = \sum_{i=0}^{n} (-1)^i c_i \sum_{i=0}^{n} c_i. \tag{2}$$

Further necessary conditions are obtained by considering cohomology operations on the characteristic classes, in particular, the Steenrod squaring and reduced power operations.

These conditions suffice to give the theorem that among the even-dimensional spheres only S^2 and S^6 are almost complex. In fact, the absence of an almost complex structure on S^{4k} was derived by Wu as a consequence of (2) and the facts: (1) $p_k = 0$; (2) $c_{2k} \cdot M = 2$. By using the formulas expressing $\mathcal{P}_p^i c_j$ as a polynomial of $c_1, \cdots, c_n$ ($\mathcal{P}_p^i$ is a Steenrod reduced power operation), Borel and Serre proved that S^{2n} is not almost complex for $n \geq 4$ [6].

The question whether an almost complex manifold can be given a complex structure remains unanswered. In particular, the existence or nonexistence of a complex structure on S^6 is still one of the urgent unsolved problems on complex manifolds. The difficulty lies in the fact that there is at present no method to find topological implications of the existence of a complex structure, which is not already true for the existence of the underlying almost complex structure. On the other hand, for a given almost complex structure, necessary conditions are known in order that it defines a complex structure (Ehresmann-Libermann-Eckmann-Frölicher [16; 17]). These conditions are sufficient, if the almost complex structure is analytic.

---

[1] For basic notions on fiber bundles we refer the reader to Steenrod [42].

[2] We shall be using the following notations, now customary: Z denotes the ring of integers, Z_p (p prime) the finite field of p elements, R the real field, and C the complex field. If X is a topological space and G an abelian group, $H^r(X, G)$ (resp. $H_r(X, G)$) denotes the r-dimensional cohomology (homology) group with coefficient group G. If G is a ring and $\gamma \in H^r(X, G)$, $c \in H_r(X, G)$, then $\gamma \cdot c \in G$ denotes the pairing of the two groups into G. When M is an oriented manifold, the same notation will be used to denote its fundamental homology class.

Two complex structures on a manifold M are called inequivalent, if there exists no homeomorphism of M onto itself which transforms one complex structure into the other. Hirzebruch proved that the manifold $S^2 \times S^2$ has an infinite number of inequivalent complex structures [20]. In his study of homogeneous complex manifolds (cf. §6), Wang [47] gave examples of manifolds, among which are products of two spheres of odd dimensions >1, which have noncountably many inequivalent complex structures. However, it is undecided whether the complex projective plane has a complex structure inequivalent to its natural one.

As a counterpart of the birational transformations in algebraic geometry, Hopf introduced the modification or the σ-process [26; 44] (cf. also the earlier work of Behnke-Stein [3]). Geometrically the process can be pictured as "blowing up" a submanifold which is complex-analytically imbedded in a complex manifold. It allows the construction of new complex manifolds from given ones. Various questions concerning the effect of this process on the invariants of complex manifolds remain to be studied.

2. **Complex analytic bundles.** Let Y be a complex manifold acted on by a complex Lie group G of complex analytic homeomorphisms. To define a complex analytic bundle over a complex manifold M with fiber Y we take a covering $\{U_i\}$ of M by coordinate neighborhoods. The bundle over U_i is homeomorphic to $U_i \times Y$ and its points have the local coordinates (z, y_i), $z \in U_i$, $y_i \in Y$, such that, if $z \in U_i \cap U_j$, the local coordinates (z, y_i) and (z, y_j) of the same point satisfy the relation $y_i = g_{ij}(z) y_j$, where g_{ij} (to be called the transition functions) defines a complex analytic mapping of $U_i \cap U_j$ into G. Perhaps the simplest case is when Y is a complex vector space of q dimensions and $G = GL(q, C)$. The bundle is then called a complex vector bundle. It is to be observed that it is the group G and the transition functions $\{g_{ij}(z)\}$, and not the fiber Y, which play a basic role in the properties of the bundle. With a notion of equivalence introduced in a natural way, a fundamental problem would be to determine all classes of

bundles over M with a given group G. So far little progress has been made toward this "classification problem." In the case of complex line bundles, that is, complex vector bundles with $q=1$ (cf. §3), the abelian character of the group $GL(1, C)$ makes it possible to introduce a group operation in the set of complex line bundles and thus allows a complete enumeration of the complex line bundles over a complex manifold. Another contribution to this problem was recently made by Grothendieck, who classified the complex vector bundles over the complex projective line (not yet published).

The topological theory of fiber bundles furnishes two tools which are of importance: the characteristic classes and the universal bundle theorem. The former are defined in terms of the underlying topological bundle of the analytic bundle. In spite of this there are problems concerning them which are not entirely of topological nature. For instance, it has been proved that, for an algebraic variety, the dual homology classes of the characteristic classes of the tangent bundle contain representative cycles which are algebraic [11]. This theorem can be proved by using the homology theory of fiber bundles, a method which will undoubtedly find further applications in the study of analytic bundles. On the other hand, the important homotopy methods in topological fiber bundles seem to have too much disregard for the complex structure and have so far not been found useful.

The Grassmann manifold $G(q, N)$ of all q-dimensional linear spaces through a point of a complex Euclidean space of dimension $q+N$ is, in an obvious way, the base space of a bundle of complex vector spaces of dimension q. The Grassmann manifold itself can be identified with the manifold of all $(q-1)$-dimensional linear spaces in a complex projective space of dimension $q+N-1$ and is a complex algebraic variety. As a result our bundle is an analytic bundle. A complex analytic mapping of a complex manifold M into $G(q, N)$ gives rise to a complex analytic vector bundle over M. In contrast to the topological case, it is not true that every complex vector bundle over M can be defined in this way, even if N is allowed to be sufficiently large.

If the base manifold M is an algebraic variety, a related theorem was proved by Nakano and Serre [37]: Let W be an analytic bundle of complex vector spaces of dimension q, with the transition functions $\{g_{ij}(z)\}$, $z \in U_i \cap U_j$, relative to a covering $\{U_i\}$ of M, $g_{ij}(z)$ being $q \times q$ nonsingular matrices. Let E be the complex line bundle defined by a generic hyperplane section of M, whose transition functions relative to the same covering of M are $\{f_{ij}(z)\}$. Then, for a sufficiently large positive integer m, the bundle $W(-E)^m$, with the transition functions $\{g_{ij}(z)f_{ij}^{m}(z)\}$, is equivalent to one induced by a complex analytic mapping of M into a Grassmann manifold $G(q, N)$.

Two other groups appear prominently as the structure groups of analytic bundles: the nonhomogeneous complex linear group $G'(q)$ in q variables and the linear fractional group $K(q)$ in q variables, considered to be acting on the complex affine space E_q and the complex projective space P_q respectively. The corresponding bundles are called the affine and projective bundles. Projective bundles are of importance in algebraic geometry. These bundles are related to the vector bundles. In fact, the group $T(q)$ of translations is a normal subgroup of $G'(q)$ and $G'(q)/T(q)$ is isomorphic to $GL(q)$. Similarly, the group S of all scalar matrices λI ($\lambda \neq 0$, I = identity matrix) is a normal subgroup of $GL(q)$, and $GL(q)/S$ is isomorphic to $K(q-1)$. By taking projections of the transition functions into quotient groups, we get a vector bundle from an affine bundle and a projective bundle of one less dimension from a vector bundle. These bundles shall be called the derived bundles. A necessary condition for two affine bundles (or two vector bundles) to be equivalent is that their derived bundles are equivalent.

It turns out that the property for a projective bundle to be the derived bundle of a vector bundle of one more dimension is equivalent to another important property. The Grassmann manifold $G(q, N)$ is the base space of a bundle of projective spaces of dimension $q-1$. We say that a projective bundle is regular if it can be induced by a complex analytic mapping of M into $G(q, N)$. It follows from the theorem of Nakano and Serre that a projective bundle is regular if and only if it is the derived bundle of a vector bundle of one higher dimension.

Because of our present limited knowledge of vector bundles the classification of affine and projective bundles has to be restricted to line bundles ($q=1$). The first invariant of an affine line bundle is the derived complex line bundle. Over a compact complex manifold M consider all the affine line bundles which have the same derived complex line bundle F, but are inequivalent to it.[4] These affine line bundles can be set in a natural way into a one-one correspondence with the complex projective space derived from the vector space $H^1(M, \Omega(F))$, where $\Omega(F)$ is the sheaf of germs of holomorphic cross-sections of F [27; 36; 48]. If M is an algebraic variety, F is defined by a divisor class[5] D and $\Omega(F) = \Omega(D)$ is the sheaf of germs of meromorphic functions $> -D$. The dimension $i = i(D) = \dim H^1(M, \Omega(D))$ is then known as the index of specialty of D, if $\dim M = 1$, and as the superabundance of D, if $\dim M = 2$.

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[4] Relative to a covering $\{U_i\}$ of M let (z, y_i), $z \in U_i$, $y_i \in A$ (=affine line) be the local coordinates in the affine line bundle. Then, in $U_i \cap U_j$, the transition of coordinates is given by $y_i = a_{ij}(z)y_j + b_{ij}(z)$, where $a_{ij}(z)$, $b_{ij}(z)$ are holomorphic functions in $U_i \cap U_j$. The derived line bundle has by definition the transition functions $\{a_{ij}(z), z \in U_i \cap U_j\}$. The affine line bundle is said to be inequivalent to its derived complex line bundle, if it is not possible to make all $b_{ij}(z) = 0$ by an "analytic change of coordinates."

The study of a projective line bundle begins with the question whether it is equivalent to an affine line bundle. A necessary and sufficient condition for this is that the projective line bundle has an analytic cross-section. If this is the case, the projective bundle is regular. The converse is true if the base manifold is an algebraic curve A projective line bundle over an algebraic curve is always regular (unpublished result of Kodaira, Serre). The classification of projective line bundles over an algebraic curve reduces to finding conditions that the reduced affine line bundles are projectively equivalent. This problem was solved by Atiyah [2]. Atiyah is able to apply his theory to ruled surfaces with a high degree of success. Many classical results become easily accessible, and he is able to add new ones.

3. **Sheaves (faisceaux)**[6]. The cohomology groups of a manifold with a coefficient sheaf furnish the algebraic tool to formulate globally the properties of its local structure [9; 35; 41]. Its usefulness is based on the fact that the 0-dimensional cohomology group $H^0(M, \mathfrak{f})$ ($\mathfrak{f}$ = coefficient sheaf) has a simple interpretation: It is the group of all cross sections $\Gamma(M, \mathfrak{f})$. But the consideration of the high-dimensional cohomology groups is important, because of the following fundamental property: Let $\mathfrak{g}$ be a subsheaf of $\mathfrak{f}$, and $\mathfrak{f}/\mathfrak{g}$ be the quotient sheaf. Then the cohomology groups are related by an exact sequence of homomorphisms

$$\begin{aligned} &0 \to H^0(M, \mathfrak{g}) \xrightarrow{i^0} H^0(M, \mathfrak{f}) \xrightarrow{j^0} H^0(M, \mathfrak{f}/\mathfrak{g}) \xrightarrow{\delta^0} H^1(M, \mathfrak{g}) \to \cdots \\ &\quad \to H^q(M, \mathfrak{g}) \xrightarrow{i^q} H^q(M, \mathfrak{f}) \xrightarrow{j^q} H^q(M, \mathfrak{f}/\mathfrak{g}) \xrightarrow{\delta^q} H^{q+1}(M, \mathfrak{g}) \to \cdots . \end{aligned} \tag{3}$$

As an example we consider the case that $\mathfrak{f} = \mu$ is the sheaf of germs of meromorphic functions and $\mathfrak{g} = \Omega$ is the subsheaf of germs of holomorphic functions. A section of the quotient sheaf μ/Ω is a system of principal parts. The classical additive Cousin problem consists in deciding whether such a system of principal parts is that of a global meromorphic function. In our terminology the problem is that of characterizing the image $j^0H^0(M, \mu)$ in $H^0(M, \mu/\Omega)$, or, since the sequence is exact, the kernel of δ^0 in $H^0(M, \mu/\Omega)$. It follows that the additive Cousin problem always has a solution if $H^1(M, \Omega)=0$.

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[5] This is essentially a part of the theorem of Lefschetz, cf. §3.

[6] For a more complete discussion cf. Part III.

The most important cohomology groups of a complex manifold M with a coefficient sheaf are the groups $H^q(M, \Omega^p)$, where Ω^p is the sheaf of germs of holomorphic differential forms of type $(p, 0)$. Cartan-Serre and Kodaira [10; 28] proved that the dimension $h^{p,q}$ of $H^q(M, \Omega^p)$ is finite if M is compact. For a Kähler manifold we have $h^{p,q}=h^{q,p}$. In general this relation is not true. Little is known about the combinations of $h^{p,q}$ which give topological invariants of M. For a Kähler manifold $\sum_{p+q=r} h^{p,q}$ is equal to the r-dimensional Betti number.

There already exist many applications of sheaves to the study of complex manifolds and classical algebraic geometry. Among them we mention the works of Kodaira-Spencer on the identification of different definitions of the arithmetic genera of algebraic varieties, on the lemma of Enriques-Severi-Zariski, the characteristic deficiency, etc. [31; 32; 33]. Hodge and Atiyah [24] applied sheaves to generalize a theorem of Picard-Lefschetz by proving that the maximum number of independent two-forms of the second kind on an algebraic variety is equal to $R_2-\rho$, where R_2 is the second Betti number and ρ is the Picard number.

As an illustration let us dwell a little bit more on the work of Kodaira-Spencer on the classification of complex line bundles [32]. All the complex line bundles over a complex manifold form an abelian group $\mathfrak{F}$. It contains as a subgroup the divisor-class group (because a divisor defines a line bundle in an obvious way). The study of the group of complex line bundles is based on the exact sequence of sheaves:

$$0 \to Z \xrightarrow{j} \Omega \xrightarrow{e} \Omega^* \to 0, \tag{4}$$

where Ω^* is the sheaf of germs of nonzero holomorphic functions, j is the inclusion mapping, and e is defined by $ef(z)=\exp(2\pi(-1)^{1/2}f(z))$. This exact sequence of sheaves gives rise to the following exact sequence of cohomology groups:

$$0 \to H^1(M, Z) \xrightarrow{j^*} H^1(M, \Omega) \xrightarrow{e^*} H^1(M, \Omega^*) \xrightarrow{\delta} H^2(M, Z) \to \cdots$$

It is easy to see that $\mathfrak{F}$ is isomorphic to $H^1(M, \Omega^*)$. If we identify these two groups, δ maps a complex line bundle into its characteristic class. The image $\delta H^1(M, \Omega^*)$ is the subgroup $H^2_{(1,1)}(M, Z)$ of $H^2(M, Z)$ consisting of all cohomology classes with integer coefficients which can be represented by a real closed form of type (1, 1). In case M is a Kähler manifold, this last condition is equivalent to saying that the harmonic part of the class is of type (1, 1). The kernel of δ or the image of ϵ^* consists of those complex line bundles which are topologically product bundles. It is a complex abelian Lie group and is called the Picard variety $\mathfrak{p}$ of M. If M is a compact Kähler manifold, its Picard variety is a complex torus of dimension $R_1/2$, where R_1 is the first Betti number of M.

The quotient group $\mathfrak{F}/\mathfrak{p}$ which, according to the above discussion, is isomorphic to $H^2_{(1,1)}(M, Z)$ (or to the subgroup $H^{(1,1)}_{2n-2}(M, Z)$ of $H_{2n-2}(M, Z)$ by duality in M), has another important interpretation when M is a compact algebraic variety. In this case it is isomorphic to G/G_a, where G is the group of all divisors on M and G_a the subgroup of all divisors which are homologous to zero with integer coefficients. The resulting isomorphism between G/G_a and $H^{(1,1)}_{2n-2}(M, Z)$ is known as Lefschetz's Theorem. It can also be stated as a criterion for a $(2n-2)$-cycle to be algebraic (criterion of Lefschetz-Hodge, $n = \dim M$): An integral $(2n-2)$-cycle on an algebraic variety of dimension n is homologous to a divisor if and only if its harmonic part is of type (1, 1). It is a conjecture that this theorem has a generalization to $(2n-2q)$-cycles, to the effect that an integral $(2n-2q)$-cycle is homologous to an algebraic cycle if and only if its harmonic part is of type (q, q).

The sheaf Ω^p has a natural and important generalization $\Omega^p(F)$, which is the sheaf of germs of holomorphic p-forms in a complex manifold M with values in a line bundle F. The corresponding cohomology groups $H^q(M, \Omega^p(F))$ are again finite-dimensional if M is compact. Among these cohomology groups we have the following duality theorem of Serre [40]:

$$H^q(M, \Omega^p(F)) = H^{n-q}(M, \Omega^{n-p}(-F)). \tag{5}$$

In this isomorphism $-F$ denotes the line bundle whose transition functions are the reciprocals of those of F. Actually Serre proved his theorem for the more general case of vector bundles.

It is important in applications to find sufficient conditions for certain cohomology groups $H^q(M, \mathfrak{f})$ to vanish. For Stein manifolds there are the famous theorems A and B of Cartan and Serre (cf. §4). Serre found the analogues of these theorems in the cases of the complex projective space and of algebraic varieties in projective space. In the case of an algebraic variety M he showed that, for sufficiently large m, $H^q(M, \mathfrak{f} \otimes \Omega(mE)) = 0$, $q \geq 1$, where $\mathfrak{f}$ is a coherent analytic sheaf, E is the divisor defined by a hyperplane section, and Ω is the sheaf of germs of holomorphic functions in M.

While these theorems are adequate for many applications, Kodaira, by adopting a differential-geometric method originated from Bochner, obtained other sufficient conditions for the vanishing of the groups $H^q(M, \Omega^p(F))$, $q \geqq 1$, where F is a complex line bundle. We say that F is ample, if its characteristic class contains a representative closed quadratic differential form of type $(1, 1)$ whose corresponding Hermitian differential form is positive definite. Similarly, one defines the notion for a complex line bundle to be sufficiently ample. Kodaira proved that $H^q(M, \Omega^0(F))=0$, $q \geqq 1$, if $F-K$ is ample, K being the canonical bundle of M [29]. Moreover, if F is sufficiently ample, then $H^q(M, \Omega^p(F))=0$. On the other hand, Akizuki and Nakano proved that $H^q(M, \Omega^p(F))=0$, $p+q \leqq n-1$, if $F-K$ is ample [1]. These results are generalized by Spencer to vector bundles [35].

The interest in Kähler manifolds lies in the fact that many theorems on algebraic varieties are valid for compact Kähler manifolds. It is natural to ascertain the scope of this notion. The complex torus which does not satisfy the Riemann conditions gives an example of a nonalgebraic Kähler manifold. An important theorem of Kodaira states that, if a compact Kähler manifold is of restricted type, that is, if its fundamental two-form has integral periods over integral cycles, it is an algebraic variety [30]. The conjecture is not true that every compact Kähler manifold can be transformed by monoidal transformations into an analytic bundle of complex tori over an algebraic variety (A. Blanchard [4]).

4. **Stein manifolds.** The Stein manifolds (of dimension >0) are noncompact complex manifolds which generalize the domains of holomorphy and which possess a sufficiently large number of holomorphic functions. Precisely speaking, a Stein manifold M is a complex manifold with a countable base, satisfying the following conditions:

(1) To any two points $p, q \in M$, $p \neq q$, there exists a holomorphic function f in M, such that $f(p) \neq f(q)$;

(2) To every point $p \in M$ there exist n functions, holomorphic in M, which form a local coordinate system at p;

(3) M is holomorphically convex.

The following fundamental theorem accounts for most of the properties of Stein manifolds:

A complex manifold M is a Stein manifold if and only if the following two properties hold:

(A) For any coherent sheaf $\mathfrak{f}$ over M, the module of global cross-sections of $\mathfrak{f}$ over M generates at every point $p \in M$ the module of the local cross-sections $\mathfrak{f}_p$.

(B) For any coherent sheaf $\mathfrak{f}$ over M, $H^q(M, \mathfrak{f})=0$, $q \geqq 1$.

This theorem is due to Oka and Cartan [9; 39]. The proof of the direct part, that is, that a Stein manifold has the properties (A) and (B), is difficult. The theorem has many consequences of which we mention the following:

(1) The additive Cousin problem (cf. §3) always has a solution on a Stein manifold;

(2) The second Cousin problem, the problem whether a given divisor is the divisor of a meromorphic function, has a solution on a Stein manifold if $H^2(M, Z)=0$;

(3) Every meromorphic function on a Stein manifold is the quotient of two holomorphic functions.

A first topological implication of a Stein manifold (of complex dimension n, and hence of real dimension $2n$) is that its p-dimensional homology groups $H_p(M, Z)$ with integer coefficients are, for $p>n$, torsion groups. It is not known whether they are all zero.

Another unsolved problem is to characterize the open submanifolds of a Stein manifold which are again Stein manifolds. It is also not known whether a covering manifold of a Stein manifold is a Stein manifold.

5. **Riemann-Roch theorem.** It has been known that the Riemann-Roch theorem can be formulated as a relation between the dimensions of certain cohomology groups with coefficient sheaves and the characteristic classes. Its exact formulation and proof were recently achieved by Hirzebruch [22; 51]: Let M be an algebraic variety of dimension n, and W an analytic vector bundle over M, with fiber E_q and structural group $GL(q, C)$. Let c_i, $1 \leqq i \leqq n$, be the Chern classes of the tangent bundle of M, and d_j, $1 \leqq j \leqq q$, be the Chern classes of the bundle W. Denote by $\Omega(W)$ the sheaf of germs of holomorphic cross sections of W over M, and put

$$\chi(M, W) = \sum_{i=0}^{n} (-1)^i \dim H^i(M, \Omega(W)). \tag{6}$$

Introduce formally the quantities γ_i, $1 \leqq i \leqq n$, δ_j, $1 \leqq j \leqq q$, by the relations

$$\begin{aligned} 1 + \sum_{i=1}^{n} c_i x^i &= \prod_{i=1}^{n} (1 + \gamma_i x^i), \\ 1 + \sum_{j=1}^{q} d_j x^j &= \prod_{j=1}^{q} (1 + \delta_j x^j). \end{aligned} \tag{7}$$

Then the function

$$(8)\qquad P(c_1,\cdots,c_n;d_1,\cdots,d_q) = (e^{\delta_1}+\cdots+e^{\delta_q})\prod_{i=1}^{n}\frac{-\gamma_i}{e^{-\gamma_i}-1}$$

is symmetric in γ_i and δ_j, and can be expressed as a power series in c_i, d_j. It can therefore be considered as a rational cohomology class of M. Following Hirzebruch, we put the symbol κ_{2n} before it to denote its value over the fundamental homology class of M. Then the Riemann-Roch-Hirzebruch theorem can be given as the formula

$$(9)\qquad \begin{aligned}\chi(M,W) &= \kappa_{2n}\left[(e^{\delta_1}+\cdots+e^{\delta_q})\prod_{i=1}^{n}\frac{-\gamma_i}{e^{-\gamma_i}-1}\right]\\ &= \kappa_{2n}\left[e^{c_1/2}(e^{\delta_1}+\cdots+e^{\delta_q})\prod_{i=1}^{n}\frac{\gamma_i/2}{\sinh\gamma_i/2}\right].\end{aligned}$$

It is worth remarking that the product

$$\prod_{i=1}^{n}\frac{\gamma_i/2}{\sinh(\gamma_i/2)}$$

can be expressed as a power series in the Pontrjagin classes of M, which depend only on the differentiable structure, and not on the complex structure, of M. This leads to the guess that the formula may be valid for any compact complex manifold, but Hirzebruch's proof makes essential use of the fact that M is an algebraic variety.

If $q=1$, that is, if W is a line bundle which we now denote by F, the above formula reduces to

$$(10)\qquad \chi(M,F) = \kappa_{2n}\left[e^{c_1/2+f}\prod_{i=1}^{n}\frac{\gamma_i/2}{\sinh(\gamma_i/2)}\right],$$

where f is the characteristic class of F. If, moreover, F is a product bundle, then $f=0$, and we have, by writing $\chi(M)$ for $\chi(M,F)$,

$$(11)\qquad \chi(M) = \kappa_{2n}\left[e^{c_1/2}\prod_{i=1}^{n}\frac{\gamma_i/2}{\sinh(\gamma_i/2)}\right].$$

The number in the right-hand side, which we shall denote by $T(M)$, was first introduced by Todd [46] and is called the Todd genus. On the other hand, we have $H^i(M,\Omega)\cong H^0(M,\Omega^i)$, so that the dimension g_i of $H^i(M,\Omega)$ is the dimension of the complex vector space of holomorphic i-forms, and we have

$$(12)\qquad \chi(M) = 1-g_1+g_2+\cdots+(-1)^n g_n.$$

$\chi(M)$ is related to the arithmetic genus $p_a(M)$ of M by the formula

$$(12a)\qquad 1+(-1)^n p_a(M) = \chi(M).$$

Formula (11) thus expresses a relationship between the arithmetic genus and the Todd genus of an algebraic variety and implies in particular that the latter is an integer.

Another particular case of (10) is the case when M is an algebraic curve. In this case we get from (10),

$$\dim |D| = \dim H^0(M, \Omega(D)) - 1 = d - p + i, \tag{13}$$

where D is a divisor, d its degree, p the genus of M, and

$$i = \dim H^1(M, \Omega(D)) = \dim |K - D| + 1$$

is the index of specialty of the divisor class $|D|$. (13) is the classical Riemann-Roch theorem. Similarly, it can be seen that (9) includes as special cases other known versions of the Riemann-Roch theorem for algebraic varieties of two and three dimensions.

An essential tool in Hirzebruch's proof of (9) is the so-called index theorem. For a manifold whose dimension is not a multiple of 4 we define its index $\tau(M)$ to be zero. If M is of dimension $4k$, the cup product $u \cup u$, for $u \in H^{2k}(M, R)$, gives rise to a nonsingular symmetric quadratic form in the vector space $H^{2k}(M, R)$, and the excess of the number of its positive eigenvalues over the number of its negative eigenvalues is called the index $\tau(M)$ of M. By using the theory of "cobordisme" of Thom, [45], Hirzebruch proved that

$$\tau(M^{4k}) = \kappa_{4k}(L_k(p_1, \cdots, p_k)), \tag{14}$$

where $p_1, \cdots, p_k$ are the Pontrjagin classes of M^{4k}, and L_k are certain polynomials with rational coefficients which, for $k = 1, 2, 3$, are given by

$$\begin{aligned} L_1 &= \frac{1}{3} p_1, \qquad L_2 = \frac{1}{45} (7p_2 - p_1^2), \\ L_3 &= \frac{1}{945} (62p_3 - 12p_1p_2 + 2p_1^3). \end{aligned} \tag{15}$$

The proof of the index theorem (14) is based on the observation that both sides of (14) depend only on the cobordisme class in the sense of Thom. Thom's theory makes use of deep results of homotopy theory in algebraic topology. As the full force of his results is not needed here, it would be desirable to give a more direct proof of (14). But even for a four-dimensional manifold such a proof is not known.

6. **Homogeneous complex manifolds.** An important class of complex manifolds consists of the homogeneous ones, that is, those which admit a transitive group of complex automorphisms. If the manifold

M is compact, the group of all its complex automorphisms is a complex Lie group; it is semi-simple if the Euler-Poincaré characteristic of M is >0. If M is a bounded domain, the group of all its complex automorphisms is a real Lie group. Since M has then the Bergmann metric, which is a homogeneous Kählerian metric, the group of isotropy of M is compact. For a general noncompact complex manifold it is well-known that the group of all its complex automorphisms is not necessarily a Lie group. It is not known whether there exists a Lie group of complex automorphisms which acts transitively on M.

Wang determined all the compact homogeneous complex manifolds with a finite fundamental group [47; also 18]. In particular, those which are simply connected can be described as follows: Let K be a compact semi-simple Lie group, and T^s a toroidal subgroup of dimension s. The centralizer Z of T^s is locally a product of a toroid and a compact semi-simple group Q. A connected closed subgroup U of K is called a C-subgroup if there is a toroidal subgroup T^s with the property that $Q \subset U \subset Z$. A simply connected compact homogeneous complex manifold is homeomorphic to a real coset space K/U, where K is a semi-simple compact Lie group and U a C-subgroup of K. Conversely, every such coset space K/U, if even-dimensional, can be given a homogeneous complex structure.

Wang's work gives many examples of compact complex manifolds, besides the algebraic varieties. These include the product of two spheres of odd dimensions >1, as first given by Calabi-Eckmann [7], and the even-dimensional compact Lie groups. The question of determining the simply-connected compact homogeneous complex manifolds which are algebraic is simplified by a theorem of Lichnerowicz [34], which states that a compact simply connected homogeneous Kähler manifold has Euler-Poincaré characteristic >0. Goto, and independently Borel and Weil, proved that this condition is sufficient [19]. Goto also proved that the algebraic variety is rational.

Much less is known about noncompact homogeneous complex manifolds. Borel studied the coset spaces of a real semi-simple Lie group which admit an invariant Kählerian structure [5]. He proved that, if the coset space is noncompact, it is an analytic fiber bundle, with Kählerian manifolds as fibers, over an Hermitian symmetric space.

Perhaps the most important noncompact homogeneous complex manifolds are the homogeneous bounded domains. In this sense the question raised by E. Cartan is of importance, as to whether a homogeneous bounded domain is always symmetric. Borel [5] and Koszul proved independently that this is the case if the group of all complex automorphisms is semi-simple. But the general question remains unanswered.

7. **Structures defined by infinite continuous pseudo-groups.** It has been observed recently that the notion of an infinite continuous pseudo-group in the sense of Lie and Cartan gives rise to structures which generalize the complex structure. The significance of this generalization remains to be seen, and we restrict ourselves to a few general remarks [12; 13].

A transformation of an infinite pseudo-group is a coordinate transformation $(x^1, \cdots, x^n) \rightarrow (x'^1, \cdots, x'^n)$ which represents a general integral of a system of partial differential equations in n independent and n dependent variables, with the property that the inverse of a transformation and the composition of two transformations are general integrals of the same differential system. They form a pseudo-group, and not a group, because the domain and the range of these transformations are allowed to vary. If the system of partial differential equations is supposed to be of the first order, the transformations can be defined to be those which reproduce n linearly independent Pfaffian forms up to a linear transformation of a given linear group. The latter is called the group of stability. The local theory of such pseudo-groups was developed by Lie and E. Cartan. It is useful to remark that these pseudo-groups are numerous.

Corresponding to every such pseudo-group there exists a type of structure on a class of differentiable manifolds, with the property that the transition of local coordinate systems is given by a transformation of the pseudo-group. The complex structure and complex manifolds constitute a notable example of this general notion. There are, however, other structures which exist in a natural way. We mention the leaved structure of Ehresmann-Reeb and the structure on an odd-dimensional manifold with a Pfaffian equation of the maximum class given everywhere. The latter structure exists on the manifold of tangent covariant directions of a differentiable manifold. (A covariant direction is the class of all nonzero covariant vectors which differ from each other by a factor.) A structure-preserving automorphism on such an odd-dimensional manifold is essentially what is known as a contact transformation.

Many notions which have been developed in the theory of the complex structure can be generalized. Thus manifolds for which the structural group of the tangent bundle can be reduced to the group of stability of an infinite pseudo-group generalize the almost complex manifolds. There is also a natural generalization of the Kähler property [13]. We can also define a class of manifolds analogous to the algebraic varieties. In fact, according to a theorem of Chow, the latter are complex manifolds which can be complex analytically imbedded in a complex projective space. The complex projective space is unique among the complex manifolds in the sense that it is the only simply connected, compact complex manifold with a group of complex automorphisms which is transitive on the tangent directions. If certain "universal spaces" of our generalized structure can be defined, universal in the sense that they admit a "large" group of structure-preserving automorphisms, then the analogues of algebraic varieties will be the manifolds imbeddable in universal spaces with structures preserved.

Finally, we remark that the sheaves will again be a useful tool in the study of such manifolds. Since the structural group of the tangent bundle can be restricted to the group of stability G and the representation of G in the space of alternating tensors may be reducible, we can speak of exterior differential forms of a type corresponding to any invariant subspace of such a representation. The sheaf of germs of differential forms of a given type is thus well defined. Moreover, from the exterior differentiation operator and projections of a differential form into one of a given type, various differential operators can be defined. This leads to the definition of cohomology groups whose study should be of importance in the theory of manifolds with structures in the sense we have explained.

BIBLIOGRAPHY

(This Bibliography includes only papers referred to in the article: no attempt is made to achieve completeness.)

1. Y. Akizuki, and S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. Japan Acad. vol. 30 (1954) pp. 266–272.

2. M. Atiyah, *New methods in the theory of algebraic varieties*, Smith and Rayleigh Prize Essay, Cambridge, England, 1954; also, *Complex fibre bundles and ruled surfaces*, Proc. London Math. Soc. (2) vol. 5 (1955) pp. 407–434.

3. H. Behnke and K. Stein, *Modifikationen komplexer Mannigfaltigkeiten und Riemannscher Gebiete*, Math. Ann. vol. 124 (1951) pp. 1–16.

4. A. Blanchard, *Un théorème sur les automorphismes d'une variété algébrique projective. Application aux fibrés et à une conjecture d'Igusa*, C. R. Acad. Sci. Paris vol. 240 (1955) pp. 2198–2201.

5. A. Borel, *Kählerian coset spaces of semi-simple Lie groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 1147–1151.

6. A. Borel and J-P. Serre, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. vol. 75 (1953) pp. 409–448.

7. E. Calabi and B. Eckmann, *A class of compact complex manifolds which are not algebraic*, Ann. of Math. vol. 58 (1953) pp. 494–500.

8. H. Cartan, *Séminaire*, mimeographed notes, Ann. École Norm. (1950–1951) (1951–1952) (1953–1954).

9. ———, *Variétés analytiques complexes et cohomologie*, Colloque de Bruxelles (1953) pp. 41–55.

10. H. Cartan and J-P. Serre, *Un théorème de finitude concernant les variétés analytiques compactes*, C. R. Acad. Sci. Paris vol. 237 (1953) pp. 128–130.

11. S. S. Chern, *On the characteristic classes of complex sphere bundles and algebraic varieties*, Amer. J. Math. vol. 75 (1953) pp. 565–597.

12. ———, *Pseudo-groupes continus infinis*, Colloques Strasbourg (1953) pp. 119–136.

13. ———, *A generalization of Kähler geometry*, to appear in Lefschetz Jubliee Volume.

14. W. L. Chow, *On compact complex analytic varieties*, Amer. J. Math. vol. 71 (1949) pp. 893–914.

15. P. Dolbeault, *Sur la cohomologie des variétés analytiques complexes*, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 175–177.

16. B. Eckmann and A. Fröhlicher, *Sur l'intégrabilité des structures presque complexes*, C. R. Acad. Sci. Paris vol. 232 (1951) pp. 2284–2286.

17. C. Ehresmann and P. Libermann, *Sur les structures presque hermitiennes isotropes*, C. R. Acad. Sci. Paris vol. 232 (1951) pp. 1281–1283.

18. A. Fröhlicher, *Zur Differentialgeometrie der komplexen Strukturen*, Math. Ann. vol. 129 (1955) pp. 50–95.

19. M. Goto, *On algebraic homogeneous spaces*, Amer. J. Math. vol. 76 (1954) pp. 811–818.

20. F. Hirzebruch, *Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten*, Math. Ann. vol. 124 (1951) pp. 77–86.

21. ———, *On Steenrod's reduced powers, the index of inertia, and the Todd genus*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 951–956.

22. ———, *Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 110–114.

23. ———, *Some problems on differentiable and complex manifolds*, Ann. of Math. vol. 60 (1954) pp. 213–236.

24. W. V. D. Hodge, and M. F. Atiyah, *Integrals of the second kind on an algebraic variety*, Ann. of Math. vol. 62 (1955) pp. 56–91.

25. H. Hopf, *Zur Topologie der komplexen Mannigfaltigkeiten*, Studies presented to R. Courant, 1948, pp. 167–185.

26. ———, *Über Komplex-analytische Mannigfaltigkeiten*, Rend. Mat. e Appl. vol. 10 (1951) pp. 169–182.

27. Y. Kawada, *On analytic line bundles with the affine structural group*, Scientific Papers of College of General Education, Univ. of Tokyo vol. 4 (1954) pp. 85–91.

28. K. Kodaira, *On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 865–868.

29. ———, *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 1268–1273.

30. ———, *On Kähler varieties of restricted type*, Ann. of Math. vol. 60 (1954) pp. 28–48.

31. K. Kodaira and D. C. Spencer, *On arithmetic genera of algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 641–649.

32. ———, *Groups of complex line bundles over compact Kähler varieties. Divisor class groups on algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 868–877.

33. ———, *On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 1273–1278.

34. A. Lichnerowicz, *Espaces homogènes Kählériens*, Colloques Strasbourg (1953) pp. 172–184.

35. J. C. Moore, H. K. Nickerson and D. C. Spencer, *Mimeographed notes on sheaves and differentiable manifolds*, Princeton, 1954.

36. S. Nakano, *On a certain type of analytic fiber bundles*, Proc. Japan Acad. vol. 30 (1954) pp. 542–547.

37. ———, *On complex analytic vector bundles*, J. Math. Soc. Japan vol. 7 (1955) pp. 1–12.

38. G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Scientifiques et Industrielles, no. 1183, 1952.

39. J-P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein*, Colloques de Bruxelles (1953) pp. 57–68.

40. ———, *Un théorème de dualité*, Comment. Math. Helv. vol. 29 (1955) pp. 9–26.

41. ———, *Faisceaux algébriques cohérents*, Ann. of Math. vol. 61 (1955) pp. 197–278.

42. N. Steenrod, *Fibre bundles*, Princeton University Press, 1951.

43. K. Stein, *Analytische Funktionen mehrer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem*, Math. Ann. vol. 123 (1951) pp. 201–222.

44. W. Stoll, *Über meromorphe Modifikationen*, Math. Zeit. vol. 61 (1954) pp. 206–234.

45. R. Thom, *Quelques propriétés globales des variétés differentiables*, Comment Math. Helv. vol. 28 (1954) pp. 17–86.

46. J. A. Todd, *The geometrical invariants of algebraic loci*, Proc. London Math. Soc. (2) vol. 43 (1937) pp. 127–138.

47. H. C. Wang, *Closed manifolds with homogeneous complex structure*, Amer. J. Math. vol. 76 (1954) pp. 1–32.

48. A. Weil, *Fiber spaces in algebraic geometry*, Notes by A. Wallace, University of Chicago, 1952.

49. Wen-Tsun Wu, *Sur les classes caractéristiques des structures fibrées sphériques*, Actualités Scientifiques et Industrielles, no. 1183, 1952.

50. O. Zariski, *Algebraic surfaces*, Berlin, Springer, 1935.

51. F. Hirzebruch, *Neue topologische methoden in der algebraischen Geometrie*, Springer, 1956.

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PART III. ALGEBRAIC SHEAF THEORY

The cohomological methods, ***in conjunction with the powerful tool of harmonic integrals***, were remarkably effective in the solution of global complex-analytic problems in general, and of problems of classical algebraic geometry in particular (Chern, Hirzebruch, Kodaira-Spencer, Serre, and others). It is natural to ask whether the cohomological methods can be equally effective in abstract algebraic geometry where the method of harmonic integrals is no longer available.

The purely algebraic theory of sheaves developed by Serre [3][1] represents the first systematic application of cohomological algebra to abstract algebraic geometry. The theory developed by Serre includes the "generalized lemma of Enriques-Severi" proved and so named by Zariski [4], and contains also the first algebraic proof of Severi's conjecture $P_a = p_a$. These are highly encouraging indications of the power and potentialities of the cohomological method in abstract algebraic geometry.[2]

1. **The general concept of a sheaf.** A *sheaf* $\mathcal{F}$ is a composite concept consisting of two topological spaces F and X and a continuous mapping (*projection*) π of F *onto* X satisfying the following conditions: (1) π is a local homeomorphism; (2) for each $x \in X$ the set $\mathcal{F}_x = \pi^{-1}\{x\}$ is an abelian (additive) group; (3) the group structure of $\mathcal{F}_x$ varies continuously with x. The space X is called the *base space* of the sheaf, and $\mathcal{F}_x$ is called the *stalk* over x. If U is an open subset of X, a *section* of $\mathcal{F}$ over U is a continuous mapping f of U into F such that πf is the identity on U. The sections of $\mathcal{F}$ over U form in an obvious way an abelian (additive) group, denoted by $\Gamma(\mathcal{F}, U)$. As U ranges over an open basis of X and f ranges over $\Gamma(\mathcal{F}, U)$, the sets $f(U)$ yield an open basis of F, as follows easily from the definitions. An important property of sections is the following: if $f, g \in \Gamma(\mathcal{F}, U)$ and $f(x) = g(x)$ for some $x \in U$, then $f = g$ on some open neighborhood of x. In practice, one often introduces a sheaf $\mathcal{F}$ as the union F of its stalks $\mathcal{F}_x$ $(x \in X)$, and the topology of F is then defined by assigning the local sections of $\mathcal{F}$.

The general notions of a subsheaf of a given sheaf $\mathcal{F}$, factor sheaf, sheaf homomorphism, direct sum and tensor product of sheaves (with the same base space) are defined in a natural fashion (Cartan [3], Serre [3]). It is important to bear in mind that, by definition, a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ is always open in F and that the canonical sheaf homomorphism $\mathcal{F}\rightarrow\mathcal{F}/\mathcal{F}'$ is always a continuous *open* mapping. A sheaf epimorphism[3] j: $\mathcal{F}\rightarrow\mathcal{F}''$ induces a homomorphism j_x of $\mathcal{F}_x$ onto $\mathcal{F}_x''$ for each x in X and also a homomorphism j_U of $\Gamma(\mathcal{F}, U)$ into $\Gamma(\mathcal{F}'', U)$ for each open subset U of X. Furthermore, each section of $\mathcal{F}''$ is *locally* the j-image of a section of $\mathcal{F}$. More precisely: if $f''\in\Gamma(\mathcal{F}'', U)$ and $x\in U$, there exists a neighborhood W of x, contained in U, and there exists a section f in $\Gamma(\mathcal{F}, W)$ such that $jf=f''$ on W.

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[1] Numbers in brackets refer to the bibliography at the end of this part of the report.

[2] The participants of the seminar on algebraic sheaf theory had in their possession a short manuscript of Serre (Serre [2]) in which he gave the definitions and stated (almost always without proof) the basic results of the theory. It seemed fruitful to take this manuscript as the subject of the seminar and to try to reconstruct the proofs, this being the best way of acquainting ourselves with the new algebraic methods, ideas and results. The present report is based:

(1) on the work of that seminar;

(2) on manuscripts given to the writer of this report by J. Igusa and S. Lang (who, together with G. Washnitzer, were the most active members of the seminar);

(3) on Serre's Annals paper [3] which became subsequently available and which formed the subject of a seminar conducted at Harvard in 1954–1955 by Igusa and the writer.

The manuscript of Part III was read by S. Lang, J. Igusa, J-P. Serre and D. C. Spencer. The author gratefully acknowledges the helpful criticisms and comments made by these mathematicians.

In applications to algebraic geometry the stalks $\mathcal{F}_x$ of a sheaf possess, as a rule, much more structure than that of mere abelian groups. The stalks may be rings, and in that case the ring structure of $\mathcal{F}_x$ is assumed to vary continuously with x. Or, each $\mathcal{F}_x$ may be a module over some fixed field. Or finally—and this is the basic setup in applications to algebraic geometry—we may deal with sheaves $\mathcal{F}$ which admit a fixed basis sheaf $\mathcal{O}$ as *sheaf of operators*: each stalk $\mathcal{O}_x$ is then a ring, and $\mathcal{F}_x$ is a module over $\mathcal{O}_x$, the ring structure of $\mathcal{O}_x$ and the module structure of $\mathcal{F}_x$ both varying continuously with x.

2. **Examples of sheaves.** (a) Let X be a complex analytic manifold and let $\bar{\mathcal{O}}_x$ be the ring of germs of holomorphic functions at x ($x\in X$). A section over an open set U is then described by a function f which is holomorphic on U (i.e., by the mapping $x\rightarrow f_x$, where $x\in U$ and f_x is the germ of f at x). This is the fundamental sheaf of the classical theory.

(b) Let us take for X an irreducible algebraic variety V in an affine or projective space, over some algebraically closed ground field k. We must first decide on the choice of a topology in V. In the classical case the choice is obvious: we take the ordinary Hausdorff topology of V. However, V also carries another topology, introduced by Zariski. In this "Zariski topology," the closed sets are the algebraic subvarieties of V. The Zariski topology is much weaker than the usual Hausdorff topology but it has a perfect meaning in the abstract case. Serre took the Zariski topology for his algebraic sheaf theory. At the same time he took as fundamental sheaf the sheaf $\mathcal{O}=\mathcal{O}_V$ whose stalks $\mathcal{O}_x$ are the local rings of V at the various points x of V. Local sections of $\mathcal{O}$ are then described by locally regular functions f which are meromorphic on V; any such function is an element of the function field K of V. The points x of V where f is not regular (i.e., the points x such that $f\notin\mathcal{O}_x$) form an algebraic subvariety W of V, and thus f defines a section of $\mathcal{O}$ over $U=V-W$ (U is open in the Zariski topology).

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[3] We shall use the following terminology: a homomorphism j: $A\rightarrow A'$ is an *epimorphism* if j maps A *onto* A'; it is a *monomorphism* if its kernel is zero; and it is an *isomorphism* if it is both an epimorphism and a monomorphism.

A sheaf on V (i.e., a sheaf having V as base space) is *algebraic* if it admits $\mathcal{O}$ as sheaf of operators. These are the sheaves studied by Serre. A homomorphism j between algebraic sheaves $\mathcal{J}$ and $\mathcal{J}''$ is called *algebraic* if it is $\mathcal{O}$-linear, i.e., if for every x in V the local homomorphism j_x is $\mathcal{O}$-linear.

The sheaf $\mathcal{O}$ itself is algebraic. What are the global sections of $\mathcal{O}$? The answer depends on whether V is an affine or projective variety. If V is an affine variety in r-space, then $\Gamma(\mathcal{O}, V)$ is the (nonhomogeneous) coördinate ring $R=k[x_1, x_2, \cdots, x_r]$ of V/k. If V is a projective variety then $\Gamma(\mathcal{O}, V)=k$.

Another example of an algebraic sheaf is the p-fold direct product of $\mathcal{O}$, denoted by $\mathcal{O}^p$. The elements of $\mathcal{O}^p_x$ are then the ordered p-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_p)$, $\alpha_i\in\mathcal{O}_x$. These sheaves play an important role in the theory of coherent sheaves (see next section).

It should be noted that if k is the field of complex numbers we have associated two types of sheaves with a variety V, namely $\tilde{\mathcal{O}}$ and $\mathcal{O}$. They differ in two important respects: in the topology of their base space V and in the fact that $\tilde{\mathcal{O}}_x$ contains $\mathcal{O}_x$ but is a much bigger ring then $\mathcal{O}_x$. Therefore in the classical case we have *a priori* two sheaf theories, or two cohomology theories of V: one is analytic, the other is algebraic. Serre has established an "isomorphism" between these two theories (see end of §7).

(c) Let V be again an irreducible algebraic variety and let W be a fixed affine or projective subvariety of V. Zariski has defined in his memoir [3] the notion of a function on V which is holomorphic along an open subset U of W. *We take W as base space* and we take as stalk $\mathcal{O}_x^*(x\in W)$ the set of germs, at x, of functions f on V which are holomorphic along some open subset of W containing x. If we assume that W is irreducible and that V is analytically irreducible at each point of W (this latter condition is satisfied, for instance, if V is normal), then it is known that the germ of a function f at x determines the function uniquely (Zariski [3]). This enables us to define a sheaf $\mathcal{O}^*_{V,W}$ by taking the union of the stalks $\mathcal{O}_x^*$. It may be of interest to examine the results and the unsolved questions of Zariski's memoir [3] from the standpoint of sheaf theory, by developing the properties of sheafs on W which admit $\mathcal{O}^*_{V,W}$ as sheaf of operators.

3. Coherent sheaves. An algebraic sheaf $\mathcal{J}$ on a variety V is *coherent* if locally, i.e., in the neighborhood of each point x of V, $\mathcal{J}$ can be written in the form $\mathcal{O}^p/\phi(\mathcal{O}^q)$, where p and q are non-negative integers and ϕ is an algebraic homomorphism of $\mathcal{O}^q$ into $\mathcal{O}^p$ (p, q and ϕ depend on x). In other words, in the neighborhood of any point x of V we have an exact sequence:

$$(1)\qquad \mathcal{O}^q \xrightarrow{\phi} \mathcal{O}^p \xrightarrow{j} \mathcal{F} \to 0,$$

all sheaf homomorphisms being assumed algebraic. If U denotes a neighborhood in which (1) holds and if $\mathcal{F}_U$ denotes the sheaf $\bigcup \mathcal{F}_x$, $x \in U$, with U as base space ($\mathcal{F}_U$ = *restriction of* $\mathcal{F}$ *to* U), then the precise way of writing (1) is the following:

$$(1')\qquad \mathcal{O}_U^q \xrightarrow{\phi} \mathcal{O}_U^p \xrightarrow{j} \mathcal{F}_U \to 0.$$

Recalling from §1 that the sections of a homomorphic image of a given sheaf (in this case, of the image of $\mathcal{O}_U^p$) are locally images of sections of the given sheaf, the existence of the homomorphism j of $\mathcal{O}_U^p$ onto $\mathcal{F}_U$ is equivalent with the following property of $\mathcal{F}$: *for each point* x *of* V *there exists an open set* W *containing* x *and there exists a finite set of sections* $f_1, f_2, \cdots, f_p$ *of* $\mathcal{F}$ *over* W, *such that for each point* y *of* W *the elements* $f_1(y), f_2(y), \cdots, f_p(y)$ *form a module basis of* $\mathcal{F}_y$ *over* $\mathcal{O}_y$ (or briefly: the sections $f_1, f_2, \cdots, f_p$ generate the stalk $\mathcal{F}_y$ for all y in W). An algebraic sheaf $\mathcal{F}$ is therefore coherent if and only if both $\mathcal{F}$ *and* the kernel of j have this property. This property shows that the theory of coherent sheaves is an additive theory, and that the modules entering in this theory are expected to be of finite type. It turns out that most sheaves which one encounters in applications to algebraic geometry (for instance, the sheaves which one must consider in connection with the Riemann-Roch theorem) are indeed coherent. It should be noted, however, that there are interesting sheaves which are not coherent, for instance the sheaf of local units: the stalk at each point x of V consists of the units of $\mathcal{O}_x$. In the classical theory this sheaf plays an important role which up to now has no counterpart in the abstract theory.

If $f_1, f_2, \cdots, f_m$ are sections of a sheaf $\mathcal{F}$ over a set U, one denotes by $\mathrm{Rel}_x\,(f_1, f_2, \cdots, f_m)(x \in U)$ the set of m-tuples $(a_1, a_2, \cdots, a_m)$, $a_i \in \mathcal{O}_x$, such that $a_1f_1(x)+a_2f_2(x)+\cdots+a_mf_m(x)=0$, and by $\mathrm{Rel}_U\,(f_1, f_2, \cdots, f_m)$ that subsheaf of $\mathcal{O}_U^m$ whose stalks are the modules $\mathrm{Rel}_x\,(f_1, f_2, \cdots, f_m)$. Thus, returning to (1′) and with the same notations as above, $\mathrm{Rel}_U\,(f_1, f_2, \cdots, f_p)$ is the kernel of j.

The three lemmas stated below are auxiliary results which are frequently used in the theory of coherent sheaves:

LEMMA 1. *If* $\mathcal{F}$ *is a coherent sheaf on* V *and* $f_1, f_2, \cdots, f_m$ *are sections of* $\mathcal{F}$ *over an open set* U, *then the sheaf* $\mathrm{Rel}_U\,(f_1, f_2, \cdots, f_m)$ *is coherent.*

LEMMA 2. *If j is an algebraic homomorphism of a coherent sheaf $\mathcal{F}$ into a coherent sheaf $\mathcal{G}$, then the kernel $\mathcal{F}'$, the image $\mathcal{F}''=j(\mathcal{F})$ and the co-kernel $\mathcal{G}/\mathcal{F}''$ of j are coherent sheaves.*

Lemma 1 is analogous to the theorem of Oka-Cartan (Cartan [3, pp. 7–18, 1951–1952]). Lemma 2 is a simple consequence of Lemma 1.

By a ***principal open*** subset U of an affine variety V we mean a subset of V whose complement (boundary) $V-U$ is the set of zeros of a function Q belonging to the coördinate ring R of V. It is easily seen that a principal open set is again biregularly equivalent to an affine variety and that the set of coverings of a variety (projective or affine) with principal open sets is cofinal with the set of all open coverings of the variety.

LEMMA 3. *If a global section f of a coherent sheaf $\mathcal{F}$ on an affine variety V is zero at each point of a principal open subset U of V and if $Q=0$ is an equation of the boundary of U, then for some integer m we have $Q^m f=0$ everywhere on V.*

The self-evident local character of this lemma allows a reduction of the proof to the special case in which $\mathcal{F}$ is isomorphic with $\mathcal{O}_V^q/\phi(\mathcal{O}_V^m)$ (ϕ, an algebraic homomorphism) and f is the image of a global section of $\mathcal{O}_V^q$. In this special case, the lemma is an easy consequence of the Hilbert Nullstellensatz.

The above definitions and lemmas hold both for affine and projective varieties, and indeed they remain valid for "abstract varieties" in the sense of Weil. While the ultimate goal of the cohomological theory of sheaves lies in the direction of projective varieties (or even "abstract varieties" in the sense of Weil), a preliminary study of sheaves on affine varieties is essential, since the set of open coverings of a projective variety V by affine varieties is cofinal with the set of all open coverings of V. The first part of the seminar was therefore entirely devoted to coherent sheaves on affine varieties.

4. Cohomology groups and exact cohomology sequences. Returning to the general theory of an arbitrary topological space X and an arbitrary sheaf $\mathcal{F}$ over X, one can define cohomology groups $H^q(X,\mathcal{F})$ of X, with coefficients in $\mathcal{F}$, by the Čech methods, as follows:

If $\mathfrak{U}=\{U_i\}$ is a finite open covering of X by open sets U_i, a q-cochain f, with coefficients in $\mathcal{F}$, is a function which associates with each q-simplex $\sigma=\{U_{i_0}, U_{i_1}, \cdots, U_{i_q}\}$ of the covering $\mathfrak{U}$ a section $f_{i_0,i_1,\cdots,i_q}$ of $\mathcal{F}$ over the support (sup σ) $U_{i_0}\cap U_{i_1}\cap\cdots\cap U_{i_q}$ of σ (it is assumed that f is an alternating function of the indices $i_0, i_1, \cdots, i_q$). The set of all q-cochains (relative to the given covering

$\mathfrak{U}$) is denoted by $C^q(\mathfrak{U}, \mathcal{F})$. One defines the coboundary δf by the usual formula

$$(\delta f)_{i_0 i_1 \cdots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j f_{i_0 i_1 \cdots \hat{i}_j \cdots i_{q+1}},$$

where the sections of $\mathcal{F}$ on the right must be intended actually as restriction maps of $U_{i_0} \cap U_{i_1} \cap \cdots U_{i_{q+1}}$. This yields cochains, cocycles and cohomology groups $H^q(\mathfrak{U}, \mathcal{F})$ depending on the covering $\mathfrak{U}$, *except that it is immediately seen that* $H^0(\mathfrak{U}, \mathcal{F})$ *is isomorphic with* $\Gamma(\mathcal{F}, X)$ and is thus independent of the covering $\mathfrak{U}$. One then passes to the inductive limit of the groups $H^q(\mathfrak{U}, \mathcal{F})$ on the directed set of all finite open coverings of X and one thus obtains the cohomology groups $H^q(X, \mathcal{F})$.

Suppose that we have an exact sequence of sheaves (over the same base space X):

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0. \tag{1}$$

Then for any open covering $\mathfrak{U}$ of X we find immediately a corresponding exact sequence

$$0 \to C^q(\mathfrak{U}, \mathcal{F}') \xrightarrow{i} C^q(\mathfrak{U}, \mathcal{F}) \xrightarrow{j} C^q(\mathfrak{U}, \mathcal{F}''). \tag{2}$$

In general, j is not necessarily onto, so that we cannot complete the sequence (2) with a 0 at the end.

In the classical case, where we are dealing with spaces X which are paracompact and Hausdorff, one can nevertheless obtain a corresponding exact sequence of the cohomology groups $H^q(X, \mathcal{F})$:

$$\begin{aligned} 0 \to H^0(X, \mathcal{F}') \to \cdots \to H^q(X, \mathcal{F}') \to H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}'') \\ \to H^{q+1}(X, \mathcal{F}') \to \cdots . \end{aligned} \tag{3}$$

A proof can be found in mimeographed Princeton notes, Chapter 1, *Sheaves*, by J. C. Moore. The following outline of a simplified proof has been communicated by D. C. Spencer and is based on the treatment of Godement-Serre (mimeographed notes of Godement's lectures at the University of Illinois, 1954–1955, and Serre [3]).

We shall use the notion of a "pre-sheaf" (Leray sheaf). A presheaf assigns to each open set U of the topological space X a K-module $A(U)$, K—a principal ideal ring. If U and V are open sets and $V \subset U$, there is a K-homomorphism ρ_{VU}: $A(U) \to A(V)$ such that for $W \subset V \subset U$ we have the transitivity condition $\rho_{WV}\rho_{VU} = \rho_{WU}$. Note that any pre-sheaf defines, by passing to limts, a sheaf over X.

A homomorphism $\phi: A \to B$ of the pre-sheaf A into the pre-sheaf B is, for each open U, a K-homomorphism $A(U) \to B(U)$ which commutes with the restriction homomorphism ρ_{VU}. A sequence $0 \to A' \to A \to A'' \to 0$ of pre-sheaf homomorphism is said to be exact if and only if $0 \to A'(U) \to A(U) \to A''(U) \to 0$ is exact for every open U. It follows that if we have an exact sequence $0 \to A' \to A \to A'' \to 0$ of pre-sheaf homomorphisms and if $\mathfrak{U}$ is an open covering of X, then we also have the following exact sequence

$$(4) \qquad 0 \to C^q(\mathfrak{U}, A') \xrightarrow{i} C^q(\mathfrak{U}, A) \xrightarrow{j} C^q(\mathfrak{U}, A'') \to 0,$$

where the group $C^q(\mathfrak{U}, A)$ of q-cochains with coefficients in a pre-sheaf is defined in the same way as in the case of sheaves. Then by a standard construction of algebraic topology there follows from the exactness of (4) the existence of an exact cohomology sequence:

$$(5) \qquad \begin{aligned} &0 \to H^0(\mathfrak{U}, A') \xrightarrow{i^*} H^0(\mathfrak{U}, A) \xrightarrow{j^*} \cdots \to H^q(\mathfrak{U}, A') \\ &\xrightarrow{i^*} H^q(\mathfrak{U}, A) \xrightarrow{j^*} H^q(\mathfrak{U}, A'') \xrightarrow{\delta^*} H^{q+1}(\mathfrak{U}, A') \to \cdots . \end{aligned}$$

Passing to the inductive limit we obtain a corresponding exact sequence of cohomology groups $H^q(X, A)$:

$$(6) \qquad \begin{aligned} &0 \to H^0(X, A') \to \cdots \to H^q(X, A') \to H^q(X, A) \to H^q(X, A'') \\ &\to H^{q+1}(X, A') \to \cdots . \end{aligned}$$

One proves quite simply the following result:

Let X be paracompact and Hausdorff and let N be a pre-sheaf such that: (a) $N(U) = 0$ if $U = \emptyset$; (b) the sheaf defined by N is zero. Then $H^q(X, N) = 0$ for all $q \geqq 0$. As a corollary we obtain the result that, if $\phi: A \to A'$ is a homomorphism of pre-sheaves which induces an isomorphism of the corresponding sheaves $\mathcal{F}$, $\mathcal{F}'$ then $H^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F}')$ for all $q \geqq 0$.

Now consider an exact sequence $0 \to \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{j} \mathcal{F}'' \to 0$ of homomorphisms of *sheaves*. The modules $\Gamma(\mathcal{F}', U)$, $\Gamma(\mathcal{F}, U)$ and $\Gamma(\mathcal{F}'', U)$ define pre-sheaves A', A and A'', and we have an exact sequence

$$0 \to \Gamma(\mathcal{F}', U) \to \Gamma(\mathcal{F}, U) \to \Gamma_0(\mathcal{F}'', U) \to 0,$$

where $\Gamma_0(\mathcal{F}'', U)$ denotes the image of $\Gamma(\mathcal{F}, U)$ in $\Gamma(\mathcal{F}'', U)$. Clearly, also the modules $\Gamma_0(\mathcal{F}'', U)$ define a pre-sheaf, which we shall denote by A_0'', so that we have the exact sequence $0 \to A' \to A \to A_0'' \to 0$. We have furthermore the exact sequence $0 \to A_0'' \to A'' \to Q \to 0$, i.e.,

$$(7) \qquad 0 \to \Gamma_0(\mathcal{F}'', U) \to \Gamma(\mathcal{F}'', U) - Q(U) \to 0,$$

where the *pre-sheaf Q defines in the limit the zero sheaf* (this is so because locally each section of $\mathcal{F}''$ is the j-image of a section of $\mathcal{F}$). From the exactness of the pre-sheaf homomorphisms (7), we find, using (6) and observing that $H^q(X, A'')\cong H^q(X, \mathcal{F}'')$:

$$H^q(X, A_0'') \cong H^q(X, A'') \cong H^q(X, \mathcal{F}'').$$

Therefore, applying (6) to the exact sequence $0\to A'\to A\to A_0''$, we find the desired cohomology sequence (3), since $H^q(X, A')\cong H^q(X, \mathcal{F}')$ and $H^q(X, A)\cong H^q(X, \mathcal{F})$.

Since the existence of the exact sequence (3) is the key to all the applications of sheaf theory, it is essential to establish (3) for *coherent sheaves* also in the abstract case. In the abstract case, however, it turns out that *even the sequence* (2) *can be completed with a zero at the end*. The problem then is to prove that j maps $C^q(\mathfrak{U}, \mathcal{F})$ *onto* $C^q(\mathfrak{U}, \mathcal{F}'')$, if $\mathcal{F}$ and $\mathcal{F}''$ are coherent sheaves, and it is sufficient to prove this only for coverings $\mathfrak{U}$ by principal open sets. If $\mathfrak{U}$ is such a covering, then the support $U_{i_0}\cap U_{i_1}\cap \cdots \cap U_{i_q}$ of every simplex of the nerve of $\mathfrak{U}$ is again a principal open set, hence is biregularly equivalent to an affine variety. Thus, our problem is therefore to prove the following theorem:

THEOREM 1. *If j is an algebraic homomorphism of a coherent sheaf $\mathcal{F}$ onto a coherent sheaf $\mathcal{F}''$ on an affine variety V, then j maps $\Gamma(\mathcal{F}, V)$ onto $\Gamma(\mathcal{F}'', V)$.*

In the proof of this theorem there are three distinct stages which can be summarized as follows:

1. First one proves directly Theorem 1 in the special case $\mathcal{F}=\mathcal{O}_V^m$. The proof in this case is a simple application of Lemma 3 (§3), in conjunction with the fact that *locally* the sections of $\mathcal{F}''$ are j-images of sections of $\mathcal{F}$ (§1).

2. Using the above special case of Theorem 1 one obtains easily the following important auxiliary result:

LEMMA 4. *If $\mathcal{F}$ is the homomorphic image of $\mathcal{O}_V^m$ (V—an affine variety) and f is a section of $\mathcal{F}$ over a principal open subset U of V having boundary $Q=0$ (Q—an element of the coordinate ring of V), then for sufficiently high integers q the section Q^qf can be extended to a section of $\mathcal{F}$ over V.*

Actually, Lemma 4 already has to be used at Step 1, but only in the special case $\mathcal{F}=\mathcal{O}_V^m$, and in that case the lemma is trivial.

3. Finally, using Lemma 4 and the fact that *locally* every coherent sheaf is of the form $\mathcal{O}^m/\phi(\mathcal{O}^q)$, one proves the following basic result:

THEOREM 2. *If $\mathcal{F}$ is a coherent sheaf on an affine variety V, then there exists an exact sequence:*

$$\mathcal{O}_V^q \to \mathcal{O}_V^m \to \mathcal{F} \to 0.$$

Thus, for affine varieties, the local conditions which define coherent sheaves (see (1), §3) can be replaced by *one* similar *global* condition. At any rate, every coherent sheaf on an affine variety V is thus a homomorphic image of $\mathcal{O}_V^m$, for some m.

Theorem 1 now follows easily. For, we have a homomorphism α of $\mathcal{O}_V^m$ onto $\mathcal{F}$, for some m, and thus $j\alpha$ is a homomorphism of $\mathcal{O}_V^m$ onto $\mathcal{F}''$. If $f'' \in \Gamma(\mathcal{F}'', V)$, we have, by the special case of Theorem 1, proved in step 1, that $f'' = (j\alpha)(g)$, for some g in $\Gamma(\mathcal{O}_V^m, V)$. If we set $f = \alpha(g)$, then $f'' = j(f)$ and $f \in \Gamma(\mathcal{F}, V)$.

The so-called "partition of unity" which played an important role in the classical theory has the following algebraic analogue which is also used in the abstract case (and in the proofs of Theorems 1 and 2): *If $\mathfrak{U}$ is a finite covering of an affine variety V by principal open subsets U_i and if $Q_i = 0$ is an equation of the boundary of $U_i (Q_i \in R)$, then there exist elements G_i in R such that* $\sum G_iQ_i = 1$. This is another formulation of the Hilbert Nullstellensatz.

5. **Cohomological properties of coherent sheaves on affine varieties.** A sheaf $\mathcal{F}$ on a variety V is said to be *semifine* (Serre [1]) if it satisfies the following two conditions:

SF.1. $H^0(V, \mathcal{F})$ (i.e., $\Gamma(\mathcal{F}, V)$) generates the stalk $\mathcal{F}_x$ over $\mathcal{O}_x$ at every point x of V.

SF.2. $H^q(V, \mathcal{F}) = 0$ if $q > 0$.

Theorem 2 (§4) shows that every coherent sheaf $\mathcal{F}$ over an *affine* variety V satisfies condition SF.1. In fact, if $\mathcal{F}$ is the image of $\mathcal{O}_V^m$, under a homomorphism j, and $E_1, E_2, \cdots, E_m$ are the basic unit sections of $\Gamma(\mathcal{O}_V^m, V)$, the m global section $j(E_i)$ of $\mathcal{F}$ generates $\mathcal{F}_x$ over $\mathcal{O}_x$ at every point x. Note that $E_1, E_2, \cdots, E_m$ generate $\Gamma(\mathcal{O}_V^m, V)$ over the coördinate ring R of $V (R = \Gamma(\mathcal{O}_V, V))$. Since, by Theorem 1, j induces a R-homomorphism of $\Gamma(\mathcal{O}_V^m, V)$ *onto* $\Gamma(\mathcal{F}, V)$, it follows that $j(E_1), j(E_2), \cdots, j(E_m)$ generate $\Gamma(\mathcal{F}, V)$ over R. Thus, Theorems 1 and 2 show that $H^0(V, \mathcal{F})$ *is finitely generated over* R (V—affine, $\mathcal{F}$—coherent).

If we apply the above argument to any subset U of V such that U itself is biregularly equivalent to an affine variety, we find that the restrictions of the $j(E_i)$ to U generate $\Gamma(\mathcal{F}, U)$ over $\Gamma(\mathcal{O}_V, U)$. We have thus found that every coherent sheaf $\mathcal{F}$ on an affine variety V enjoys the following property: *there exists a finite number of global sections $f_1, f_2, \cdots, f_m$ of $\mathcal{F}$ such that for any open subset U of V which* *is biregularly equivalent to an affine variety (in particular, for any principal open subset U of V) the restrictions of the f_i to U generate* $\Gamma(\mathcal{F}, U)$ *over the coördinate ring* $\Gamma(\mathcal{O}_V, U)$ *of* U. From this property, and using the algebraic analogue of the "partition of unity" for affine varieties (§4), it follows by a well-known argument (Weil [2]) that *if $\mathfrak{U}$ is a finite covering of an affine variety V by principal subsets U_i, then* $H^q(\mathfrak{U}, \mathcal{F}) = 0$ *for* $q > 0$. Since the set of finite coverings of V by principal subsets is co-final with the set of all open coverings of V, it follows that $H^q(V, \mathcal{F}) = 0$ if $q > 0$, and this is condition SF.2. We have therefore

THEOREM 3. *Every coherent sheaf $\mathcal{F}$ on an affine variety is semi-fine, and for such a sheaf we have that $H^0(V, \mathcal{F})$ is finitely generated over the coördinate ring R of V* (cf. Cartan [3, Exp. 18–19, 1951–1952]).

If we set $M = H^0(V, \mathcal{F})$, then M is a finitely generated R-module, and it generates $\mathcal{F}_x$ at every point x of V. Conversely, starting with any finite R-module M we can define a sheaf $\mathcal{F}$ over V by setting $\mathcal{F}_x = M_R \times \mathcal{O}_x$ (= tensor product, over the common ring of operators R) and by defining local sections in a natural manner, by means of the local sections of $\mathcal{O}_V$. One finds then that there is a (1, 1) correspondence between finite R-modules and coherent sheaves over V.

All these results (except the property of $H^0(V, \mathcal{F})$ of being finitely generated) are analogous to well-known statements concerning Stein varieties (see Part I of this report) and coherent sheaves over them. However, the proofs for affine varieties are much easier, and, as we have seen, are essentially of elementary nature.

Using a well-known theorem on "double complexes" (Leray [1]; H. Cartan [3, 1953–1954]), one derives at once from Theorem 3 the following result (Serre [3]; for the classical case see Leray [1], Weil [2]):

THEOREM 4. *If $\mathfrak{U}$ is a finite covering of a variety V by affine varieties U_i and $\mathcal{F}$ is a coherent sheaf on V, then the canonical homomorphism of $H^q(\mathfrak{U}, \mathcal{F})$ into $H^q(V, \mathcal{F})$ is an isomorphism.*

Theorem 4 shows that to catch the cohomology group of an arbitrary variety V, relative to a given coherent sheaf, it is not necessary to pass to the inductive limit: those groups which are obtained from the nerve of a covering of V by affine varieties are already isomorphic to the inductive limit. The cohomology groups of V (in the case of coherent sheaves) can therefore be found by purely algebraic processes.

6. **Coherent sheaves on projective varieties.** A first consequence of Theorem 4 is the following: *if $\mathcal{F}$ is a coherent sheaf on a projective*

variety V, *of dimension* m, *then* $H^q(V, \mathcal{F})=0$ *for* $q>m$. In view of Theorem 4 it is sufficient to show that $H^q(\mathfrak{U}, \mathcal{F})=0$, $q>m$ and for some finite covering $\mathfrak{U}$ of V by affine varieties. We fix $m+1$ hyperplanes H_i which have no point in common on V. Then $\mathfrak{U}=\{V-H_i\}$ is a covering of V by principal open sets, and since the nerve of $\mathfrak{U}$ has only $m+1$ vertices it follows that $H^q(\mathfrak{U}, \mathcal{F})=0$ for $q>m$.—Before Serre, this result was derived from *Dolbeault's isomorphism* (Dolbeault [1]), but for locally free sheaves only.

The treatment of sheaves on projective varieties is greatly facilitated by the operation of extension of sheaves. This operation allows us to replace any sheaf $\mathcal{F}$ over a variety V by a sheaf $\mathcal{G}$ carried by the ambient projective space X of V: we simply set $\mathcal{G}_x=\mathcal{F}_x$ for $x\in V$ and $\mathcal{G}_x=0$ if $x\notin V$. Conversely, if a sheaf $\mathcal{G}$ over X is such that $\mathcal{G}_x=0$ outside of a variety V, then $\mathcal{G}$ gives rise to a unique sheaf $\mathcal{F}$ carried by V. We shall often identify sheaves $\mathcal{F}$ and $\mathcal{G}$ in such a relation, since it is easily seen that $H^q(V, \mathcal{F})=H^q(X, \mathcal{G})$ for all q. *If* $\mathcal{F}$ *is algebraic then* $\mathcal{G}$ *is always algebraic* (in a natural way), for the local ring $\mathcal{O}_{x,V}$ of any point x of V is a homomorphic image of the local ring $\mathcal{O}_{x,X}$ of this point, when x is regarded as a point of X, whence the $\mathcal{O}_{x,V}$-module $\mathcal{F}_x$ is also an $\mathcal{O}_{x,X}$-module. The converse is not always true. However, if both $\mathcal{F}$ and $\mathcal{G}$ are algebraic and if one of them is coherent then also the other is coherent.

Now let us restrict ourselves to sheaves $\mathcal{F}$ over the projective r-space X. In practice, $\mathcal{F}$ will vanish outside the irreducible projective variety V which is the object of study, and in that case we shall say that $\mathcal{F}$ *is carried by* V.

Let $\mathcal{F}$ be an algebraic sheaf over X. A basic operation considered by Serre is the one which associates with $\mathcal{F}$, and with any integer n, another algebraic sheaf $\mathcal{F}(n)$. This sheaf $\mathcal{F}(n)$ is defined as follows: Let $y_0, y_1, \cdots, y_r$ be homogeneous coördinates in X and let U_i be the open set defined by $y_i\neq 0$. We consider the $r+1$ sheaves $F^i=\mathcal{F}_{U_i}$ ($=$restriction of $\mathcal{F}$ to U_i) and we patch them up together again, but by a different rule: if $x\in U_i\cap U_j$ and $f_i\in\mathcal{F}^i_x$, $f_j\in\mathcal{F}^j_x$, then we identify f_i and f_j if

$$f_j = \left(\frac{y_i}{y_j}\right)^n f_i. \tag{1}$$

(Note that (1) makes sense since y_i/y_j is a unit in $\mathcal{O}_x$ if $x\in U_i\cap U_j$.) Every element of the stalk of $\mathcal{F}(n)_x$ shall have a unique representative in $\mathcal{F}^i_x$, if $x\in U_i$, and two representatives f_i, f_j of f (if $x\in U_i\cap U_j$) shall be related by (1). A sheaf structure is then introduced in $\mathcal{F}(n)$ in a natural way, so that $\mathcal{F}(n)$ and $\mathcal{F}$ are locally isomorphic everywhere.

It follows that $\mathcal{J}(n)$ is coherent if $\mathcal{J}$ is coherent. Note that $\mathcal{J}(0)=\mathcal{J}$.

The geometric meaning of the sheaf $\mathcal{J}(n)$ becomes clear in the important special case in which $\mathcal{J}$ is a sheaf carried by V and associated with a divisor D on V (we shall assume that V is normal). Then, for $x\in V$, $\mathcal{J}_x$ is the set of rational functions f on V such that $(f)+D>0$ locally at the point x. The global sections of $\mathcal{J}$ are given by the functions f such that $(f)+D>0$ globally on V. These functions f form a finite-dinensional vector space $\Gamma(\mathcal{J}, V)$ over k, say of dimension $\rho+1$, and the set of integral cycles $(f)+D, f\in\Gamma(\mathcal{J}, V), f\neq 0$, is the complete linear system $|D|$, which is then of dimension ρ. The sheaf $\mathcal{J}$ associated with a divisor D will be denoted by $\mathcal{L}(D)$. Note the relation

$$\dim H^0(V, \mathcal{L}(D)) = 1+\dim|D|. \tag{2}$$

If D and D' are linearly equivalent divisors, so that $D-D'=(\alpha)$, $\alpha\in K$, then $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic sheaves, the isomorphism being obtained by multiplying the sheaf elements of $\mathcal{L}(D)$ by α.

It is now not difficult to see that *if C denotes a hyperplane section of V and C_n is any divisor linearly equivalent to nC, then $\mathcal{L}(D)(n)$ is isomorphic with $\mathcal{L}(D+C_n)$*. It is sufficient to prove this for one particular C_n, and we shall take for C_n the divisor nC^0, where C^0 is the section of V with the hyperplane $y_0=0$ (we may assume, without loss of generality, that this hyperplane does not contain the variety V). If $f\in\mathcal{L}(D)(n)_x(x\in V)$ and f_i is a representative of f in $\mathcal{L}(D)_{U_i}$ $(0\leq i\leq r; x\in U_i)$, then $f^{(n)}=f_i(y_i/y_0)^n$ depends only on f, by (1), and belongs to $\mathcal{L}(D+nC^0)_x$. It is then immediately seen that the mapping $f\rightarrow f^{(n)}$ is an isomorphism of $\mathcal{L}(D)(n)$ with $\mathcal{L}(D+nC^0)$.

The study of the complete linear systems $|D+nC|$ is fundamental in theory of linear systems on V, and the sheaf-theoretic analogue of this is the study of the sheaves $\mathcal{L}(D)(n)$.

If D is null divisor, then $\mathcal{L}(D)$ is the sheaf $\mathcal{O}_V$. It follows that $\mathcal{O}_V(n)$ is isomorphic with $\mathcal{L}(nC^0)$.

Let $R=k[z_0, z_1, \cdots, z_r]$ be the homogeneous coördinate ring of V, where $z_0, z_1, \cdots, z_r$ are therefore strictly homogeneous coördinates of the general point of V/k. Then $R=\sum_{n=0}^{+\infty} R_n$ is graded ring, and also its integral closure $I=\sum_{n=0}^{\infty} I_n$ is a graded ring (Zariski [1]); here R_n and I_n are the sets of homogeneous elements (of R and I, respectively) which are of degree n. From known theorems on normal varieties it follows that

$$\Gamma(\mathcal{O}_V(n), V)\cong I_n, \tag{3}$$

$$\Gamma(\mathcal{O}_V(n), V)\cong R_n(=I_n), \text{ if } n \text{ is large,} \tag{3'}$$

all the isomorphisms being canonical.

If V is not normal then the following can be easily proved: *there exists a greatest graded ring* $J=\sum_{n=0}^{\infty} J_n$ *between* R *and* I *such that the conductor of* R *relative to* J *contains a power of the irrelevant ideal* $(z_0, z_1, \cdots, z_r)$($J=I$ if and only if V is normal; $J_n=J\cap I_n$), *and relations* (3), (3′) *continue to hold if we replace* I *by* J.

Another representation of the sheaf $\mathcal{O}_V(n)$ is the following: *the stalk* $\mathcal{O}_V(n)_x$ ($x\in V$) *consists of the quotients of the form*

$$f(z_0, z_1, \cdots, z_r)/g(z_0, z_1, \cdots, z_r),$$

where f *and* g *are forms of degree* $j+n$ *and* j *respectively* (j-*arbitrary*) *and where the form* g *does not vanish at the point* x. From this representation of $\mathcal{O}_V(n)$ it follows that the product of an element of $\mathcal{O}_V(n)_x$ by a form of degree s in $y_0, y_1, \cdots, y_r$ can be identified with a well-defined element of $\mathcal{O}_V(n+s)_x$.

If $\mathcal{F}$ is an arbitrary sheaf then a representation of $\mathcal{F}(n)$ is the following: $\mathcal{F}(n)$ *is isomorphic to the tensor product* $\mathcal{F}\otimes\mathcal{O}_V(n)$, i.e.,

$$\mathcal{F}(n)_x \cong \mathcal{F}_x \underset{\mathcal{O}_x}{\otimes} \mathcal{O}_x\mathcal{O}(n)_x.$$

REMARK. From this representation of $\mathcal{F}$ it follows that the product of any element of $\mathcal{F}(n)_x$ by any form of degree h in $y_0, y_1, \cdots, y_r$ can be identified with a well-defined element of $\mathcal{F}(n+h)_x$. Consequently, if S denotes the polynomial ring $k[y_0, y_1, \cdots, y_r]$ and S_h is the set of forms of degree h then we can write $S_hH^0(V, \mathcal{F}(n)) \subset H^0(V, \mathcal{F}(n+h))$. Later we shall make use of this remark.

Before going further with our summary we shall make some general remarks about the methods of proof. There are essentially two principal methods:

a. The inductive, algebro-geometric method.

b. The abstract method of cohomological algebra of functors. We shall describe briefly these two methods.

The inductive method. Let $\mathcal{F}$ be a coherent sheaf over our variety V. The operation $\mathcal{F}\rightarrow\mathcal{F}(n)$ and the extension of $\mathcal{F}$ to the whole projective space are commutative operations. Hence we may assume that $\mathcal{F}$ is defined over the whole of X. Let $L(y)$ be an arbitrary linear form in the y's. If $f\in\mathcal{F}_x$, the collection of elements $fL(y)/y_i$, where i ranges over those indices $0, 1, \cdots, r$ for which $x\in U_i$, represents an element $\phi(f)$ of $\mathcal{F}(1)_x$. Thus we get an algebraic homomorphism of $\mathcal{F}$ into $\mathcal{F}(1)$, and by Lemma 2 (§3) the kernel $\mathcal{L}$ and cokernel $\mathcal{M}$ of ϕ are both coherent. If we replace here $\mathcal{F}$ by $\mathcal{F}(n)$, then $\mathcal{F}(1)$, $\mathcal{L}$ and $\mathcal{M}$ must be replaced by $\mathcal{F}(n+1)$, $\mathcal{L}(n)$, $\mathcal{M}(n)$. Hence we have an exact sequence of the form:

$$0 \to \mathcal{L}(n) \to \mathcal{F}(n) \to \mathcal{F}(n+1) \to \mathcal{M}(n) \to 0. \tag{4}$$

The equation $L(y)=0$ defines a hyperplane Y, and Y is of course a projective space of dimension $r-1$. It is not difficult to see that the sheaves $\mathcal{L}(n)$ and $\mathcal{M}(n)$ are zero outside of Y, hence are carried by Y. Moreover it can be verified that they are algebraic, hence coherent, as sheaves with Y as base space. The sequence (4) offers therefore a general method for applying induction with respect to the dimension m of the carrier V of $\mathcal{F}$, for the sheaves $\mathcal{L}(n)$ and $\mathcal{M}(n)$ are zero outside $V\cap Y$, and $V\cap Y$ is of dimension $m-1$. If we wish to prove a theorem which contains, among its assumptions, the assumption that V satisfies a certain condition (say, that V is irreducible, or nonsingular or normal), the success of the induction will depend, among other things, on the possibility of choosing the hyperplane Y in such a manner that also $V\cap Y$ satisfies that same condition (if the condition is one of those which we have mentioned above in parenthesis, such a choice is always possible).

The method just outlined is standard in the classical theory, and many arguments involving only this method in the analytical case can be made abstract without too great difficulty. The following outline of the proofs of the two fundamental theorems given below (Theorems 5 and 6) will illustrate this method.

THEOREM 5. *If $\mathcal{F}$ is a coherent sheaf over X then the sheaf $\mathcal{F}(n)$ is semi-fine for all sufficiently large n.*

THEOREM 6. *If $\mathcal{F}$ is a coherent sheaf over X then all the cohomology groups $H^q(X, \mathcal{F})$ are finite dimensional (as vector spaces over the ground field k).*

It is easy to see that $\mathcal{F}(n)$ satisfies condition SF.1 (see §5) if n is large. In fact, we know that each of the sheaves $\mathcal{F}_{U_i}$, $i=0, 1, \cdots, r$, is semifine. Since $\mathcal{F}(n)_x$ is essentially $\mathcal{F}_x$, for all $x\in X$, it is sufficient to show that if f is any section of $\mathcal{F}$ over one of the U_i, say over U_0, then, for large n, f can be extended to a global section g of $\mathcal{F}(n)$. From Lemma 4 and Theorem 2 (§4) it follows that $(y_0/y_i)^{n\nu}$ can be considered as the restriction to $U_0\cap U_i$ of a section $f_i^{(n)}$ of $\mathcal{F}$ over U_i, provided n is sufficiently large. The desired global section g of $\mathcal{F}(n)$ is the one defined by the $r+1$ partial sections $f_i^{(n)}$, $i=0, 1, \cdots, r$.

We now show that $\mathcal{F}(n)$ also satisfies condition SF.2 if n is large, and we begin with the case $\mathcal{F}=\mathcal{O}_X^q$. In this case we have $\mathcal{L}=0$ (essentially because each $\mathcal{O}_x$ is an integral domain) and the exact sequence (4) now specializes to $0 \to \mathcal{O}^p(n) \to \mathcal{O}^p(n+1) \to \mathcal{O}^p_Y(n+1) \to 0$. If we assume by induction that $\mathcal{O}^p_Y(n+1)$ has the property SF.2, we get canonical isomorphisms: $H^q(X, \mathcal{O}^p(n)) \to H^q(X, \mathcal{O}^p(n+s))$ for $q>1$ and for $s>0$. Since the mapping: $H^0(X, \mathcal{O}^p(n+1)) \to H^0(Y, \mathcal{O}^p_Y(n+1))$ is an epimorphism, we get a similar isomorphism also for $q=1$. If we combine with this the remark that every coherent sheaf over $X-Y$ is semifine, we can prove that $H^q(X, \mathcal{O}^p(n))=0$, i.e., $\mathcal{O}^p(n)$ has the property SF.2. We can now treat the general case. Since $\mathcal{F}(n)$ has the property SF.1 for n sufficiently large, we have an algebraic homomorphism from $\mathcal{O}^p$ onto $\mathcal{F}(n)$. Let $\mathcal{R}$ be the kernel of this homomorphism. Then for any s we have an exact sequence of the form $0 \to \mathcal{R}(s) \to \mathcal{O}^p(s) \to \mathcal{F}(n+s) \to 0$, and therefore $H^q(X, \mathcal{F}(n+s)) = H^{q+1}(X, \mathcal{R}(s))$ by the special case treated above. After $r-q+1$ steps we get an isomorphism between $H^q(X, \mathcal{F}(n))$ and an $(r+1)$th cohomology group of a coherent sheaf over X, provided n is sufficiently large. However this is zero by the result stated in the beginning of this section. Thus Theorem 5 is proved.

From this theorem it follows that

$$0 \to \mathcal{R}(-n) \to \mathcal{O}^p(-n) \to \mathcal{F} \to 0, \tag{5}$$

for n sufficiently large, and it is therefore sufficient to prove Theorem 6 for $\mathcal{O}^p(-n)$ and $\mathcal{R}(-n)$. Moreover, since $H^0(X, \mathcal{R}(-n))$ is a subspace of $H^0(X, \mathcal{O}^p(-n))$, and since the latter is of dimension p or zero according as $r=0$ or positive, the former is of finite dimension. We shall denote by $\mathcal{G}$ either $\mathcal{R}(-n)$ or $\mathcal{O}^p(-n)$, and we shall show that $H^q(X, \mathcal{G})$ is of finite dimension for $q>0$. If we take $\mathcal{G}$ as $\mathcal{F}$ in (4), we get $0 \to \mathcal{G}(s) \to \mathcal{G}(s+1) \to \mathcal{M}(s) \to 0$. If we apply an induction on r, we may assume that $H^q(Y, \mathcal{M}(s))$ is of finite dimension for all s. Then $H^q(X, \mathcal{G}(s))$ and $H^q(X, \mathcal{G}(s+1))$ have or have not finite dimensions at the same time. However, since $H^q(X, \mathcal{G}(s))=0$ for s sufficiently large, $H^q(X, \mathcal{G})$ must be of finite dimension.

The abstract method of functorial algebra. If $\mathcal{F}$ is an algebraic sheaf over the projective space X we set

$$M(\mathcal{F}) = \sum_n H_0(V, \mathcal{F}(n)), \qquad M(\mathcal{F})_n = H_0(V, \mathcal{F}(n)),$$

where n ranges over the set of all integers and where the sum is direct. By a remark made earlier in this section, $M(\mathcal{F})$ is a graded S-module. It turns out that $\mathcal{F}$ *is coherent if and only if there exists an integer* n_0 *such that* $\sum_{n=n_0}^{+\infty} M(\mathcal{F})_n$ *if a finite S-module and that the mapping* $\mathcal{F} \to \sum_{n \geq n_0} M(\mathcal{F})_n$ *is a* (1, 1) *correspondence between coherent sheaves* *and finite S-modules, provided two such modules which differ only in their components of low degrees are identified.* Geometric theorems can then be translated into algebraic theorems concerning certain functors derived from such modules (in particular, the so-called *extension functors*), and the powerful technique of modern functorial algebra can then be used. This method, developed in full by Serre in [3], has not been considered in the seminar, for the lack of knowledge and time. However, by using this method, *in conjunction with the inductive method described above*, Serre was able to prove abstractly his fundamental duality theorem (see below, §8) and therefore also the equality $P_a = p_a$ conjectured by Severi (see Zariski [4]; cf. also §8). So it would seem that the deeper results of theory of sheaves over algebraic varieties depend on the possibility of applying this second abstract method of functor algebra.

7. The arithmetic genera. We have now enough tools to reproduce, by sheaf-theoretic methods, the algebraic theory of arithmetic genera given by Zariski in [4]. The method used by Serre is largely the same as the one used by Kodaira and Spencer in [1]. If $\mathcal{F}$ is a coherent sheaf on our m-dimensional variety V, in the projective space, then the Euler-Poincaré characteristic of V, with coefficients in $\mathcal{F}$, is defined by

$$\chi(V, \mathcal{F}) = \sum_{q=0}^{m} (-1)^q \dim H^q(V, \mathcal{F}).$$

(Recall Theorem 6 and note that $\dim H^q(V, \mathcal{F}) = 0$ if $q > m$).

If we apply an induction using the fundamental exact sequence (4) of §6, we can show easily that $\chi(V, \mathcal{F}(n))$ is a polynomial in n for all n, of degree at most equal to m. Also, for n sufficiently large, this polynomial coincides with $\dim H^0(V, \mathcal{F}(n))$, by the property SF.2 of $\mathcal{F}(n)$ (Theorem 5). In particular, if we remark that, for large n, $H^0(V, \mathcal{O}(n))$ is canonically isomorphic with the module R_n of homogeneous elements of degree n in the homogeneous coördinates ring R of V (see §6; also Zariski [1]), we see that $\chi(V, \mathcal{O}(n))$ is nothing but the Hilbert polynomial of V. Therefore

$$\chi(V, \mathcal{O}) = \sum_{q=0}^{m} (-1)^q \dim H^q(V, \mathcal{O})$$

is the constant term of this polynomial, and consequently

$$\chi(V, \mathcal{O}) = 1 + (-1)^m p_a, \tag{1}$$

where p_a is the arithmetic genus of V. Since the sheaf theory is invariant under biregular transformations, $\chi(V, \mathcal{O})$ is a biregular invariant of V. This theorem was proved by Muhly and Zariski [1] for normal varieties, but under more general transformations. On the other hand, Kodaira and Spencer [1] used the above expression of the arithmetic genus to prove its *birational* invariance for nonsingular varieties over the complex field. Firstly, $H^q(V, \mathcal{O})$ is isomorphic to the space of "harmonic forms of type $(0, q)$," by Dolbeault's isomorphism and then by the decomposition formula of harmonic forms. Secondly, if we pass to the complex conjugates of those forms, we get harmonic forms of type $(q, 0)$, i.e., algebraic differential forms of degree q of the first kind. Finally, these forms behave nicely under birational transformations, and, e.g., the number g_q of linearly independent forms is birationally invariant for every q. Done!

It is worthwhile to remark here the following: Kodaira and Spencer considered only analytic sheaves over nonsingular varieties with the usual Hausdorff topology. Thus, in the classical case, there are two cohomology groups, one is analytic, and the other is algebraic. Also, theorems concerning these cohomology groups are similar. Since the Zariski topology is weaker than the Hausdorff topology, we can define a mapping from the algebraic cohomology groups into the corresponding analytic cohomology groups which commutes with homomorphisms in both theories. However, since the mapping is an isomorphism for zero-dimensional cohomology groups and since the $\mathcal{F}(n)$ have the property SF.2 for coherent sheaves $\mathcal{F}$ in both theories, the mapping gives an "isomorphism" of the two theories.

Let us now assume that V is nonsingular. For every divisor D on V we define a numerical character $\chi_V(D)$ of D by

$$\chi_V(D) = \chi(V, \mathcal{O}) - \chi(V, \mathcal{L}(-D)). \tag{2}$$

Then $\chi_V(D)$ has the characteristic properties which Zariski [4] proved for the "virtual arithmetic genus $p_a(D)$ of D with respect to V," and from this it follows easily that

$$\chi_V(D) = 1 + (-1)^{n-1}p_a(D). \tag{3}$$

This we can show by taking $L(D)$ as F in the fundamental exact sequence (4) of §6 (cf. Kodaira-Spencer [1]). Thus the virtual arithmetic genera are connected with sheaf theory.

8. Theorem of Riemann-Roch and related topics. If D is a divisor on V, we shall write $h^q(D)$ for $\dim H^q(V, \mathcal{L}(D))$ and $\chi(V, D)$ for $\chi(V, \mathcal{L}(D))$. We shall also write $\chi(V)$ for $\chi(V, \mathcal{O})$. If C denotes a hyperplane section of V, then it is known (Zariski [4]) that

$$1 + \dim |D + nC| = (-1)^n\{p_a(V) + p_a(-D - nC)\}, \tag{1}$$

for n large. This result can also be derived, by sheaf-theory, from (1) and (3) of §7 and from the fact that the sheaf $\mathcal{L}(D+nC)$ $(=\mathcal{L}(D)(n))$ is semifine if n is large (Theorem 5, §6). Namely, if n is large, then the left-hand side of (1) coincides with $\chi(V, D+nC)$, while the right-hand side is equal to $\chi(V)-\chi_V(-D-nC)$, and (1) now follows from the definition of the symbol $\chi_V(-D-nC)$ (see (2), §7).

The expression $(-1)^m\{p_a(V)+p_a(-D)\}+1$, which is not in general equal to the dimension of the complete linear system $|D|$, may be said—to use a time-honored expression of the Italian geometers—to represent the *virtual dimension of* $|D|$. Relation (1) tells us that for large n the virtual dimension of $|D+nC|$ coincides with the effective dimension. We have, by (1) and (3) of §7:

$$(-1)^m\{p_a(V) + p_a(-D)\} = \chi(V, D). \tag{2}$$

This precise equality can be regarded, in a limited sense, as *a generalization of the Riemann-Roch theorem to varieties*. Sheaf theory tells us that *the virtual dimension of* $|D|$, *increased by* 1, *equals the Euler-Poincaré characteristic of* V, *with coefficients in the sheaf* $L(D)$.

In this connection it may be pointed out that the following formula is implicitly contained in Zariski [4]:

$$(-1)^m\{p_a(V) + p_a(-D)\} = \sum_{i=0}^{m} (-1)^i s_i(D), \tag{2'}$$

where $s_0(D)=1+\dim |D|$;

$s_1(D)=$ deficiency of the linear system $\{D^{(1)}\}$ cut out on a generic C_n by the complete linear system $|D+C_n|$; here n is a sufficiently high integer, and C_n is the section of V with a hypersurface of order n.

$s_2(D)=$ deficiency of the linear system $\{D^{(2)}\}$ cut out on $C_n\cdot C'_{n'}$ by the complete linear system $|D^{(1)}|$; here n' is sufficiently large with respect to n, and $C_{n'}'$ is generic with respect to C_n; and so on, except that $s_m(D)$ is defined as the index of specialty of the zero-dimensional cycle $D^{(m-1)}$ cut out on the curve $C_n\cdot C'_{n'}\cdots C_{n^{(m-2)}}^{(m-2)}$ by $C_{n^{(m-1)}}^{(m-1)}$. It is clear that $s_0(D)=h^0(D)$. Using the duality relation (8) established below it is possible to show that $s_m(D)=h^m(D)$.

In the case of curves $(m=1)$, $p_a(V)$ is the genus of the curve V, and $p_a(-D)=-\deg D-1$, and thus we have by (2):

$$\dim |D| = \deg D - g + h^1(D).$$

It follows by the classical Riemann-Roch theorem that

$$h^1(D) = h^0(K - D), \tag{3}$$

where K is a canonical divisor on V.

For $m>1$ a satisfactory algebro-geometric characterization of the cohomological dimension $h^q(D)$, $q>0$, is still missing, except for $q=m$, in which case we have a fundamental duality relation similar to (3):

$$h^m(D) = h^0(K - D), \tag{4}$$

where K is again a canonical divisor on V, i.e., the divisor of an m-fold differential on V. This equality follows readily from the so-called "lemma of Enriques-Severi-Zariski" proved by Zariski [4], which states that "*if D is any divisor on V and C_n is a general section of V by a hypersurface of order n, then for n sufficiently large the linear system* $\mathrm{Tr}_{C_n}|D|$, *cut out on C_n by the complete linear system $|D|$, is itself complete.*" An equivalent formulation is the following equality:

$$\dim |D| = \dim |D \cdot C_n|, \tag{5}$$

if n is large, since for large n we have $\dim |D| = \dim \mathrm{Tr}_{C_n} |D|$. If C_n' is another member of $|C_n|$, different from C_n, then—always under the assumption that C_n is a general section of V—it can be shown that the following sequence is exact:

$$0 \to \mathcal{L}(D) \xrightarrow{i} \mathcal{L}(D + C_n') \xrightarrow{j} \mathcal{L}(C_n, (D + C_n')\cdot C_n) \to 0$$

(where the quotient sheaf $\mathcal{L}(C_n, (D+C_n')\cdot C_n)$ denotes the sheaf *on* C_n, defined by the divisor $(D+C_n')\cdot C_n$), and the corresponding exact cohomology sequence

$$\begin{aligned} \cdots &\to H^q(V, \mathcal{L}(D + C_n')) \xrightarrow{j^*} H^q(C_n, \mathcal{L}((D + C_n')\cdot C_n)) \\ &\xrightarrow{\delta^*} H^{q+1}(V, \mathcal{L}(D)) \xrightarrow{i^*} H^{q+1}(V, \mathcal{L}(D + C_n')) \to \cdots \end{aligned} \tag{6}$$

shows that if $q>0$ and n is sufficiently large, then the two middle terms are isomorphic (since $\mathcal{L}(D+C_n')$ is then semifine) and hence

$$h^q(C_n, (D + C_n')\cdot C_n) = h^{q+1}(D), \tag{7}$$

(n—large, $q>0$). In view of (3), we can prove (4) by induction with respect to m: Writing (7) for $q=m-1$ (we may assume $m>1$) and using our induction hypothesis we find

$$h^m(D) = h^0(C_n, \bar{K} - (D + C_n')\cdot C_n),$$

where $\bar{K}$ is a canonical divisor on C_n. It is known (Zariski [4]) that $\bar{K} \equiv (K+C_n')\cdot C_n$. Hence $h^m(D) = h^0(C_n, (K-D)\cdot C_n)$, i.e.,

$$h^m(D) = 1 + \dim |(K - D)\cdot C_n|, \tag{8}$$

n large, and now (4) follows in view of the lemma of Enriques-Severi-Zariski (5), as applied to the divisor $K-D$.

Conversely, the Zariski result (5) follows from the duality formula (4), in view of (8).

For $m=2$ the Riemann-Roch formula (2) together with the duality formula (4) yields the relation

$$\dim |D| = p_a(V) + p_a(-D) - \dim |K - D| + h^1(D),$$

and hence the well-known *Riemann-Roch inequality* $\dim |D| \geqq p_a(V) + p_a(-D) - \dim |K-D|$, and $h^1(D)$ therefore coincides with the so-called *superabundance* of $|D|$.

Another result that is equivalent with the lemma of Enriques-Severi-Zariski is the following:

$$h^1(D - C_n) = 0, \tag{9}$$

if n is large. In the proof of (9) we may replace D by $D+C_h$, h arbitrary. Therefore we may assume that $h^1(D)=0$. If in (6) we replace D by $D-C_n$, we find, for large n:

$$\begin{aligned} 0 \to H^0(V, \mathcal{L}(D - C_n)) &\xrightarrow{i^*} H^0(V, \mathcal{L}(D)) \xrightarrow{j^*} H^0(C_n, \mathcal{L}(D\cdot C_n)) \\ &\to H^1(V, \mathcal{L}(D - C_n)) \to 0 \to \cdots . \end{aligned} \tag{10}$$

Now, Zariski's result (5) signifies that j^* is an epimorphism if n is large. Since (10) is exact, we must have $H^1(V, \mathcal{L}(D-C_n))=0$, i.e., $h^1(D-C_n)=0$. Conversely, if $h^1(D-C_n)=0$, then (10) shows that j^* is an epimorphism.

We have thus three basic but equivalent statements: (4), (5) and (9). We have outlined the proofs of (4) and (9) by using the relation (5) which we have proved in [4] by a direct algebro-geometric argument (and for normal, not only nonsingular, varieties). Serre gave two sheaf-theoretic proofs of these results. The first proof (unpublished; cf. Serre [4]) uses explicitly the m-fold differentials on V and is—in part—an adaptation of an argument used by Weil [1] in the one-dimensional case. This proof is applicable only to nonsingular varieties. The second proof (Serre [3]) uses the algebra of functors, goes much further, and is applicable also to normal varieties. The first proof was discussed in detail in the seminar and will now be briefly outlined. After that, we shall say a few words about the second proof.

We denote by $\Omega^p(D)$ the sheaf of germs of p-fold differentials on V

which are (locally) multiples of $-D$. This is a coherent sheaf, since it is locally isomorphic with the $\binom{m}{p}$-fold direct product of the sheaf $\mathcal{O}$. Note that $\Omega^0(D)=\mathcal{L}(D)$. We write Ω^p for $\Omega^p(0)$: this is the sheaf of germs of holomorphic differentials.

First of all one proves that $H^m(V, \Omega^m)$ is of dimension 1 over k. For $m=1$ this is a consequence of the residue theorem for abelian differentials on an algebraic curve V ("there exists a differential with preassigned poles of order $\leqq 1$ and preassigned residues at these poles, provided the sum of residues is zero"). The general case can be reduced to the case $m=1$ by the inductive procedure described in §6. The fundamental exact sequence (4) of §6 to be used in this case is the one in which $\mathcal{J}$ is replaced by Ω^m and C is replaced by C_n, n large. A similar argument shows that $H^m(V, \Omega^m(D))$ is zero for any strictly positive divisor D. These two assertions can be regarded as generalizations of the "residue theorem" for abelian differentials.

Now, if ω is any element of $H^0(V, \Omega^m(-D))$ (i.e., ω is an m-fold differential which is globally a multiple of D), then the multiplication by ω of each stalk of $\mathcal{L}(D)$ defines an element of Hom $[\mathcal{L}(D), \Omega^m]$. Therefore ω also defines an element $J_D(\omega)$ of Hom$_k$ $[H^m(V, \mathcal{L}(D)), H^m(V, \Omega^m)]$, and since $H^m(V, \Omega^m)\cong k$, $J_D(\omega)$ is an element of the dual space of $H^m(V, \mathcal{L}(D))$. We thus have *a mapping* J_D *of* $H^0(V, \Omega^m(-D))$ *into the dual space of* $H^m(V, \mathcal{L}(D))$. We shall denote this dual space by $W(-D)$. Since $H^0(V, \Omega^m(-D))$ is obviously isomorphic with $H^0(V, K-D)$, the duality relation (4) will follow if it is proved *that* J_D *is an isomorphism.*

If D is a multiple of D', the cokernel of the inclusion map i: $\mathcal{L}(D')\to\mathcal{L}(D)$ is carried by a variety of dimension $m-1$. Therefore i^*: $H^m(V, \mathcal{L}(D'))\to H^m(V, \mathcal{L}(D))$ is an epimorphism, hence the dual of i^* is a monomorphism as a mapping of $W(-D)$ into $W(-D')$. We can consider the inductive limit W of $W(-D)$ by these maps. It is easy to introduce in W a structure of a vector space over the function-field K of V such that the inductive limit J of J_D is a K-linear mapping of the space of all global m-forms on V into W. Here,—*and this is the key point of the proof—if* $J(\omega)$ *is contained in* $W(-D)$, *then* ω *is necessarily contained in* $H^0(V, \Omega^m(-D))$. The proof is indirect and is as follows:

If we assume that ω is not contained in $H^0(V, \Omega^m(-D))$, the reduced expression of $(\omega)-D$ is of the form C_1-C_2 where C_1 is nonnegative and C_2 is strictly positive. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be cokernels of the inclusion maps $\mathcal{L}((\omega)-C_1)\to\mathcal{L}(D)$ and $\mathcal{L}((\omega))\to\mathcal{L}((\omega)+C_2)$, respectively. Then we get the following commutative diagram of exact sequences:

$$\begin{tikzcd}
{} \arrow[r] & H^{m-1}(V, \mathfrak{M}_1) \arrow[r] \arrow[d] & H^m(V, \mathcal{L}((\omega) - C_1)) \arrow[r] & H^m(V, \mathcal{L}(D)) \arrow[r] & 0 \\
{} \arrow[r] & H^{m-1}(V, \mathfrak{M}_2) \arrow[r] \arrow[d] & H^m(V, \mathcal{L}((\omega))) \arrow[r] & 0 \\
& 0
\end{tikzcd}$$

We note that $J_{(\omega)}(\omega)\colon H^m(V, \mathcal{L}(\omega)) \to H^m(V, \Omega^m)$ is an isomorphism. Moreover, since $J(\omega)$ is contained in $W(-D)$, by definition, $J_{(\omega)-C_1}(\omega)$, i.e., the product of $H^m(V, \mathcal{L}((\omega)-C_1)) \to H^m(V, \mathcal{L}((\omega)))$ and $J_{(\omega)}(\omega)$ vanishes on the image of $H^{m-1}(V, \mathfrak{M}_1)$ in $H^m(V, \mathcal{L}((\omega)-C_1))$. However the commutativity and the exactness of the diagram show the opposite of this assertion, and this gives us the desired contradiction.

As a simple consequence, we see that J is an isomorphism. On the other hand, *W is of dimension one over K*. In fact, let C be a general hyperplane section of V. Then, by an induction on m, we can show that ord $(V)\cdot n^m/m!$ is the dominant term of dim $H^m(V, \mathcal{L}(D-nC))$ as a function of n, this function being equal to dim $W(-D+nC)$. Since ord $(V)\cdot n^m/m!$ is the highest term of dim $|nC|$ as a polynomial in n for n sufficiently large, we can readily get a contradiction if W is not of dimension one over $F(V)$. Since J is a monomorphism, it is now shown that J is an isomorphism. If we apply the key result which we have established earlier, we conclude that J_D is an isomorphism. Thus (4) is proved.

In [3] Serre proves, by using extension functors, the following fundamental result:

THEOREM 7. *Let $\mathcal{J}$ be a coherent sheaf over the projective r-space X and let p be an integer $\geqq 0$. In order that $H^q(X, \mathcal{J}(-n))$ be zero for large n and for $0 \leqq q < p$ it is necessary and sufficient that for every $x \in X$ the $\mathcal{O}_x$-module $\mathcal{J}_x$ have cohomological dimension $\leqq r-p$ (i.e., that there exist an exact sequence $0 \to L_{r-p} \to L_{r-p-1} \to \cdots \to L_0 \to \mathcal{J}_x \to 0$, where each L_i is a free $\mathcal{O}_x$-module; here $\mathcal{O}_x$ denotes the local ring of x).*

Now, suppose that $\mathcal{J}$ is zero outside of our m-dimensional variety V (i.e., that $\mathcal{J}_x = 0$ for all $x \notin V$). For $x \in V$ let $\mathfrak{o}_x$ be the local ring of x on *the variety V. Assume furthermore that $\mathcal{J}_x$ is a free module over $\mathfrak{o}_x$.* If V is nonsingular then it is known that $\mathfrak{o}_x$ has cohomological dimension $r-m$ over $\mathcal{O}_x$, $x \in V$. If V is normal, then it can be shown that for all $x \in V$ the $\mathcal{O}_x$-module $\mathfrak{o}_x$ has dimension $\leqq r-2$ (Serre [3]). Hence by Theorem 7 we find that for large n

$$H^q(V, \mathcal{J}(-n)) = 0, \qquad 0 \leqq q \leqq m-1, \text{ if } V \text{ is nonsingular.} \tag{11}$$

and

$$H^q(V, \mathcal{J}(-n)) = 0, \qquad 0 \leq q \leq 1, \text{ if } V \text{ is normal.} \tag{12}$$

Relation (12), for $q=1$, includes the lemma of Enriques-Severi-Zariski as a special case. If V is nonsingular, every sheaf $\mathcal{L}(D)$ defined by a divisor D is free over $\mathfrak{o}_x$ for every x in V. Hence, we have by (11):

$$h^q(D - C_n) = 0, \qquad 0 \leq q \leq m-1, \; n\text{—large.} \tag{13}$$

For $q=1$ we find relation (9). Essentially the same proof which enabled us to derive from (9) the duality relation (4) enables one to derive from (13) the duality relations

$$h^q(D) = h^{m-q}(K - D), \qquad 0 \leq q \leq m. \tag{14}$$

The following is a noteworthy application of (14). We have, by (14): $\chi(V,D) = (-1)^m\chi(V,K-D)$. For $D=0$, we find $\chi(V) = (-1)^m\chi(V,K)$, and this yields, by (2), §7:,

$$\chi_V(-K) = \begin{cases} 2\chi(V) & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases} \tag{15}$$

Relation (15) is equivalent with the equality $P_a = p_a$ (Zariski [4]), and thus we have a proof of Severi's conjecture $P_a = p_a$.

In conclusion we mention the following unsolved problem: do the symmetry relations $h^{p,q} = h^{q,p}$ hold? Here $h^{p,q} = \dim H^p(V, \Omega^q)$. It is known that these equalities hold in the complex domain, but the proof depends on properties of harmonic integrals and cannot be therefore algebraicized.

As was pointed in §7, the symmetry relations in the classical case, and in particular the relations $h^{p,0} = h^{0,p}$, yield the following expression of the arithmetic genus: $p_a = h^{m,0} - h^{m-1,0} + \cdots + (-1)^{m-1}h^{1,0}$, where $h^{p,0}$ is the number of linearly independent p-fold differentials of the first kind on V. In the case of surfaces this leads to the equality $p_g - p_a = h^{1,0}$, where $p_g = h^{2,0}$ is the geometric genus of the surface. In the abstract case it is highly probable that the above expression for the genus may still be valid even if the symmetry relations turned out to be false.

BIBLIOGRAPHY

S. BOCHNER

1. *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. vol. 52 (1946).

H. CARTAN

1. *Sur les matrices holomorphes de n variables complexes*, J. Math. Pures Appl. vol. 19 (1940).
2. *Idéaux de fonctions analytiques de n variables complexes*, Ann. École Norm. vol. 61 (1944).
3. Seminar, 1950–1951, 1951–1952, 1953–1954.

H. CARTAN-J-P. SERRE

1. *Un théorème de finitude concernant les variétés analytiques compactes*, C. R. Acad. Sci. Paris vol. 237 (1953).

P. DOLBEAULT

1. *Sur la cohomologie des variétés analytiques complexes*, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 175–177.

F. HIRZEBRUCH

1. *Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954).

K. KODAIRA

1. *On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).
2. *On Kähler varieties of restricted type*, Ann. of Math. vol. 60 (1954).

K. KODAIRA-D. C. SPENCER

1. *On arithmetic genera of algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).
2. *Divisor class groups on algebraic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).
3. *On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953).

J. LERAY

1. *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, J. Math. vol. 29 (1950).

H. T. MUHLY-O. ZARISKI

1. *Hilbert's characteristic function and the arithmetic genus of an algebraic variety*, Trans. Amer. Math. Soc. vol. 69 (1950).

K. OKA

1. *Sur les fonctions analytiques de plusieurs variables* VII, Bull. Soc. Math. France vol. 78 (1950).

J-P. SERRE

1. *Faisceaux analytiques cohérents sur une variété algébrique*, written in the fall, 1953. (Cf. Sém. Bourbaki, mars 1954).
2. *Faisceaux algébriques*, written in the spring, 1954.
3. *Faisceaux algébriques cohérents*, Ann. of Math. vol. 61 (1955).
4. *Cohomologie et géométrie algébrique*, written in the fall, 1954. (Cf. Proc. Amsterdam Congress).

A. WEIL

1. *Zur algebraischen Theorie der algebraischen Funktionen*, J. Reine Angew. Math. vol. 175 (1938).
2. *Sur les théorèmes de de Rham*, Comment. Math. Helv. vol. 26 (1952).

O. ZARISKI

1. *Some results in the arithmetic theory of algebraic varieties*, Amer. J. Math. vol. 61 (1939).
2. *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc. vol. 50 (1944).
3. *Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields*, Memoirs of the American Mathematical Society, no. 5, 1951.
4. *Complete linear systems on normal varieties and a generalized lemma of Enriques-Severi*, Ann. of Math. vol. 55 (1952).

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THE ANNUAL MEETING IN HOUSTON

The sixty-second Annual Meeting of the American Mathematical Society was held at The Rice Institute, Houston, Texas, on Tuesday, Wednesday, and Thursday, December 27–29, 1955, in conjunction with the Annual Meeting of the Mathematical Association of America on Friday, December 30. The attendance was nearly 400, including 265 members of the Society.

The twenty-ninth Josiah Willard Gibbs Lecture was delivered by Professor J. E. Mayer of the University of Chicago on Tuesday at 8:00 P.M. Professor Mayer's lecture was entitled *The structure of simple fluids*. Presiding officer at the session, held in the Auditorium of the Prudential Insurance Company, was President William V. Houston of The Rice Institute.

The Presidential Address was delivered by Professor G. T. Whyburn of the University of Virginia on Wednesday at 10:30 A.M. with Professor R. L. Wilder presiding. Professor Whyburn's topic was *Topological analysis*. Professor R. L. Moore was in the audience and Professor Wilder took the opportunity to pay him appropriate tribute.

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, Professor S. S. Chern of the University of Chicago addressed the Society on *Complex vector bundles*. Professor Deane Montgomery presided at the lecture.

There were a total of fourteen sessions at which contributed papers were presented. Presiding officers at these various sessions were Professors H. M. MacNeille, J. J. Gergen, W. T. Scott, F. B. Jones, E. F. Beckenbach, L. C. Young, R. H. Bing, H. F. Bohnenblust, H. E. Bray, C. B. Allendoerfer, R. M. Thrall, G. R. MacLane, Dr. John Todd and Mr. Michael Goldberg.

There was a banquet in The Rice Institute Commons on Wednesday at 6:30 P.M., a banquet unique in the recent history of the Society both for the caliber of the food and the quality of the speeches. Professor Wilder, the President of the Society, acted as Toastmaster and introduced President William V. Houston of The Rice Institute. President Houston was followed by President W. L. Duren, Jr. of the Mathematical Association of America. A resolution of thanks to The Rice Institute was read by Dr. L. W. Cohen of the National Science Foundation.

On Thursday afternoon there was a trip to a refinery of the Humble Oil and Refining Company and a number of the visiting mathematicians got a glimpse of the manufacturing aspect of the oil business. In the evening there was a concert by the Lyric Art String Quartet.

The ladies of The Rice Institute Department of Mathematics presided daily at teas served for the mathematicians in the lounge of the Faculty Club.

The Council met on Wednesday afternoon, December 28, 1955.

The Secretary announced the election of the following forty persons to ordinary membership in the Society:

Mrs. Fortunata Vitanza Altmayer, Ford Instrument Co., Long Island City, N. Y.;
Mr. William Moses Ashley, Douglas Aircraft Company, Santa Monica, California;
Dr. Roger Harris Bender, Datamatic Corporation, Newton Highlands, Massachusetts;
Professor Joseph Buffington Roberts, Reed College;
Professor Donald Lyman Burkholder, University of Illinois;
Mr. William Virgil Caldwell, University of Michigan;
Mr. John Cocke, Duke University;
Dr. Thomas H. Crowley, Bell Telephone Laboratories, Murray Hill, New Jersey;
Dr. John Warren Dettman, Bell Telephone Laboratories, Whippany, New Jersey;
Mr. Lionel David Dureau, Tulane University;
Dr. Walter James Feeney, Weston College;
Dr. Bobby J. Hollingsworth, United Gas Corporation, Shreveport, Louisiana;
Dr. James Lucien Howland, Computing Devices of Canada, Ltd., Ottawa, Ontario, Canada;
Mr. Robert Shepard Johnson, University of Pennsylvania;
Dr. John Killeen, Bell Telephone Laboratories, Murray Hill, New Jersey;
Mr. Naoki Kimura, Tokyo Institute of Technology;
Dr. Vernon Albert Kramer, University of California, Riverside;
Mr. James Lanier Knott, Miles College;
Professor Lester E. Laird, Kansas State Teachers College;
Mr. Muniswamyroddy Lakshmanan, Maryland College, Liberia, West Africa;
Mr. Henry Curt Lefkovits, Shell Development Co., Houston, Texas, and Rice Institute;
Mr. Harry W. Lew, Hughes Aircraft Company, Culver City, California;
Professor Carlos B. de Lyra, University de Sao Paulo and Instituto de Matematica Pura e Aplicada, Brazil;
Mr. Stanley Walter Malinowski, Moore School, University of Pennsylvania;
Dr. Michael Anthony Martino, National Security Agency, Washington, D. C.;
Professor Robert T. McLean, College of Steubenville;
Dr. Trevor James McMinn, University of California, Berkeley;
Mr. Fritz Walter Mezger, General Electric Company, Cincinnati, Ohio;
Professor Taro Morishima, Tokyo College of Science, Tokyo, Japan;
Sister Irene Morvan, Annhurst College;
Professor William J. Pervin, University of Pittsburgh;
Mr. Robert Charles Pfeilsticker, Frankford Arsenal, Philadelphia, Pennsylvania;
Dr. Bayard Rankin, Massachusetts Institute of Technology;
Mr. Neal Jules Rothman, Burroughs Corporation, Paoli, Pennsylvania;
Professor Robert Winter Royston, Washington and Lee University;
Mr. Joseph Julius Schoderbek, Lockheed Aircraft Corporation, Van Nuys, California;
Professor Ivar Stakgold, Harvard University;

Mr. V. Thillainayagam, Annamalai University, Annamalainagar, South India;
Mr. Frederick Leonard Zarnfaller, Olin Mathieson Chemical Corporation, New Haven, Connecticut;
Mr. Fred Selwyn Zusman, John Hopkins University.

It was reported that the following forty-five persons had been elected to membership on nomination of institutional members as indicated:

University of British Columbia: Mr. George Elliot Cross.

The University of Cincinnati: Mr. Roger Chalkley.

Columbia University: Mr. Joseph Francis Manogue, Mr. Lee Paul Neuwirth, Miss Helen Lynn Paisner, Mr. Jyri Erkki Paloheimo, Mr. John Wolfgang Smith, and Mr. Joe Fred Traub.

Cornell University: Mr. David Norman Freeman, Mr. Benoit Vincent Lachapelle, Mr. Shing-Ming Lee, Mr. George Fitzgerald Lowerre, Mr. Richard Felix McCoart, Mr. Edward Norman, and Miss Marion Ilse Walter.

University of Georgia: Mr. James J. Andrews.

Iowa State College: Mr. William Duane Montgomery.

University of Kansas: Mr. Robert Brainard Buckley.

Massachusetts Institute of Technology: Mr. William Bernard Houston, Jr., Mr. Millard Wallace Johnson, Mr. Robert Richard Dingle Kemp, Mr. Donald L. Kreider, Mr. Albert Louis Rabenstein, Mr. Anthony Ralston, and Dr. Leonard Roberts.

University of Michigan: Mr. Michael Barr, Mr. James Oliver Brooks, Mr. Ross Lee Finney, III, Mr. Gordon McCrea Fisher, Mr. Carlos Imaz, Mr. Richard Patterson Jerrard, Mr. Howard Earl Reinhardt, Mr. Roger Wolcott Richardson, Jr., Mr. Neal McCalla Speake, Mr. Joseph Gail Stampfli, and Mr. William Blauvelt Woolf.

University of New Hampshire: Mr. Bruce Cale McQuarrie.

University of Wisconsin: Mr. Richard Francis De Mar, Mr. Robert Irving Jennrich, Mr. William John Kammerer, Mr. William Charles Lordan, Mr. Thomas La Rue McCoy, Mr. Clark Thompson Miller, Mr. Hiram Paley, and Mr. Raymond Warren Rishel.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker Vereinigung: Dr. Hans Konrad Schuff, Institut für Spektrochemie und angewandte Spektroskopie, Bau, Germany; Indian Mathematical Society: Dr. Alladi Ramakrishnan, University of Madras, Madras, India; Islenzka Staerðfraeðafelagið: Professor Leifur Asgeirsson, University of Iceland; London Mathematical Society: Professor Reuben Louis Goodstein, University College of Leicester; Schweizerische Mathematische Gesellschaft: Dr. Henrich Guggenheimer, The Hebrew University, Jerusalem, Israel; Svenska Matematikersamfundet: Mr. Olof Edgar Asplund, Royal Institute of Technology, Stockholm, Sweden; Wiskundig Genootschap te Amsterdam: Professor Jacob Ridder, State University of Utrecht, Utrecht, Netherlands.

The following appointments by the President were reported: to the joint committee on the Employment Register: Mr. George Patterson and Professor Ray Berkowitz. (Committee now consists of J. S. Frame, Chairman, H. M. Bacon, Ray Berkowitz, W. M. Hirsch, Morris Ostrofsky, George Patterson, and J. A. Ward); as a program committee for the Symposium in Applied Mathematics to be held in April 1956 in conjunction with the Society's Chicago meeting: William Prager, Chairman, L. M. Graves, Norman Levinson, Marston Morse, J. J. Stoker, and L. H. Thomas; as tellers for the 1955 election: Professors Arlen Brown, David Gale, and John Wermer; as a Committee to Recommend the Award of the Cole Prize in Number Theory: Professors Richard Brauer, Chairman, Leonard Carlitz, and Morgan Ward; as a Committee to Recommend to the Council Desirable Extensions of the Society's Activities: A. A. Albert, Wilfred Kaplan, J. L. Kelley, William Prager, and A. W. Tucker; as members of Committees to Select Hour Speakers (terms to expire December 31, 1957): Summer and Annual Meetings: G. P. Hochschild. (Committee now consists of E. G. Begle, Chairman, D. H. Lehmer, and G. P. Hochschild); Eastern Sectional Meetings: E. R. Kolchin. (Committee now consists of R. D. Schafer, Chairman, Lipman Bers, and E. R. Kolchin); Western Sectional Meetings: E. H. Spanier. (Committee now consists of J. W. T. Youngs, Chairman, P. V. Reichelderfer, and E. H. Spanier); Far Western Sectional Meetings: Arthur Erdélyi. (Committee now consists of V. L. Klee, Chairman, Ivan Niven, and Arthur Erdélyi); Southeastern Sectional Meetings: E. E. Floyd. (Committee now consists of J. H. Roberts, Chairman, G. B. Huff, and E. E. Floyd); as Chairman of the Committee on Places of Meetings for 1956: W. M. Whyburn. As a new member of this Committee: R. M. Thrall. (Committee now consists of W. M. Whyburn, Chairman, C. B. Morrey, and R. M. Thrall).

The following appointments to represent the Society were reported: at the 100th anniversary of the founding of the Polytechnic Institute of Brooklyn, October 8, 1955: Professor Samuel Borofsky; at the Centennial Homecoming Convocation of Albright College on October 22, 1955: Professor D. W. Western; at the inauguration of Paul S. Bachman as President of the University of Hawaii on November 9, 1955: Professor Christopher Gregory; at the inauguration of Howard Rothmann Bowen as the 7th President of the College at Grinnell College, Grinnell, Iowa, on November 13, 1955: Professor L. A. Knowler; and at the Academic Convocation held by Tufts University on the theme "The Role of the Small University in American Higher Education," on December 8, 1955: Professor C. T. Bumer.

The Secretary reported that Professor S. S. Chern had accepted an invitation to deliver an hour address at the Annual Meeting of 1955 in Houston, Texas; that Dr. H. C. Wang had accepted an invitation to deliver an hour address at the April 1956 meeting in Monterey, California; and that Professor J. T. Schwartz had accepted an invitation to deliver an hour address at the April 1956 meeting at Columbia University.

The Council voted to grant permission to the Editors of *Mathematical Tables and Other Aids to Computation* to reproduce reviews from *Mathematical Reviews.*

The Council voted to request the Trustees to increase the total amount of subsidies to other journals by approximately 10% to allow for the increase in costs of publication.

The Bulletin Editorial Committee reported that 608 pages had been published in 1955, not including a supplement of 66 pages containing a ten-volume index. The Council voted to recommend to the Board of Trustees that the Bulletin be authorized to print 625 pages in 1956.

The Transactions and Memoirs Editorial Committee reported that the interval between receipt of a paper and publication has been reduced to about eight months. This reduction is largely due to a third volume of the Transactions published in 1955. The Council voted to recommend to the Board of Trustees that three volumes of 550 pages each be published in the Transactions during 1956.

The Proceedings Editorial Committee reported that the interval between the receipt of a manuscript and publication was approximately ten or eleven months. The Council voted to recommend to the Board of Trustees that 1006 pages be authorized for the 1956 Proceedings. Professors Raoul Bott, Paul Civin, Edwin Hewitt, and R. M. Thrall were reported as new Associate Editors of the Proceedings.

The Mathematical Reviews Editorial Committee reported that 1190 pages, not including the index number, would be published in volume 16.

The Council voted to accept an invitation from the Pennsylvania State University to hold the 1957 Summer Meeting at University Park, Pennsylvania. The Council voted to hold the Annual Meeting for 1957 in January 1958, as an experiment.

The Council approved the following times and places of sectional meetings: The University of Chicago, April 12–14, 1956; Massachusetts Institute of Technology, October 27, 1956; California Institute of Technology, November 17, 1956; University of Kentucky, November 1956; University of Missouri, November 1957; and Wayne University, November 1959.

The Council voted to approve in principle a plan whereby any member of the Society may receive either the Transactions or Mathematical Reviews instead of the Proceedings on payment of a premium which will be a function of the member's subscription rates to these journals.

On the grounds that such liaison was no longer needed, the Council voted to eliminate the position of liaison officer to the Quarterly of Applied Mathematics.

The Council voted to co-sponsor a joint meeting of the Western Section of the Operations Research Society of America and the Southern California Chapter of the Institute of Management Sciences to be held at the University of California at Los Angeles, May 30–31, 1956.

The Council voted to request the Trustees to provide a subvention of $1000 to the Institute of Mathematics of the National University of Mexico for support of a Symposium in Algebraic Topology to be held in the summer of 1956.

The Annual Business Meeting of the Society was held on Thursday, December 29, 1955. The Secretary reported that at this time the ordinary membership of the Society is now 4892, including 468 nominees of institutional members and 33 life members. The membership of the Society continues to show a substantial annual increase. There are also 133 institutional members. The total attendance at all meetings in 1955 was 2213; the number of papers read was 760; there were 13 hour addresses; one Gibbs Lecture; one Retiring Presidential address; and 9 papers at the Applied Mathematics Symposium. The number of members attending at least one meeting was 1525.

At the annual election, in which over 1300 votes were cast, the following officers were elected:

President Elect, Professor Richard Brauer.

Vice Presidents, Professor A. A. Albert and Nathan Jacobson.

Associate Secretaries, Professors V. L. Klee, Jr., J. H. Roberts, and J. W. T. Youngs.

Members of the Editorial Committee of the Bulletin, Professor G. B. Price for a term of two years and Professor J. C. Oxtoby for a term of three years.

Members of the Editorial Committee of the Proceedings, Professors R. P. Boas, Jr. and S. S. Chern.

Members of the Editorial Committee of the Transactions and Memoirs, Professors A. H. Clifford and Mark Kac.

Member of the Editorial Committee of the Colloquium Publications, Professor Deane Montgomery.

Member of the Editorial Committee of the Mathematical Reviews, Professor W. S. Massey.

Member of the Editorial Committee of the Mathematical Surveys, Professor I. J. Schoenberg.

Member of the Committee on Printing and Publishing, Professor A. W. Tucker.

Members of the Board of Trustees, Professor H. F. Bohnenblust for a term of two years and Professor W. T. Martin for a term of five years.

Representative on the Board of Editors of the American Journal of Mathematics, Professor Andre Weil.

Members-at-large of the Council, Professors Einar Hille, J. L. Kelley, Hans Samelson, A. H. Taub, and A. D. Wallace.

Professor R. H. Cameron urged a return to the former practice of printing abstracts in the next issue of the Bulletin after receipt rather than with the report of the meeting at which the paper was presented. There was a lively discussion of this and of other plans for making the abstracts more useful to members of the Society. The Secretary agreed to have a careful study made of these plans.

Abstracts of the papers presented follow. Those having the letter "t" after their numbers were read by title. Where a paper has more than one author, the paper was presented by that author whose name is followed by "(p)." Mr. L. W. Anderson was introduced by Professor A. D. Wallace, Mr. Baker by Professor H. A. Arnold, Dr. Krickeberg by Professor J. L. Doob, and Dr. Thorne by Dr. J. H. Curtiss.

ALGEBRA AND THEORY OF NUMBERS

163. W. W. Boone: *The equivalence of the word problem and Magnus' extended word problem.*

An arbitrary finitely presented group $\mathfrak{G}$ can be extended to a finitely presented group $\mathfrak{G}'$ such that a solution of the word problem for $\mathfrak{G}'$ implies the solution of the extended word problem (Magnus, Math. Ann. vol. 106, p. 297) for $\mathfrak{G}$. Let $\mathfrak{G}$ have generators $g_1, g_2, \cdots, g_n$ and relations $R_1, R_2, \cdots, R_s$; let m be an integer such that $0 \leq m \leq n$. We define $\mathfrak{G}'$ to be the group with generators $g_1, g_2, \cdots, g_n, k$ and relations $R_1, R_2, \cdots, R_s$, $kg_i = g_ik$ for $1 \leq i \leq m$. Let Σ (Let Γ) be a variable for words made up of $g_1, g_2, \cdots, g_n$ (of $g_1, g_2, \cdots, g_m$). THEOREM: *There is a Γ such that $\Sigma = \Gamma$ in $\mathfrak{G}$ if and only if $k\Sigma = \Sigma k$ in $\mathfrak{G}'$.* This theorem should be compared with the unsolvability result, Part I, p. 232. (References are to the parts of *Certain simple unsolvable problems of group theory*, Proc. K. Akad. Wetensch. A, Parts I and II, vol. 57; Part III, vol. 58; Part IV, vol. 58.) A proof, in $\mathfrak{G}'$, that $k\Sigma = \Sigma k$ can be altered (Part I, Section 3 and Part II, p. 493, line 14—but k here is treated like q there) as follows: (1) Eliminate all uses of k^{-1}. (The methods of Part II, Lemma 7 and Theorem III, Case 1, suffice but are unnecessarily complicated.) (2) Eliminate all replacements of $g_i^{+1}k$ by kg_i^{+1} and $g_i^{-1}k$ and kg_i^{-1}. (Amend the marker convention so as to regard $kg_i^{-1} = g_i^{-1}k$ as an independent relation. Then for $\cdots$, $Bg_i^{\pm 1}kC$, $Bkg_i^{\pm 1}C, \cdots$ use $\cdots$, $Bg_i^{\pm 1}kC$, $Bg_i^{\pm 1}kg_i^{\mp 1}g_i^{\pm 1}C$, $Bg_i^{\pm 1}g_i^{\mp 1}kg_i^{\pm 1}C$, $Bkg_i^{\pm 1}C, \cdots$.) (3) Interchange operations until U, the subsequence of operations performed on k or right of k precedes the operations left of k. The word resulting from the last operation of U is of form Γk. (Received December 30, 1955.)

164t. J. L. Brenner: *Some free groups of matrices.*

In Doklady Akad. Nauk SSSR (N.S.) vol. 57 (1947) pp. 657–659 [Math. Reviews vol. 9 (1948) p. 224], Sanov proved that the matrices $I+2e_{12}$, $I+2e_{21}$ generate a free group, which consists of all $(a_{ij})_{0<i,j<3}$ such that det $(a_{ij})=1$, $a_{11}\equiv a_{22}\equiv 1$ (mod 4); $a_{12}\equiv a_{21}\equiv 0$ (mod 2). The generalization obtained is as follows. If $m\geqq 2$, the matrices $I+me_{12}$, $I+me_{21}$ generate a free group, which consists of all (a_{ij}) such that det $(a_{ij})=1$, $a_{11}\equiv a_{22}\equiv 1$ (mod m^2), $a_{12}\equiv a_{21}\equiv 0$ (mod m), and the ratio $|a_{11}/a_{12}|$ is not strictly between $m/2+(m^2-4)^{1/2}/2$ and $m/2-(m^2-4)^{1/2}/2$; m need not be an integer. (Received November 4, 1955.)

165. J. L. Brenner: *The linear homogeneous group.* III. Preliminary report.

Let R be the ring of rational integers. The normal and characteristic subgroups of the general linear group $GL_n[R]$ are determined ($n>2$). The method is extended to the case in which the underlying ring is euclidean. It is proved that every subgroup of $GL_n[R]$ of finite index contains a congruence subgroup, verifying a conjecture communicated by M. Newman. Some miscellaneous results and examples are given for the case $n=2$. This article will be published in the Annals of Mathematics. Research sponsored by Office of Ordnance Research, U. S. Army. (Received November 4, 1955.)

166. A. H. Copeland (p) and Frank Harary: *Periodic sets of ordinals.*

A subset A of the positive integers is said to have period n provided $T_{\beta n}(\alpha)=\alpha+\beta n$ is in A if and only if α is in A for every positive integer β. We extend this definition to sets A of transfinite ordinals by replacing the usual sum by the natural sum and allowing α and β to range over all values preceding some ordinal of the form ω^μ. In the case of the integers it is sufficient to restrict β to be 1. Associated with every order type ω^μ and every integer $n\geqq 2$ there is a minimal set F (called a free set) such that any set with period n is a union of translations $T_{\beta n}$ of some subset of F. The representation space of an arbitrary boolean algebra can be regarded as a free set F of a properly chosen order type. The representation of the boolean algebra as subsets of F induces a representation as subset of ω^μ with period n. By adding sets with various periods, one can extend this boolean algebra in such a manner that it becomes implicative. (Received November 14, 1955.)

167t. R. H. Crowell: *Forests and determinants.*

Various forms of an important combinatorial identity relating determinants to maximal trees of a connected graph exist in the literature. The purpose of this paper is: (i) To present a generalization, called the matrix-forest theorem, of previous results, in which forests replace trees and the corresponding extension is to arbitrary principal minors. (II) To give a precise proof exhibiting the result as a formal identity of ring addition and multiplication. Let V be the set of n vertices, and E the set of edges, of an arbitrary, oriented (each edge has an initial point and a terminal point) graph $\mathfrak{G}$. Where $\mathfrak{R}$ is an associative, commutative ring with a nonzero multiplicative identity, form $\mathfrak{R}(E)$ equal to the ring of polynomial forms over $\mathfrak{R}$ in the edges E. Define $\phi: V\times V\rightarrow\mathfrak{R}(E)$ by: for any $u, v\in V$, if $u\neq v$, then $\phi(u, v)=$ the sum of the edges in E whose initial point is u and terminal point is v, $\phi(u, u)=-\sum_{w\in V-u}\phi(w, u)$. The set $\phi(V\times V)$ is "the matrix" of the theorem, and for any subset $R\subset V$ of cardinality r, the corresponding principal minor Δ of order $n-r$ is easily defined: $\Delta=\sum_{\pi\in\mathfrak{S}^{n-r}}\epsilon_\pi\prod_{u\in V-R}\phi(u, \pi u)$. $\mathfrak{S}^{n-r}$ is the group of permutations π of $V-R$ and ϵ_π is the parity of π. A maximal *rooted forest* F of $\mathfrak{G}$ with *roots* $=R$ is any acyclic subgraph of $\mathfrak{G}$ such that $V\subset F$, no vertex is the terminal point of more than one edge of F, and the set R constitutes precisely those vertices which are the terminal points of no edge of F. The set of all such rooted forests is denoted $\mathfrak{F}_R$ and the form $\rho\in\mathfrak{R}(E)$ is defined: $\rho=\sum_{F\in\mathfrak{F}_R}\prod\{e: e\in F\cap E\}$. *The matrix-forest theorem* is the identity $\rho=(-1)^{n-r}\Delta$. The proof is based on the proof of the result for $r=1$ given in (R. Bott and J. P. Mayberry, *Matrices and trees*, "Economic Activity Analysis," O. Morgenstern, Ed., New York, Wiley, 1954). (Received October 24, 1955.)

168t. W. E. Deskins: *On bound algebras.*

A linear associative algebra A with radical N is ***right bound*** if all elements of A which annihilate N on the right are contained in N. A is ***strictly right bound*** if all of its homomorphic images are right bound. *Theorem.* An algebra A is expressible as a (vector space) direct sum of left ideals, $A_0+A_1+\cdots+A_n$, where A_0 is semisimple and A_i, $i=1, \cdots, n$, is strictly right bound. *Theorem.* If A is strictly right bound, if the radical N of A is of index r, and if $N-N^{r-1}$ is of order n_r^2, then the order of $\bar{A}=A-N$ is at most n_r^2. These theorems and their duals, applied to the bound algebras of M. Hall (Trans. Amer. Math. Soc. vol. 48 (1940) pp. 391–404), yield new results concerning the relative sizes of an algebra and its radical. (Received November 14, 1955.)

169t. Walter Feit: *On the modular characters of finite groups.*

Let G be a finite group of order g, and let K be an algebraic number field with the property that every absolutely irreducible representation of G can be written with coefficients in K. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be prime ideals in K which divide distinct rational primes $p_1, \cdots, p_s$, let $\mathfrak{a}=\mathfrak{p}_1, \cdots, \mathfrak{p}_s$, $a=p_1, \cdots p_s$. Two matrices A and B whose coefficients are local integers with respect to each $\mathfrak{p}_i$ are said to be $\mathfrak{a}$-similar if there exists a unimodular matrix P with such coefficients, such that $A=PBP^{-1}$, an element of G is said to be a-regular if its order is relatively prime to a. The regular representation of G is $\mathfrak{a}$-similar to a direct sum $\mu_1\oplus\cdots\oplus\mu_n$ mod $\mathfrak{a}$ such that no μ_i is $\mathfrak{a}$-similar to a direct sum mod $\mathfrak{a}$. We define the character Φ_i of each μ_i as it is done for representations mod $\mathfrak{p}$ (see Brauer and Nesbitt, Ann. of Math. vol. 42 (1941) pp. 556–590) so that $\Phi_i(x)$ is an algebraic number in K for every a-regular element x of G. In analogy with the results in the above mentioned paper we show, (1) Φ_i is an ordinary character which vanishes for all elements not a-regular, (2) The number of non $\mathfrak{a}$-similar representations μ_i mod $\mathfrak{a}$ equals the number of classes of a-regular elements of G, and the corresponding Φ_i are linearly independent. Cartan invariants c_{ij} are defined by $gc_{ij}=\sum\Phi_i(x)\overline{\Phi_j(x)}$ where the sum ranges over all x in G. These can then be used to generalize a result of Brauer and Nesbitt as follows. If $g=g'p^bq^a$, where p and q are primes which do not divide g', and if there are exactly p^bq^a elements in G whose order divides p^bq^c, then these elements form a normal subgroup of G. (Received November 7, 1955.)

170*t*. Casper Goffman: *Concerning lattice homomorphisms.*

It is shown that if L is a distributive lattice with a minimal element then, for every ideal $J\subset L$, there is one and only one homomorphism of L onto a disjunctive lattice for which J is the kernel. This is the maximal homomorphism whose kernel is J. For the case where $J=[0]$, the residue classes are the "fillets" in L. For the case where L is relatively complemented, the known uniqueness theorem is obtained as a corollary. (Received August 19, 1955.)

171. Arno Jaeger: *On differentiations in commutative rings of squarefree nonzero characteristic.*

Let R be a commutative ring with identity 1 of characteristic $pq>0$ (p and q two distinct prime numbers) in which each nonzero divisor has an inverse. A countable family of mappings $D=(D^i)_{i=0,1,2}\cdots$ of R into itself is called a ***differentiation*** if $D^0x=x$, $D^i(x+y)=D^ix+D^iy$, $D^i(xy)=\sum_{\alpha+\beta=i} D^\alpha xD^\beta y D^iD^jx=C_{i+j,i}D^{i+j}x$ holds for all $x,y\in R$, $i=0, 1, 2, \cdots$, and, in particular, a ***differentiation with respect to u*** $(\in R)$ if $D^1u=1$ holds. An element v of R is called *D-regular* if Dv is a nonzero diviser and if there exists such a v in R then D is called *regular*. If D is regular and x is D-regular there exists a differentiation Δ with respect to x such that D can be derived from Δ by means of the chain rule. An element w of R is D-integrable in R if and only if $pD^{q-1}w=qD^{p-1}w=0$ holds; and if the integral exists a particular integral can be constructed by the use of operators involving differentiations with respect to elements. These theorems are the foundation of a general theory of differentiations in commutative rings of any squarefree nonzero characteristic. (Received November 14, 1955.)

172. B. W. Jones (p) and G. L. Watson: *Indefinite ternary quadratic forms.*

Using concepts closely related to the spinor genera of Eichler, the authors define a multiplicative group Γ_d of square-free integers prime to d, the determinant of the indefinite ternary quadratic form f. Further, they show that Γ_d has a subgroup $\gamma(f)$ consisting of all those elements of Γ_d which are denominators of rational automorphs of f, where by the denominator of a matrix is meant the least common multiple of the denominators of its elements. They show that the number of classes in the genus of f is equal to the order of the factor group $\Gamma_d/\gamma(f)$. Also they show that if an integer n is represented by at least one but not by all of the classes of forms in the genus of f and if $dn=n_1n_2^2$, where n_1 is square-free, then 1. $n_1>1$; n_1 divides d; if d is odd, $n_1\equiv 1$ (mod 8). 2. If p is a prime not dividing $2d$ and $(n_1|p)=-1$, then p cannot divide n_2. 3. The number of classes in the genus which represent n is equal to the number of classes that do not. (Received November 10, 1955.)

173*t*. Tosio Kato and Olga Taussky: *Normal matrices.*

It is shown that a finite matrix A is necessarily normal if $A(AA^*-A^*A)=(AA^*-A^*A)A$. The elements of A are either complex numbers or taken from a formally real field, or from a field with an involution $a\rightarrow\bar{a}$ where $\sum a_i\bar{a}_i=0$ implies $a_i=0$. The same property also holds for bounded operators in the Hilbert space. It can further be shown that $(A, (A,^* (A, (A^*, \cdots, (A, A^*) \cdots)=0$ implies $(A, A^*)=0$ where $(X, Y)=(XY-YX)$. If the finite matrix is a 2×2 matrix then (A, A, A, A^*)

$=k(A, A^*)$ where k is a constant which vanishes if A has a double characteristic root. (Received October 20, 1955.)

174t. J. H. B. Kemperman: *On complexes in a group.*

Let $A = \{a\}$ and $B = \{b\}$ be finite subsets of an arbitrary group and let $C = AB$ be the set of all the elements $c = ab$. Let $k = [A] + [B] - [AB]$, where $[D]$ denotes the number of elements in the set D. ***Theorem.*** *Each element c_0 in C admits at least k different representations of the form $c_0 = ab$.* This verifies the conjecture of the author's paper: *Complexes in arbitrary groups*, Bull. Amer. Math. Soc. Abstract 61-6-655. (Received November 14, 1955.)

175. Naoki Kimura: *On finite monothetic semigroups.*

A semigroup M will be called monothetic, if M is generated by one element. Let M be a finite monothetic semigroup, then M is characterized up to isomorphism by two positive integers one of which is called a length and the other a period. A semigroup whose length is l and whose period is p will be called of type $C(l, p)$. All types $C(l, p)$ where l, p run over all positive integers form a complete lattice isomorphic with direct product of countable number of chains of positive integers by defining a suitable order. All homomorphic images and all subsemigroups of a semigroup of type $C(l, p)$ are determined. The lattice of all subsemigroups of a semigroup of type $C(l, p)$ ordering by set-inclusion is distributive if $l \leqq 5$, but is not even modular if $l \geqq 6$. Further a notion of tensor product is introduced and discussed on application to some decomposition theorem of semigroups. (Received November 16, 1955.)

176t. Walter Ledermann and B. H. Neumann: *On the order of the automorphism group of a finite group.* II.

A function $g = g(k, p)$ of the positive integer k and the prime number p is found with the property that every finite group whose order is divisible by p^g has an automorphism group whose order is divisible by p^k. The smallest such g satisfies $g(1, p) = 2$ and $g(k, p) \leqq (k-1)^2p^{k-1} + k$ for $k \geqq 2$. Results for small values of k, or for special classes of groups, had previously been obtained by J. E. Adney, Jr. [Bull. Amer. Math. Soc. Abstract 61-2-166 (1955)]; I. N. Herstein and J. E. Adney [Amer. Math. Monthly vol. 59 (1952) p. 309], E. Schenkman [Proc. Amer. Math. Soc. vol. 6 (1955) p. 6]; W. R. Scott [Proc. Amer. Math. Soc. vol. 5 (1954) p. 23]. The method is a refinement of that used in the first part [Proc. Roy. Soc. London (A), in print, cf. Bull. Amer. Math. Soc. Abstract 61-6-657] and employs the Schur multiplicator and its properties; cf. J. Schur [J. Reine Angew. Math. vol. 127 (1904) p. 20 and vol. 132 (1907) p. 85]. (Received November 14, 1955.)

177. Mark Lotkin: *The diagonalization of skew-hermitian matrices.*

Hermitian matrices may be diagonalized by an infinite sequence of unitary transformations. Each of these transformations may be chosen in such a manner as to annihilate a pair of selected off-diagonal elements. In the diagonalized matrix, the elements represent the characteristic values of the hermitian matrix. This procedure, originally due to Jacobi, has been extended to skew-hermitian matrices. The rate of convergence is the same as that for the hermitian case. (Received October 28, 1955.)

178. Ivan Niven: *A single rational approximation to a complex number.*

Let $R(i)$ denote the field obtained by adjoining i to the rational numbers. It is proved that given any complex number θ there exists at least one pair p, $q \neq 0$ of integers of $R(i)$ such that $|\theta - p/q| \leq (2-3^{1/2})^{1/2}|q|^{-2}$. This result becomes false if $(2-3^{1/2})^{1/2}$ is replaced by any smaller constant. A similar result for the approximation of any real number by a rational number was given by A. V. Prasad, J. London Math. Soc. vol. 23 (1948) pp. 169–171. (Received November 14, 1955.)

179. R. H. Oehmke: *Shrinkable algebras of level* 2.

Let $\mathfrak{A}$ be an algebra over the field $\mathfrak{F}$ of characteristic not 2, 3 or 5 and w an element of $\mathfrak{A}$. Use the symbols R_w and L_w to denote the operations of right multiplication and left multiplication by the element w, the symbol $T(w)$ to denote either of these multiplications, and the symbol $T(x, y, z)$ to denote a multiplication $T(w)$ defined for a product $w = xyz$ with an unspecified association. Then an algebra is said to be shrinkable of level 2 if every $T(x, y, z)$ is identically equal to a finite linear combination of products $T_1(w_i, w_j)T_2(w_k)$, $T_1(w_i)T_2(w_j, w_k)$, and $T_1(w_i)T(w_j)T_3(w_k)$ over the field $\mathfrak{F}$ where w_i, w_j and w_k represent the three factors x, y and z. Two of these algebras $\mathfrak{A}$ and $\mathfrak{B}$ are said to be quasi-equivalent if the algebras are isomorphic as vector spaces and if the product $a \cdot b$ of $\mathfrak{B}$ can be expressed in terms of the product ab of $\mathfrak{A}$ by $a \cdot b = \lambda ab + (1-\lambda)ba$ for an element of $\mathfrak{A}$ that is not equal to 1/2. The class of non-commutative shrinkable algebras of level 2 with a unity element is divided into subclasses of quasi-equivalent algebras in order to facilitate the investigation for new simple power-associative algebras. (Received November 14, 1955.)

180. B. W. Volkmann: *A theorem on the set of perfect numbers.*

Let $\sigma(n)$ denote the sum of all positive divisors of the integer n and let U_i be the set of odd numbers n with $\sigma(n) = 2^i n$ $(i = 1, 2, \cdots)$. Then it is shown that the number of its elements not exceeding x is $U_i(x) = O(x^{\alpha})$ where $\alpha = 1 - (2(i+2))^{-1}$. Since U_1 is the set of odd perfect numbers, the case $i = 1$ has, by virtue of known properties of the even perfect numbers, the immediate corollary that the set V of all perfect numbers satisfies the condition $V(x) = O(x^{5/6})$. The proof is based on a lemma due to H. J. Kanold (J. Reine Angew. Math. vol. 194 (1955) pp. 218–220) stating that any perfect or multiply perfect number has a factor k^2 such $k^2 \geq (p+1)/2$, p being the largest and prime factor of n. Full proofs are given in a paper forthcoming in the J. Reine Angew. Math. (Received November 16, 1955.)

181. D. W. Wall: *Characterizations of uniserial and generalized uniserial algebras in terms of Frobenius type algebras.*

Let A be an algebra over an algebraically closed field K. It is known (Nakayama, Ann. of Math. vol. 42 (1941) pp. 1–21) that the following are equivalent: (1) A is a uniserial algebra. (2) For every two-sided ideal Z, A/Z is a Frobenius algebra. (3) For every two-sided ideal Z, A/Z is a weakly symmetric algebra. The purpose of this paper is to give an extension of this result (Theorem 2) and a parallel result for generalized uniserial algebras (Theorem 1). Theorem 1. A is a generalized uniserial algebra if and only if for every two-sided ideal Z, A/Z is a QF-2. An algebra is QF-2 if every primitive left and right ideal has a unique minimal subideal (Thrall, Trans. Amer. Math. Soc. vol. 64, pp. 173–183). By assuming this property for every A/Z, it is possible to construct a unique composition series for each primitive ideal and thus prove that A is generalized uniserial. The converse follows easily from the known properties of the algebras. Theorem 2. A is uniserial if and only if for every two-sided ideal Z, A/Z is quasi-Frobenius. Since every quasi-Frobenius algebra is QF-2, Theorem 1 implies that if every A/Z is quasi-Frobenius then A is generalized uniserial. It remains to show that A is the direct sum of two-sided ideals which are themselves primary algebras. This is done by using the dualities that exist between primitive left and right ideals in every A/Z. The converse follows from the earlier result stated above. (Received November 9, 1955.)

182. G. P. Weeg (p) and Bernard Vinograde: *The defect of completely primary algebras*. Preliminary report.

Let $A = \text{Ext}\,(R, g, N)$ be an associative extension of the finite-dimensional R-R module N by the division algebra R with g a noncobounding 2-cocycle in $C^2(R, N)$ such that $N = Rg(R, R)R$ is the radical of A. If for some left basis $n_1, \cdots, n_k$ of N it is true that $g(x, y) = \sum p_i(x, y)n_i$ for all x, y in R, where $p_i(x, y)$ is in R, then we call the p_i's projections of g with respect to the n_i's. It is shown that: (1) N is in the center of A if and only if every possible projection of g is a cocycle; (2) When N is in the center of A then all the projections of g are cobounding if and only if g is cobounding; (3) When A is considered modulo N^2, then (a) A is generated by every residue system modulo N^2 (that is, A is maximally uncleft) if and only if all projections of g are noncobounding, (b) to every set of linearly independent noncobounding cocycles there exists a maximally uncleft extension A by R. Applications are made to pure inseparable R's. (Received November 9, 1955.)

ANALYSIS

183t. Shmuel Agmon: *The Dirichlet problem for linear elliptic partial differential equations with constant coefficients and of arbitrary order.*

An approach to the existence theory is outlined which can be considered as the generalization to higher order equations of the classical procedure of representing the solution as a multiple layer potential. The fundamental solution is not used directly for this purpose. Instead, inspired by a treatment of Pleijel of the biharmonic equation (Proceed. Symp. Spectral Theory and Differential Problems, Stillwater 1951, pp. 413–437), the Poisson kernels obtained from the explicit solution of the Dirichlet problem for the half-space are employed. One first tries to get the desired solution as an average on the boundary of the domain of these Poisson kernels. Introducing the Banach space of m-vectors: $x = (\phi_0(Q), \cdots, \phi_{m-1}(Q))$, where $2m$ is the order of the equation and $\phi_k(Q)$ are functions belonging to $C_{m-1-k+\alpha}(0<\alpha<1)$ on the sufficiently smooth boundary, one first reduces the problem to solving: (*) $X+T(x) = y$ where T is a completely continuous operator and y is the vector of the prescribed boundary values. Thus the problem is certainly solvable in this way when $x+T(x) = 0$ admits no nontrivial solutions. Furthermore, even if nontrivial solutions of the homogeneous equation exist, one can always obtain the solution of the Dirichlet problem as a multiple layer potential of the above form and a linear combination of some fundamental solutions with poles outside the domain. (Received November 14, 1955.)

184. Nachman Aronszajn: *A unique continuation theorem for elliptic partial differential equations and inequalities of second order.*

The unique continuation is proved for solutions u of partial differential inequalities of the type $|Au| \leq c\sum|\partial u/\partial x^k|$ for $|x| < R$ where $c>0$ is a constant and A is elliptic of second order with variable coefficients and principal part $\sum a^{ij}\partial^2 u/\partial x^i \partial x^j$. The proof is obtained by extending to the present case an inequality proved by E. Heinz

(Nachr. Ges. Wiss. Göttingen (1955) pp. 1–12) for the case $A=\Delta$, the ordinary laplacian. To obtain this extension one considers first the riemannian metric $\sum a_{ij}dx^i dx^j$, $\{a_{ij}\}$ being the inverse matrix of $\{a^{ij}\}$. Denoting by r the geodesic distance from the origin to a variable point x, a suitably large constant $\alpha>0$ can be chosen such that by putting $\phi(r)=\exp(-\alpha r^2)$ and $A'=$ the Beltrami-Laplace (B.-L.) operator corresponding to the riemannian metric $ds^2=\phi(r)\sum a_{ij}dx^i dx^j$, Heinz's inequality can be proved when put in invariant form relative to ds^2 (i.e. ordinary laplacian replaced by A', distances, spheres, etc. replaced by geodesic distances, spheres, etc.). The proof of the inequality relies on the special form of the B.-L. operator in terms of geodesic polar coordinates, and proceeds by partial integrations in the main integral of the inequality. The role of the constant α is to render positive a crucial term in the final form of this integral. (Received November 14, 1955.)

185. J. H. Barrett: *A necessary condition for nonoscillation of a system of second order differential equations*. Preliminary report.

Consider the matrix second-order self-adjoint differential equation: (1) $(P(x)Y')'+Q(x)Y=0$, where, for $0\leq x<\infty$, $P(x)$ and $Q(x)$ are continuous real square matric functions and $P(x)$ is positive definite. Theorem: If the corresponding vector-matrix equation (2) $(P(x)\alpha')'+Q(x)\alpha=0$ is nonoscillatory (i.e. has no pair of conjugate values for large x) then there exists a matrix solution $U(x)$ of (1) such that $\int^\infty[\operatorname{tr} P(x)]^{-1}\cdot\|U(x)\|^{-2}dx<\infty$. For the scalar case this is similar to a result of Wintner and Leighton (see Leighton, J. London Math. Soc. vol. 27 (1952) pp. 46–47) but the proof is simpler since it does not involve a polar transformation. Sufficient conditions for boundednes of all solutions of (1) (or (2)) have been given by G. Borg, R. Bellman, and the author. For any such set of conditions together with the assumption that $\int^\infty(\operatorname{tr} P)^{-1}=\infty$ it follows easily from the above theorem that the system (2) is oscillatory. Also, additional results along this line are obtained. (Received November 14, 1955.)

186*t*. R. W. Bass: *A generalization of the functional relation $Y(t+s)=Y(t)Y(s)$ to piecewise-linear difference-differential equations.*

Let A, B, P, Q be real constant $n\times n$ matrices, and y, b, f real n-vectors, where f is a step-function of the scalar Σ, and b a constant. Let $\Sigma(t)=b\cdot By(t)$ (where $\cdot$ denotes scalar product) and $\Sigma_1(t)=b\cdot B(Py(t)+Qf(\Sigma_1(t-t_d))$. Consider the system $(*)$: $\dot{y}=Ay+f(\Sigma_1(t-t_d))$ during an interval $(t-2t_d, t+2t_d)$ for which it is known *a priori* that f is constant. If the general solution of $(*)$ for $y(0)=y_0$ is $y(t)=y(t)y_0+Z(t)f$, then $y(t+t_d)=Y(t_d)y(t)+Z(t_d)f$. *Corollary*: *Let* $P=Y(t_d)$, $Q=Z(t_d)$; *then* $\Sigma_1(t)=\Sigma(t+t_d)$, *so that* $(*)$ *is in* $(t, t+t_d)$ *equivalent to* $\dot{y}=Ay+f(\Sigma(t))$. There are important applications to the design of control systems. (Received November 14, 1955.)

187. R. W. Bass: *On nonlinear repulsive forces.*

Consider the real system of second order differential equations $x''=f(x, x', t)$, where x is an n-vector, and f a continuous vector function (on the half-space $t\geq 0$, x_i, x_i' unrestricted) satisfying $x\cdot f\geq 0$ ($\cdot$ is scalar product); $|f|$ is subject only to a growth restriction $|f|<C_K|x'|^2$ (for (t, x) in any compact set K). Then through each point x_0 there passes at least one unrestricted, bounded solution, and one (possibly restricted) unbounded solution. The bounded solutions $x(t)$ are "asymptotic" in the sense that the squared magnitude $r=|x(t)|^2$ is nonincreasing and convex from below; that is, $r\geq 0$, $r'\leq 0$, and $r''\geq 0$. The proof consists principally of *a priori* majorisation of the "asymptotic" solutions and application of the Leray-Schauder fixed-point

theory. The result generalizes theorems of Kneser on scalar differential equations, of Wintner on linear system (Amer. J. Math. vol. 68 (1946) pp. 173–178), and of Hartman and Wintner on linear systems. (Received November 14, 1955.)

188*t*. R. W. Bass: *On the regular solutions at a point of singularity of a system of nonlinear differential equations.*

This paper generalizes the Cauchy-Poincaré existence theorem to singular systems of the form $s^{s_i}w_i'(s) = f_i(s, w_1, \cdots, w_n)$ $(i = 1, \cdots, n)$, where the f_i are regular; $f_i(0, 0, \cdots, 0) = 0$; and s_i are non-negative integers satisfying $0 \leqq s = s_1 + \cdots + s_n < n$. It is asserted that there is a k-parameter family of solutions regular at $s = 0$, where $n - s \leqq k \leqq n$; the solutions will have algebroid dependence on the k parameters and any additional parameters in which the f_i are given regular. This theorem generalizes results of Perron and Lettenmeyer on the linear case (cf. Bieberbach's recent book "· · · *gewöhnliche Differentialgleichungen* · · · ," pp. 161–172). The technique of the proof constitutes an application of the method of undetermined coefficients, combined with Wintner's fixed-point theorem for "analytic" mappings in a separable Hilbert space. Details will appear in the Amer. J. Math. (Received November 14, 1955.)

189. E. F. Beckenbach (p) and M. O. Reade: *Generalized Laplacians and harmonic and subharmonic functions.*

For harmonic, subharmonic, and convex functions the authors discuss characteristic properties which include previously discussed properties as special cases. In particular, the validity of a strengthened form of a conjecture of Osamu Ishikawa [Proc. Japan. Acad. vol. 30 (1954) pp. 686–690] concerning the operator $(1/2\pi)\int_0^{2\pi} u(x+r \cdot \cos\theta, y+r\sin\theta)d\theta - (1/\pi r^2)\int_0^{2\pi}\int_0^r u(x+\rho\cos\theta, y+\rho\sin\theta)\rho d\rho d\theta$ is established. (Received November 14, 1955.)

190*t*. Garrett Birkhoff: *Extensions of Jentzsch's Theorem.* Preliminary report.

Let C be any convex set of "positive" rays of an (L)-space into itself, and let P be any linear transformation which carries C into itself. If some iterated transform CP^r of C under P has finite hyperbolic diameter relative to C, then, for any $f \in C$, the sequence of fP^n converges geometrically to a unique characteristic ray $c \in C$. It is shown that this geometrical result implies various generalizations of Jentzsch's Theorem. (Received November 9, 1955.)

191. Arlen Brown: *Congruence of self-adjoint operators.*

Two self-adjoint operators A and B (on a Hilbert space) are said to be congruent if there exists a bounded operator P with bounded inverse such that $B = P^*AP$. Using the congruence invariants obtained by Koethe in Math. Zeit. vol. 41 (1936) pp. 137–152 (in a somewhat modified form) the following theorem is proved: If $A \geqq 0$, $B \geqq 0$ then A and B are congruent if and only if A^k and B^k are (k any positive number). It follows from this, according to a suggestion of Kaplansky's, that necessary and sufficient conditions that two closed operators A and B be equivalent are that A^*A and B^*B be congruent and that A^* and B^* have equal nullity. Thus the equivalence problem can be satisfactorily solved in terms of congruence invariants. An analogous theorem is proved concerning the similarity of operators A and B in the special case $A^2 = B^2 = 0$. (Received November 14, 1955.)

192. H. D. Brunk: *On an inequality for convex functions.*

Let $X(t)$ be given nondecreasing, continuous from the left, and bounded on $[a, b]$, and let $G(t)$ be a given function of bounded variation continuous from the right on $[a, b]$, with $G(a)=0$, $G(b)=1$. For an interval I, define $M(I)=\int_I G(t)dX(t)/\int_I dX(t)$, and define $G^*(t)=\inf_v \sup_u M(I)=\sup_u \inf_v M(I)$, where u, v are respectively left and right end points of the interval I containing t. The function G^* can be shown to be the best nondecreasing approximant to G, in the sense of least squares, with respect to the function $X(t)$. It is shown that a necessary and sufficient condition that $\int_{[a,b]} f[X(t)]dG(t) \geqq f\left[\int_{[a,b]} X(t)dG(t)\right]$ for every continuous convex f is that G^* be a distribution function on $[a, b]$ (G^* is nondecreasing; it suffices then that it be bounded by 0 and 1). This theorem contains and is a consequence of the well-known theorem to the effect that the above inequality holds for every continuous convex f if G is a distribution function having total variation 1. It contains also the inequalities discussed by E. M. Wright (*An inequality for convex functions*, Amer. Math. Monthly vol. 61 (1954) pp. 620–622). (Received November 14, 1955.)

193. H. D. Brunk, G. M. Ewing (p), and W. R. Utz: *Minimizing integrals in certain classes of monotone functions.*

Suppose given an interval $I=[a, b]$; $\alpha(t)$, $t=(t^1, \cdots, t^n)$, a suitable function measurable over R_n with range in I; $F(u, v)$, a suitable function on $I \times I$, such that, for fixed u, F is strictly decreasing (increasing) for $a<v<u$ ($u<v<b$); a totally finite complete measure μ on a Borel field of subsets of R_n. Consider the integral over R_n, $J[\theta]=\int F[\alpha(t), \theta(t)]d\mu$, in the class M of functions $\theta(t)$, each with range in I, monotone in each coordinate t^i, and such that J exists finite. Also consider $J[\theta]$ in the subclass M^* consisting of those $\theta(t)$ in M, each of which is a bivariate distribution function (or its negative) in each pair of variables t^i, t^j when the others are fixed. The existence and uniqueness of minimizing functions in M (M^*) are established under additional hypotheses on F and representations are obtained for a minimizing function in M. Various results of Brunk, Ewing, and Reid (Bull. Amer. Math. Soc. Abstract 60-6-684) are now extended to n variables and more general integrals. (Received November 10, 1955.)

194. R. H. Cameron: *Nonlinear Volterra functional equations and linear parabolic differential systems.*

In this paper certain nonlinear functional equations of the form $y(t)=x(t)+\Lambda(x|t)$ are studied from the point of view of determining conditions under which continuous solutions x vanishing at $t=0$ exist for almost every y of the same class. [Here "almost every" means all except a set of Wiener measure zero.] This question is shown to be closely connected with the question whether certain linear parabolic differential systems have minimal positive solutions. Both questions can be answered in terms of the values of certain Wiener integrals. (Received November 14, 1955.)

195t. Paul Civin and Bertram Yood: *Semi-reflexive spaces.*

Let X be a Banach space and let π be the canonical isomorphism of X into its second conjugate space X^{**}. Call X semi-reflexive of order n if $\pi(X)$ has finite deficiency n in X^{**}. An example of a semi-reflexive space of order one was given by R. C. James (Ann. of Math. vol. 52 (1950) pp. 518–527). Let T be a continuous linear mapping of X onto a Banach space Y with kernel E. Then X is semi-reflexive if and only if Y and E are semi-reflexive. Furthermore, if X, Y, E are semi-reflexive of order α, β and γ respectively, then $\beta+\gamma=\alpha$. Properties analogous to those for reflexive spaces are also established for semi-reflexive spaces. (Received November 14, 1955.)

196*t*. Albert Edrei: *Signs of the coefficients of real entire functions, bounded on the positive axis.*

Let $f(z)=\sum a_n z^n$ be a real entire function, bounded for $z>0$. Let $\{\lambda_\nu\}$ denote the sequence, taken in its natural order, of all the subscripts λ of those coefficients a which determine a change of sign (for a precise definition, see G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 37). Then $\sum \lambda_\nu^{-1}$ diverges. Conversely, if $\{\lambda_\nu\}$ is a given series of positive increasing integers and if $\sum \lambda_\nu^{-1}$ diverges, there exists a real entire function $f(z)$, bounded for $z>0$, and such that the coefficients a of its expansion have the following properties: (i) $a_n \neq 0$ ($n=0, 1, 2, \cdots$); (ii) $a_\lambda a_{\lambda-1}<0$ if and only if λ is a member of $\{\lambda_\nu\}$. (Received November 14, 1955.)

197*t*. Jacob Feldman: *AW^* embedded AW^* algebras are weakly closed.*

Let $\mathcal{A}$ be an AW^* algebra which is AW^* embedded as operators on a Hilbert space, and let $\mathcal{B}$ be its weak closure. Then $\mathcal{A}=\mathcal{B}$. The proof proceeds as follows: when $\mathcal{B}$ possesses some separating normal state, one can use noncommutative integration theory, following a technique of Segal, to show that $\mathcal{A}=\mathcal{B}$. The theorem then easily also follows when $\mathcal{B}$ is a C^* sum of such rings of operators. In the general case, the problem is first reduced to that of showing that $\mathcal{A}$ contains the weak closure of any *sequence* of its elements. Let $A_1, A_2, \cdots$ be such a sequence; then the ring of operators $\mathcal{B}_0$ generated by this sequence has a countable dense subset. Let $\mathcal{A}_0=\mathcal{A}\cap\mathcal{B}_0$. Then $\mathcal{A}_0$ is an AW^* embedded AW^* algebra with weak closure $\mathcal{B}_0$. But it can be shown that if a ring of operators $\mathcal{C}$ has a weakly dense subset of cardinality $\aleph$, then $\mathcal{C}$ is a C^* direct sum of rings of operators $\mathcal{C}_i$ such that every set of orthogonal nonzero projections in one of the $\mathcal{C}_i$ has cardinality $\leq\aleph$. Thus $\mathcal{B}_0$ is a C^* sum of "$\aleph$-finite" rings of operators. But an $\aleph_0$-finite ring of operators has a separating normal state, as shown by E. Griffin. Therefore $\mathcal{A}_0=\mathcal{B}_0$, and $\mathcal{A}=\mathcal{B}$. (Received October 26, 1955.)

198. L. R. Ford: *Interval-additive propositions.*

A statement involving an interval will be called interval-additive if when true for each of two overlapping intervals it is true also for their union; for example, the statement that a function $f(x)$ is bounded in an interval, or continuous, or of constant sign, or strongly monotonic. The main theorem which is proved is that if an interval-additive proposition is true at each point of a closed interval I it is true for I. Here by truth at a point we mean truth in a subinterval of I enclosing the point, with the convention that at an endpoint of I the subinterval terminates at and includes the end point. Basic properties of a continuous function in a closed interval, or of sets of points, are easy applications of this theorem. (Received November 14, 1955.)

199. R. E. Fullerton: *The construction of smoothed contours of a Fréchet surface.*

Let S be a Fréchet surface with representation T defined over a disk D in the plane and let f be a Lipschitzian function defined on E_3. The contour defined by f and T for the real value t is the boundary of the open set in D for which $f(T(x))<t$. Cesari has defined smoothed contours by considering certain irreducible continua in the hyperspace of continua of constancy of T in D. In this paper a method is given for constructing smoothed contours together with a proof that they are uniquely defined. It is shown that for a smoothed contour the topology in the hyperspace of continua of constancy is equivalent to the interval topology on certain equivalence classes of

ends and prime ends corresponding to points of the contour. It is also proved that the generalized lengths of segments on the reduced contours are the same when computed from both sides of the contour and that if S is nondegenerate, there exists a representation T of S for which all smoothed contours are arcs. (Received November 14, 1955.)

200. T. M. Gallie, Jr.: *Region of convergence of Dirichlet series with complex exponents.*

Hille (Ann. of Math. (1924)) has described a convex "maximal region" D such that the Dirichlet series $\sum_{n=1}^{\infty} c_n \exp(-\lambda_n s)$, with complex exponents λ_n, does not converge outside of D and converges absolutely in the interior of D provided that $\lim (\log n)/\lambda_n = 0$. This description was simplified somewhat by Valiron (Ann. École Norm. (1929)). In the present paper it is observed that D is simply the set of points s for which $d(s) = \liminf [-|\lambda_n|^{-1} \log |c_n \exp(-\lambda_n s)|]$ is non-negative. It is proved for $s \in D$ that $d(s)$ is the distance from s to the frontier of D and that the series converges absolutely to an analytic function at every point of D whose distance from the frontier is greater than $\limsup (\log n)/|\lambda_n|$. Thus the formula for $d(s)$ contains the Cauchy-Hadamard formula for Taylor series and a well known formula relating the abscissas of simple and absolute convergence for Dirichlet series with positive exponents. (Received November 7, 1955.)

201*t*. L. W. Green: *Bounded solutions of Riccati systems.* Preliminary report.

The matrix Riccati equation associated with a system of n second order linear equations is investigated under (among others) the hypothesis of nonconjugacy. E. Hopf's results (Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 47–51) in the case $n=1$ are extended, not without modifying his method, to show that there exists a matrix solution of the Riccati system with never-vanishing determinant whose associated (symmetric) quadratic form is bounded. As Hopf (loc. cit.) found, this yields an equation on the unit tangent bundle of any compact Riemannian manifold whose geodesics contain no conjugate points. Unfortunately, and contrary to the case of surfaces, the geometric significance of this last equation is still obscure. (Received November 14, 1955.)

202*t*. D. L. Guy: *Weighted p-norms and Fourier transforms.* Preliminary report.

The function $f(x) \in L^{\alpha,p}$, $1 < p < \infty$, $-1/p < \alpha < 1-1/p$, if $\int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha p} dx < \infty$. Let $F(t)$ be the Fourier transform of $f(x)$ and $\Delta_n(x) = \int_{-\infty}^{\infty} \chi_n(t) F(t) e^{-ixt} dt$, where $\chi_n(t)$ is the characteristic function of $[2^{n-1}, 2^n]$ if $n \geqq 1$, of $[-1, 1]$ if $n=0$, and of $[-2^n, -2^{n+1}]$ if $n \leqq -1$. The following transform analogue of a theorem of I. I. Hirschman [Memoirs of the American Mathematical Society, no. 15, p. 49] is proved by methods similar to those of Hirschman [op. cit. p. 29–51], Littlewood and Paley [Proc. London Math. Soc. vol. 42, p. 52] and Zygmund [Bull. Amer. Math. Soc. vol. 51, p. 439 and Fund. Math. vol. 30, p. 170]: if $f \in L^{\alpha,p}$, then there are constants A_1 and A_2 such that $A_1 \leqq \int_{-\infty}^{\infty} (\sum_{n=-\infty}^{\infty} |\Delta_n(x)|^2)^{p/2} |x|^{\alpha p} dx / \int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha p} dx \leqq A_2$. By means of this result the transform analogue of the theorem of Marcinkiewicz [Studia Math. vol. 8, p. 78] is proved. (Received September 20, 1955.)

203. Guy Johnson, Jr.: *Functions which have harmonic support.*

A real valued function $v(z)$ of the complex variable z belongs to the class (hs) in a domain D if and only if for each $z_0 \in D$ there exists a function $h(z)$ harmonic in D such that $h(z) \leq v(z)$ and $h(z_0) = v(z_0)$. These functions are continuous and subharmonic. A subclass of the negatives of these functions was used by Carathéodory (Amer. J. Math. (1937)) and Brelot observed (J. Math. Pures Appl. (1945) footnote p. 6) that there are subharmonic functions having continuous derivatives of order p which are not in the class (hs). If $\mathcal{J}$ denotes a subuniformly (i.e. uniformly on each compact subset of D) bounded family of functions $f(z)$ holomorphic in D, then $v(z) = \sup_{\mathcal{J}} |f(z)|$ belongs to (hs). If D is simply connected, let (pl) denote the class of functions $v(z)$ such that $\log v(z) \in (hs)$, then $(pl) \subset (hs)$. Also $v(z) \in (pl)$ if and only if there exists a subuniformly bounded family $\mathcal{J}$ of functions $f(z)$ holomorphic and nonzero in D such that $v(z) = \sup_{\mathcal{J}} |f(z)|$. If $v(z) \in (hs)$ then there exists a sequence of functions of (hs) having continuous derivatives of second order which converges subuniformly to $v(z)$ in D. A subuniformly bounded family $\mathcal{U}$ of functions of the class (hs) is normal. (Received November 14, 1955.)

204t. Jacob Korevaar: *Fourier transforms of generalized functions.*

A distribution ϕ is given by a fundamental sequence $\{f_n\}$, that is, a sequence of integrable functions such that for every finite interval $[a, b]$ there is a number p and an associated sequence of p times iterated indefinite integrals $\{f_n^{(-p)}\}$ which is uniformly convergent on $[a, b]$. ϕ is said to be of slow growth when one of its defining sequences $\{f_n\}$ is of slow growth: $|f_n^{(-p)}(t)| \leq A(|t|+1)^k$, $-\infty < t < \infty$, $n = 1, 2, \cdots$, for some associated sequence $\{f_n^{(-p)}\}$. When ϕ is of slow growth it may be given by a special sequence $\{g_n\}$ of slow growth such that $g_n(t)$ is continuous, equal to zero for $|t| \geq n$, and of bounded total variation. Setting $\mathcal{J}g_n = (2\pi)^{-1/2}\int_{-\infty}^{\infty} g_n(t)e^{-ixt}dt$ the sequence $\{\mathcal{J}g_n\}$ is fundamental and of slow growth. It defines a distribution ψ of slow growth which may be called $\mathcal{J}\phi$. Indeed, when $\{g_n^*\}$ is any other sequence of slow growth defining ϕ for which $\{\mathcal{J}g_n^*\}$ is fundamental, then the latter sequence defines the same ψ. In obvious notation, $\overline{\mathcal{J}}\psi$ also exists. It may be given by the fundamental sequence $\{\overline{\mathcal{J}}\mathcal{J}g_n\} = \{g_n\}$, so $\overline{\mathcal{J}}\psi = \phi$. One next proves that $\mathcal{J}\phi' = ix\mathcal{J}\phi$ for every ϕ of slow growth, that $\mathcal{J}1 = (2\pi)^{1/2}\delta$, $\mathcal{J}\delta = (2\pi)^{-1/2}$, etc. Thus the stage is set for various applications. (Received November 14, 1955.)

205. Stephen Kulik: *Variations of a method for evaluating the zeros of analytic functions.*

Let $f(z)$ and $\phi(z)$ be two analytic functions inside and on the circle C of radius r; let a be any zero of the function $f(z)$ of multiplicity m inside the circle C, which may also be a zero of the function $\phi(z)$, but multiplicity not higher than $m-1$; the function $\phi(z)$ is otherwise arbitrary. Then it is shown that $a - z = \lim f(z)Q_{n-1}(z)/Q_n(z)$ and $a - z = \lim f(z)/(Q_n(z))^{1/n}$, $n \to \infty$, if z is any number which is closer to a than to any other zero of $f(z)$ or to the circle C, and $Q_n(z)$ is the determinant of nth order, the kth line of which is $|\phi^{(k-1)}(z)/(k-1)!,\ f(z)^{(k-1)}/(k-1)!,\ f(z)^{(k-2)}/(k-2)!, \cdots, f'(z), f(z), 0, \cdots, 0|$ and can be calculated by the recurrence formula $Q_n(z) = f'(z)Q_{n-1}(z) - f''(z)f(z)Q_{n-2}(z)/2! + \cdots + (-1)^{n-2}f^{(n-1)}(z)f(z)^{n-2}Q_1/(n-1)! + (-1)^{n-1}\phi^{(n-1)}(z) \cdot f^{n-1}(z)Q_0/(n-1)!$ with $Q_0(z) = 1$, $Q_1(z) = \phi(z)$, $Q_2(z) = \phi(z)f'(z) - \phi'(z)f(z)$. Two particular cases are of special interest, $\phi(z) = 1$ and $\phi(z) = f'(z)$. If $f(z)$ is a polynomial with real coefficients and the roots of the equation $f(z) = 0$ are all real, it is possible to obtain the upper and lower bounds for the roots, as close to each other as necessary, by using the two formulas above with a finite value of n instead of $n \to \infty$ and $\phi(z)$ appropriately chosen. (Received November 7, 1955.)

206. E. K. McLachlan: *Extremal elements of a convex cone of subadditive functions.*

Let E be an arbitrary set of elements, and let $\mathcal{G}$ be a class of subsets of E such that the empty set and E belong to $\mathcal{G}$ and such that X and Y belong to $\mathcal{G}$ implies $X\cup Y$ belongs to $\mathcal{G}$. The set of non-negative, nondecreasing, and subadditive maps f from $\mathcal{G}$ into the real line form a convex cone $\mathcal{C}$. A set X of $\mathcal{G}$ is said to be f-equivalent to a set Y of $\mathcal{G}$ if for a given f of $\mathcal{C}$ there exists a finite sequence of elements, $X=X_0$, $X_1, \cdots, X_k=Y$ of $\mathcal{G}$ such that $X_i\subset X_{i+1}$ or $X_i\supset X_{i+1}$ and $f(X_i)=f(X_{i+1})$ for $i=0, 1, \cdots, k-1$. Let $\mathcal{C}'$ be the subcone of $\mathcal{C}$ which consists of those functions f of $\mathcal{C}$ which have only a finite number of f-equivalence classes. Using a certain system of equations associated with each function f of $\mathcal{C}'$ a necessary and sufficient condition is given for f to be an extremal element of $\mathcal{C}'$. The terms *convex cone* and *extremal element* are used as they are defined in Bourbaki, XV. (Received November 14, 1955.)

207. G. R. MacLane: *Meromorphic functions with specified asymptotic behavior.*

Previously (Bull. Amer. Math. Soc. Abstract 60-2-252) the author gave an elementary proof of a sharpened version of Roth's theorem (Comment. Math. Helv. vol. 11). This theorem may be used to prove the following much stronger theorem. Let $\{D_n\}$ be an infinite sequence of domains in the z-plane such that: (1) the boundary of D_n is a simple curve Γ_n with both ends at ∞, (2) $(D_m\cup\Gamma_m)\cap(D_n\cup\Gamma_n)=0$ for $m\neq n$, (3) $(D_n\cup\Gamma_n)\cap(|z|\leq R)=0$ for $n>n_0(R)$, and (4) $(\bigcup\Gamma_n)\cap(|z|\leq R)$ is rectifiable and of length $\leq k(R+1)^c$, where k and c are constants. Let $f_n(z)$ and $\phi_n(z)$ be holomorphic in D_n, $n\geq 1$. Let $m(z)=\text{dist}\ (z, \bigcup\Gamma_n)$. Let A_n be a closed (but in general unbounded) subset of D_n, $n\geq 1$. Let a, η, and p be positive constants. Then there exists an entire function $F(z)$ such that for $z\in A_n$, $n\geq 1$, $|f_n(z)-F(z)|<\eta\{1+m(z)^{-1}\cdot(|z|+1)^{-p}\}\exp\{-a\Re\phi_n(z)\}$. An appropriate choice of ϕ_n shows that this estimate is essentially the "best possible." (Received July 11, 1955.)

208*t*. Szolem Mandelbrojt: *Composition theorem on Fourier transforms.*

Let $F_1(z)=\int_0^\infty f_1(u)\exp(iuz)du$, $F_2(z)=\int_0^\infty f_2(u)\exp(iuz)du$, with $f_1, f_2\in L$, $f_1(u)\cdot f_2(u)=0$. A relationship can be established between the singularities of $F_1(z)$ and those of $F_2(z)$ in the half-plane $y\leq 0$. For instance, if some conditions are added concerning the growth of $F_1(z)$ in the half-plane $y\geq d$ $(d<0)$, when $|x|\to\infty$, then if $F_1(z)$ has, among its singularities, a pole on the real axis, $F_2(z)$ cannot have one, and only one, singularity in this same half-plane $(y\geq d)$. (Received November 4, 1955.)

209. Szolem Mandelbrojt: *Fourier transforms and functions holomorphic in a half-plane.*

Consider the two equations (i) $\int K(y-x)F(x)dx=0$, (ii) $\int K(y-x)dF(x)=0$. In (i) suppose $K\in L$, $F\in B$, with $M=$ essent. max. of $|F|$. In (ii) suppose K continuous, with $\sum_{n>-\infty}\text{Max}_{x\in[n,n+1]}|K(x)|<\infty$, F locally of bounded variation, with $\int_n^{n+1}|dF|<C$ (C independent of n). In (i) and (ii) suppose $k(u)\neq 0$ for $u<-h$, where k is the Fourier transform of K (In (ii): $h\geq 0$). Then, in case (i), there exists a function $F_0(z)$, holomorphic for $y>0$ $(z=x+iy)$, such that for each $N>0$: (iii) $\lim_{y\to 0}\int_{-N}^{N}|F_0(x+iy)-F(x)|dx=0$, with $|F_0(z)|\leq M\cdot\exp(hy)(y>0)$. In case (ii) there exists a function $F_0(z)$, holomorphic for $y>0(z=x+iy)$, such that (iii) is satisfied, and for each

$c>0$: $\left|[(\exp(icx)-1)/x]F_0(x)\right| \leq \operatorname{Sup}_\xi \left|[(\exp(ic\xi)-1)/\xi]F(\xi)\right| \exp(by)$ $(y>0)$. (Received November 4, 1955.)

210*t*. E. P. Merkes: *A note on harmonic functions.*

Contained in this paper is a generalization of a theorem of Beckenbach on the definition of a harmonic function (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 240–245). The proof of this result is made a very simple consequence of the well-known generalizations of the Looman-Menchoff Theorem by means of the concepts and notations introduced by E. Kasner in connection with polygenic function theory. Extensions are made to q-harmonic functions. (Received November 14, 1955.)

211. E. P. Miles, Jr. (p) and Ernest Williams: *The Cauchy problem for linear partial differential equations with restricted boundary conditions.* III.

The authors apply their theorem of I (Abstract 62-1-103) to the damped wave equation (A): $[\nabla^2-(D_t^2-k^2)]u=0$ subject to the initial conditions $u(x, y, z, 0)=F(x, y, z)$, $u_t(x, y, z, 0)=0$ where $\nabla^{2j}F\neq 0$, $\nabla^{2j+2}F=0$. The solution is seen to be of the form (B), $u=\sum_{s=0}^{j}\nabla^{2s}F\cdot u_s$ where the $\{u_s\}$ are given as follows: $u_0=\cosh kt$, $u_1=(t\sinh kt)/2k$ and $u_s=(-1)^s/(2s)!!k^{2s}\{\sum_{n=1}^{[s/2]}[(2s-2n-1)!(kt)^{2n}\cosh kt/2^{s-2n}(s-2n)!(2n-1)!]-\sum_{n=1}^{[(s+1)/2]}[(2s-2n)!(kt)^{2n-1}\sinh kt/2^{s-2n+1}(s-2n+1)!(2n-2)!]\}$. For (A) with initial conditions $u(x, y, z, 0)=0$, $u_t(x, y, z, 0)=G(x, y, z)$, and G polyharmonic of order $r+1$, the same methods lead to a solution of type $V=\sum_{s=0}^{r}\nabla^{2s}G\cdot v_s$ which can be added to (B) to provide the solution of (A) for polyharmonic initial values for u and u_t. The $\{u_s\}$ given above with the factor $(-1)^s$ deleted enable us to obtain analogous solutions of the Cauchy problem for the special Helmholtz Equation $(\nabla^2-k^2)u=0$. (Received November 14, 1955.)

212. L. E. Payne and H. F. Weinberger (p): *Lower bounds for eigenvalues of elastically supported membranes.*

The eigenvalues $\lambda_1(k)\leq\lambda_2(k)\leq\cdots$ of the problem $\Delta u+\lambda u=0$ on a bounded two-dimensional domain R with $\partial u/\partial n+ku=0$ on the boundary of R are considered. Let $\mu_1(k, L)\leq\mu_2(k, L)\leq\cdots$ be the eigenvalues of the one-dimensional problem $v''+\mu v=0$, $v'(0)-kv(0)=0$, $v'(L)+kv(L)=0$. (They are the solutions of a simple transcendental equation.) Let L be the maximum length of any line segment contained in R and parallel to the x-axis (the x-direction is, of course arbitrary.) Then it is shown that $\lambda_1(k)\geq\mu_1(k, L)$. While equality is not attained except in the trivial case $k=0$, it is approximated with arbitrary accuracy for a sufficiently narrow rectangle with its long side in the x-direction. Furthermore, if R is symmetric with respect to both the x and y axes, if the intersection of R with any line in the x or y direction is simply connected, and if L is maximum length of such intersections, then $\lambda_2(k)\geq\inf\{\mu_2(k, L), \lambda_1(\infty)\}$. For $k=0$, this gives a lower bound for the principal frequency of a free membrane. Equality is attained in this case for an arbitrary rectangle. (Received October 24, 1955.)

213. G. O. Peters: *Euler polynomials and numbers of the second kind.*

Nörlund, in his *Vorlesungen über Differenzenrechnung*, defines Euler polynomials and numbers of higher order and of negative order. Interchanging the roles of Δ and D and replacing powers of x by factorials of x, the author calls a new set of polynomials, Euler polynomials of the second kind. The author first defines a new operator ℸ (Daleth) where $\daleth = 1+D/2$ then denotes Euler polynomials and numbers of the second kind, of order n and degree ν, by $e_\nu^n(x)$, e_ν^n, and c_ν^n respectively, defined by: $e_\nu^0(x) = x^{(\nu)} = x(x-1)(x-2)\cdots(x-\nu+1)$; $\daleth e_\nu(x) = x^{(\nu)}$; $\daleth^n e_\nu^n(x) = x^{(\nu)}$; $e_\nu^n(x) = \sum_{i=0}^{\nu} {}_\nu C_i (e_i^n/2^i)(x-n/2)^{(\nu-i)} = (e^n/2+x-n/2)^{(\nu)}$; $e_\nu^n(x) = \sum_{i=0}^{\nu} {}_\nu C_i (c_i^n/2^i)x^{(\nu-i)} = (c^n/2+x)^{(\nu)}$; $\nu = 0, 1, 2, \cdots$; $n = 0, \pm 1, \pm 2, \cdots$. These are Appel polynomials of the second kind since $\Delta e_\nu^n(x) = \nu e_{\nu-1}^n(x)$. Properties of these polynomials are found. Formulas, involving Sterling's numbers of the second kind, are found for evaluating the Euler numbers of the second kind of higher order. The symmetry of the polynomials is found to be given by $e_\nu^n(x) = (-1)^\nu e_\nu^n(\nu+n-1-x)$. (Received November 14, 1955.)

214. Alexander Peyerimhoff: *Summability factors for Cesàro's method.*

It is well known that a sequence $\{\epsilon_n\}$ transforms every C_α-summable series $\sum a_n$ $(\alpha \geqq 0)$ into a C_β-summable series $\sum a_n\epsilon_n$ $(\beta \geqq 0)$ if and only if $\epsilon_n = O(n^{\beta-\alpha})$, $\epsilon_n = O(1)$ and $\sum (n+1)^\alpha |\Delta^{\alpha+1}\epsilon_n| < \infty$ (Theorem of I. Schur). In this report a new characterization of these summability factors ϵ_n is given: A sequence $\{\epsilon_n\}$ has the above property if and only if (i) $\epsilon_n = O(1)$ and (ii) $\sum (n+1)^\alpha |\Delta^{\alpha+1}\epsilon_n\delta_n| < \infty$ for all sequences $\delta_n = o(1)$ and $\sum (n+1)^\beta |\Delta^{\beta+1}\delta_n| < \infty$. By use of this characterization it is possible to simplify the hitherto existing proofs of Schur's theorem. There are corresponding results for summability factors of the types $(|C_\alpha|, |C_\beta|)$ and $(C_\alpha, |C_\beta|)$. (Received November 14, 1955.)

215. T. J. Pignani (p) and W. M. Whyburn: *Differential systems with interface and general boundary conditions.*

The general linear differential system of the nth order is studied in cases where the solutions are required to satisfy certain interface conditions as well as linear boundary conditions of quite general type. Existence of nonsingular solutions for the homogeneous system is established for cases where singular coefficient matrices occur in the interface conditions. Through use of these solutions of the homogeneous system, and functions which generalize the Green's functions of classical analysis, solutions are obtained for a large class of nonhomogeneous systems. Some of the results are extended to cases where the set of interface points is infinite. (Received September 6, 1955.)

216*t*. Pasquale Porcelli: *A Stieltjes type of inversion.*

The following theorems are proved: Theorem A. Let g be a nondecreasing function on $[0, 1]$ and $F(s) = \int_0^1 (1/1+st)dg(t)$ for $s \notin [-\infty, -1]$, then $g(u)-g(v) = (-1/\pi) \cdot \lim_{y\to 0+}\int_v^u 1/x I_m F(-1/x+iy)dx$, where $0<v<u<1$, g is continuous at each of u and v, and $I_mF(s)$ is the imaginary part of $F(s)$. Theorem B. If F and g satisfy the integral equation of Theorem A, then there exists a nonnegative function f on $(0, 1)$ such that f is Lebesgue summable on each subinterval $[v, u]$ of $(0, 1)$ and $(-1/\pi) \lim_{y\to 0}+1/x \cdot I_mF(-1/x+iy) = f(x)$ almost everywhere; moreover, $g'(x) = f(x)$ almost everywhere. (Received November 14, 1955.)

217*t*. John Raleigh: *The Fourier coefficients of the invariants $j(2^{1/2}; \tau)$ and $j(3^{1/2}; \tau)$.*

Two infinite Fuchsian groups of linear transformations, $G(2^{1/2})$ and $G(3^{1/2})$, are considered: $G(2^{1/2})$ with generators $S(\tau)=\tau+2^{1/2}$, $T(\tau)=-1/\tau$; $G(3^{1/2})$ with generators $S(\tau)=\tau+3^{1/2}$, $T(\tau)=-1/\tau$. Both groups possess invariants; respectively: $j(2^{1/2};\tau)=x^{-1}+\sum_{n=0}^{\infty}c_nx^n$, $x=\exp(2\pi i\tau/2^{1/2})$, $j(3^{1/2};\tau)=x^{-1}+\sum_{n=0}^{\infty}a_nx^n$, $x=\exp(2\pi i\tau/3^{1/2})$. It is required to determine the coefficients of the above expansions. The Hardy-Littlewood method is employed for the purpose, and the procedure is an extension of the one used by H. Rademacher for the Fourier coefficients of the modular invariant $j(1;\tau)$. Convergent series for the c_n's and the a_n's are obtained: $c_n=(2\pi/n^{1/2})\sum_{\nu=1}^{\infty}A_{2\nu}(n)/2\nu\cdot I_1(2\pi n^{1/2}/\nu)+(2\pi/(2n^{1/2}))\cdot\sum_{\nu=1}^{\infty}A_{2\nu-1}((1-\nu)n)/2\nu-1\cdot I_1(2\pi(2n)^{1/2}/2\nu-1)$, $a_n=(2\pi/n^{1/2})\sum_{\nu=1}^{\infty}A_{3\nu}(n)/3\nu\cdot I_1(4\pi n^{1/2}/3\nu)+(2\pi/(3n)^{1/2})\sum_{\nu=1}^{\infty}A_{3\nu-1}(\nu n)/3\nu-1\cdot I_1(4\pi(3n)^{1/2}/3(3\nu-1))+(2\pi/(3n)^{1/2})\sum_{\nu=1}^{\infty}A_{3\nu-2}((1-\nu)n)/3\nu-2\ I_1(4\pi(3n)^{1/2}/3(3\nu-2))$, where $n\geqq 1$, $A_k(m)=\sum{}'\exp(-2\pi i(mh+h')/k)$, $hh'\equiv-1 \pmod{k}$, $\sum{}'$ indicates the sum with respect to h from 0 to $k-1$ with $(h,k)=1$ and $I_1(z)$ is the Bessel function of first order with purely imaginary argument. Moreover, algebraic equations are established relating $j(1;\tau)$ to $j(2^{1/2};\tau)$ or to $j(3^{1/2};\tau)$ respectively. Such equations may be used to calculate the c_n's and the a_n's in closed form, for the first few n's and hence to verify, in part, the results obtained from the series. (Received November 14, 1955.)

218t. M. O. Reade: *Radial limits of certain subharmonic functions.*

We prove the following theorem, which represents a generalization of a result due to Beckenbach and Radó (Trans. Amer. Math. Soc. vol. 35 (1933) pp. 648–661). *If $p(x, y)$ is bounded and of class PL in the disc $|z|=|x+iy|<1$, and if $\lim_{r\to1}p(z)=\lim_{r\to1}p(re^{i\theta})=0$ for a set of θ's of positive (linear) measure, then $p(z)\equiv 0$.* The technique used is that due to Carathéodory (*Funktionentheorie*, Basel, 1950, vol. II, p. 46). (Received November 15, 1955.)

219. W. T. Reid: *Oscillation criteria for linear differential systems with complex coefficients.*

For a self-adjoint second order linear vector differential equation $L[u]\equiv(R(x)u'+Q(x)u)'-(Q^*(x)u'+P(x)u)=0$ with complex-valued coefficients there are presented various general variational criteria for oscillation and nonoscillation, corresponding to known results for the case of such systems with real coefficients. Certain specific criteria for a scalar second order differential equation are generalized to the above vector equation. The results established for the self-adjoint equation $L[u]=0$ are used to deduce sufficient conditions for a general vector differential equation of the form $A_2(x)u''+A_1(x)u'+A_0(x)u=0$ to be nonoscillatory. (Received November 14, 1955.)

220t. H. J. Reiter: *L^1-spaces on groups.* III.

Let g be a discrete abelian group, B a Banach space. Define the space $L_B^1(g)$ as the space of all functions on g with values in B: $f=(f_k)$, $k\in g$, $f_k\in B$, with L^1-norm $\|f\|_1=\sum_{k\in g}\|f_k\|<\infty$, where $\|\cdot\|$ denotes the norm in B. *Theorem*: $\inf\|f_k-\sum_{r=1}^{R}b_rf_{k+n_r}\|_1=\|\sum_{k\in g}f_k\|$, where R ranges over all positive integers, the coefficients b_r range over all complex numbers satisfying $\sum_{r=1}^{R}b_r=0$, and the elements n_r over the group g. *Application*: B is the space of all complex-valued continuous functions $f(t)$ on $[0, 1]$, with norm $\|f\|=\max_{0\leq t\leq 1}|f(t)|$, g the additive group of integers. Then the space W of all continuous functions $f(x)$, $-\infty<x<\infty$, with norm $\sum_{n=-\infty}^{\infty}\max_{n\leq x\leq n+1}|f(x)|<\infty$, considered by N. Wiener (cf. N. Wiener, *Tauberian theorems*, Ann. of Math. vol. 33 (1932) pp. 1–100) is a (closed) subspace of $L_B^1(g)$. Let W_0 consist of all functions in W such that $\sum_{n=-\infty}^{\infty}f(x+n)=0$. Then the quotient-space W/W_0 is isomorphic and isometric to the space of all continuous functions of period 1 under the mapping $f(x) \to Tf(x)$, where $Tf(x) = \sum_{n=-\infty}^{\infty} f(x+n)$; a corresponding result holds for subspaces of W which are invariant under the translations $f(x) \to f(x+k)$, k an integer. The definition of the space W, and the result, may be extended to locally compact groups G containing a discrete abelian subgroup g such that the quotient-space G/g is compact. (Received November 14, 1955.)

221*t*. H. M. Schaerf: *The equivalence of two continuity properties of measurable functions*. Preliminary report.

A measure space (R, S, m) whose carrier is a general topological space is said to have the *property of* (a) *Lusin* or (b) *Vitali-Carathéodory* if the following statements hold, respectively: If f is any measurable function on R, then (a) for every positive number ϵ and every measurable set E there is a closed measurable set $F \subset E$ with $m(E-F) < \epsilon$ such that f is continuous on F; (b) there are two monotone sequences of semi-continuous functions $\{l_n\}$ and $\{u_n\}$ which converge to f almost everywhere and have the properties stated in the usual formulations of the Vitali-Carathéodory theorem in Euclidean spaces (e.g. in S. Saks, *Theory of the integral*, 2d edition, Theorem 7.6). *Theorem*: For every totally σ-finite measure space whose carrier is a general topological space the above two properties are equivalent. Criteria for the validity of generalizations of the Vitali-Carathéodory theorem to spaces without topological structure are also established. (Received November 14, 1955.)

222*t*. V. L. Shapiro: *The symmetric derivative on the* $(k-1)$*-dimensional hypersphere.*

Let x be a point on the unit $(k-1)$-dimensional hypersphere Ω in Euclidean k-space, $k \geqq 3$, and let μ be a completely additive set function of bounded variation defined on the Borel sets of Ω. Let $D(x, h)$ represent the spherical cap on Ω obtained by intersecting Ω with a sphere whose center is x and radius is $2 \sin h/2$, and let $|D(x, h)|$ be the $(k-1)$-dimensional volume of $D(x, h)$. Then μ will be said to have a symmetric derivative at x, designated by $\mu_s(x)$, if $|D(x, h)|^{-1}\mu[D(x, h)]$ tends to $\mu_s(x)$ as h tends to zero. Let $S[d\mu] = \sum_{n=0}^{\infty} Y_n(x)$ be the Stieltjes series of surface harmonics defined by μ. Then it is shown in this paper that if $\mu_s(x_0)$ exists and is finite and μ satisfies the global condition $|\mu|[D(x_0', \epsilon)] = 0$ for some $\epsilon > 0$, where x_0' is the point diametrically opposite to x_0 and $|\mu|$ is the total variation of μ, then $S[d\mu]$ is summable (C, δ), $\delta > (k-2/2)+1$, to $\mu_s(x_0)$. This result generalizes the well-known result for Fourier-Stieltjes series where $\delta > 1$. In case the global condition is not satisfied, it is shown that $S[d\mu]$ is summable (C, η) to $\mu_s(x_0)$ where $\eta > k-2$ for $k \geqq 4$ and $\eta > 3/2$ for $k = 3$. In the special case when μ is absolutely continuous and $Y_n(x_0) = O(n^{-1})$, a necessary and sufficient condition that $S[d\mu]$ converges at x_0 to the finite value b is obtained also in this paper, namely that $\mu_s(x_0)$ exists and equals b. This fact generalizes a result previously obtained for Fourier series by Hardy and Littlewood. (Received November 9, 1955.)

223. R. C. Thorne: *Asymptotic expansions of Legendre functions.*

Although asymptotic expansions for the Legendre functions $P_n^{-m}(x)$, $Q_n^{-m}(x)$ have been given for m fixed and $n \to \infty$ by Olver (International Mathematical Congress, Amsterdam, 1954, vol. II), no expansions have been derived for both n and m large and positive. For this case, three sets of expansions in ascending powers of $u^{-1}(u = n+1/2)$ are developed here. The ratio $0 < \alpha = m/u < 1$ is fixed and $u \to \infty$. The first expansions, involving exponential functions, are not valid over the transition point

$x = \beta = (1-\alpha^2)^{1/2}$. The second set of expansions, obtained by application of Olver's theory (Phil. Trans. Roy. Soc. London, Ser. A. vol. 247, pp. 307–327), are in terms of Airy functions and are valid over the domain D consisting of the whole x plane (cut from $+1$ to $-\infty$) except for a pear shaped domain surrounding $x = -1$ and the strip (S, say) $-\delta \leqq \text{Im}\ (x) < 0$, $|\text{Re}\ (x)| < \beta$; δ small > 0. Expansions valid in $D+S$ are obtained by using Bessel functions $I_m(ut)$, $K_m(ut)$, $t = t(x)$. This theory is new and differs from that of Cherry (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 253–257). Also, $P_n^{-m}(x)$ is exponentially small as $n \to \infty$ in a pear shaped region surrounding $x = 1$, oscillatory for x real and $|x| < \beta$, and exponentially large elsewhere. Related properties exist for $Q_n^{-m}(x)$ as $n \to \infty$. (Received December 1, 1955.)

224. H. S. Wall: *Partially bounded continued fractions.*

For each complex number sequence a, $f(a)$ denotes the continued fraction $1/1 + a_1/1 + a_2/1 + \cdots$. The statement that $f(a)$ is partially bounded means that the sequence a has a bounded infinite subsequence. Theorem. If there is a nonnegative integer k such that the subsequence $\{f_i\}_{i=k}^{\infty}$ of the sequence of approximants of the partially bounded continued fraction $f(a)$ is bounded and the even (odd) part of $f(a)$ converges and has the value v, then there exists an infinite subsequence of the sequence of approximants of the odd (even) part of $f(a)$ which converges to v. Corollary. If the even and odd parts of the partially bounded continued fraction $f(a)$ converge, then $f(a)$ converges. (Received October 31, 1955.)

APPLIED MATHEMATICS

225*t*. R. W. Bass: *Improved on-off missile stabilization.*

A new synthesis procedure for on-off control systems is presented, by means of which the delays due to time-lag and hysteresis in the relay or contactor can be completely eliminated. This is readily effected by adding feedback from the on-off element itself to the conventional error and error-derivative feedbacks. By generalization (to inhomogeneous piecewise-linear difference-differential equations) of the functional relationship $E(t+s) = E(t)E(s)$ (which is satisfied by any fundamental matrix of solutions of a system of homogeneous linear differential equations with constant coefficients), an explicit algorithm is developed, according to which the ideal feedback coefficients of a "perfectly" stabilized servo-system can be read off directly from the equations of the corresponding conventional system. The use of this algorithm is illustrated by application to three second order systems, and one third order system, of one degree of freedom, and to one second order system with three degrees of freedom. The results are applied to several specific relay servomechanisms and missile control systems. (Received November 14, 1955.)

226*t*. Stefan Bergman: *On a class of subsonic flows.*

The author considers subsonic flow patterns in a channel C bounded by $[y = 0, -\infty < x < \infty]$ on one side, the segments $(-\infty A_0) = [y = -c = \text{const.}, c > 0, -\infty < x < 0]$, $(A_{n-1}A_n) = [x = y \cos \tau_n + a_n]$, $n = 1, 2, \cdots, N$, $\tau_n = \text{const.}$, $0 < \tau_1 \cdots < \tau_{n-1} < \tau_n < \cdots \tau_N < \pi/2$, and the segment $(A_N \infty)$ of a free boundary on the other side. The image of the flow in the pseudo logarithmic plane (see Bergman, Proceedings of Symposium in Applied Mathematics, vol. 6, §4 ff.) is a half strip bounded by the segment $(a_{N-1}a_N) = [\theta = \tau_N, \lambda < \lambda_1]$, $[\lambda = \lambda_1, 0 \leqq \theta < \tau_N]$ (which is the image of the free boundary), $[\theta = 0, \lambda = \lambda^{(0)}]$, $[\theta = 0, \lambda^{(0)} \leqq \lambda \leqq \lambda_1]$, and $(N-1)$ slits $S_n = [-\infty < \lambda < \alpha_{n-1}, \theta = \tau_n]$, $n = 1, 2, \cdots, N-1$. $\lambda^{(0)}$ is the image of $[y = -\infty]$ and the slits are images of sides $(A_{n-1}A_n)$, $n=1, 2, \cdots, N-1$. The stream function $\psi(\lambda, \theta)$ satisfies a linear differential equation $L(\psi)=\psi_{\lambda\lambda}+\psi_{\theta\theta}+N(\lambda)\psi=0$ (see Bergman, Journal for Rational Mechanics and Analysis vol. 4, §2). Using the theory of linear integral equations the author shows that for every set of slits S_n and every $\lambda^{(0)}$ (satisfying certain conditions) there exists the desired stream function $\psi(\lambda, \theta)$, $L(\psi)=0$. Using the theory of orthogonal functions (see Bergman and Schiffer, Duke Math. J. vol. 15, in particular §7, p. 560 ff.) one obtains a representation for the stream function $\psi(\lambda, \theta)$. (Received December 13, 1955.)

227. L. A. Colquitt: *A modified iteration procedure.*

Suppose that the equation $x=f(x)$ has a unique root, and suppose that $f'(x)$ satisfies the inequality $-a<f'(x)<-1$ for all x, so that the conventional iteration fails to converge. The following modified iteration procedure converges for this case. Let x_0 be arbitrary, and define $x_{2n-1}=f(x_{2n-2})$; $x_{2n}=1/2n\sum_{i=0}^{2n-1} x_i$, $n=1, 2, \cdots$. In other words, substitutions alternate with averagings. If x is the desired root of the given equation, it can be shown that $x-x_{2n}=1/2n(x-x_0)[1+f'(x_0')]\prod_{i=1}^{n-1}\{1+1/2i[1+f'(x_{2i}')]\}$, where x_k' lies between x and x_k. From this relation it follows that $\lim_{n\to\infty}(x-x_{2n})=0$. Even if $f(x)$ is such that $-1<f'(x)<0$, so that the conventional iteration converges, it may do so only slowly if $f'(x)$ is close to -1. A procedure similar to that used above will accelerate the convergence in this case. (Received November 14, 1955.)

228. A. H. Diamond: *Certain classes of finite sequential machines with distinguished states.*

In a paper entitled *Gedanken-experiments on sequential machines*, which will appear in the Automata Studies, E. F. Moore develops the theory of *strongly connected* machines, that is, machines having the property that for any pair of states of the machine there exist input sequences which will take the machine from either of the two states to the other. In this paper some of Moore's results are extended to certain classes of quasi strongly connected machines with distinguished states. (Received November 14, 1955.)

229t. H. H. Goldstine and Joseph Gillis: *The stability of superposed fluids.*

Two homogeneous, incompressible, nonviscous fluids, which may have different densities, are superposed in equilibrium one above the other between a pair of infinite, parallel, vertical walls. The walls are defined by $x=0$, $x=\pi$, while the equilibrium position of the interface between the fluids, supposed horizontal, is given by $y=0$. Equilibrium is perturbed and the subsequent motion studied, with particular reference to the question of stability. The configuration of the interface at time t is expressed as a double series $y=\sum_{m,n}\eta_{m,n}(t)\alpha^m\cos nkx$ where α is a perturbation parameter and k the mode of the perturbation. The coefficients $\eta_{m,n}(t)$ satisfy linear differential equations of the second order which are obtainable by means of an algorithm. Some general results are derived relating to the functions $\eta_{m,n}(t)$ and they are calculated as explicit functions of t for $1\leqq m\leqq 5$ and all relevant n. The resulting approximation to y is compared with the known experimental facts in some special cases. (Received November 14, 1955.)

230. T. P. Higgins (p) and W. T. Guy, Jr.: *On the Hankel transform.* II.

The Hankel integral transform of the nth derivative of a function is used to establish some new relations involving Meijer's G-functions. (Received November 14, 1955.)

231. Ernest Ikenberry: *The collision integral for Maxwellian molecules; elementary spherical moments.*

In this paper there are obtained the collision integrals $C(Y_s)$ for homogeneous spherical harmonics $Y_s(o)$, for Maxwellian molecules. In the notation adopted by Ikenberry and Truesdell (Journal of Rational Mechanics and Analysis, forthcoming), $C(Q)=\int_\omega do\int_\omega do^*\int_0^\infty 2vb\,db\int_0^{2\pi}d\epsilon\{Q'-Q\}FF^*$. Let elementary spherical moments S_s of order s be defined by $S_s=m\int_\omega C^sF\,do$, where $C^s=(c_3+ic_1\cos\alpha+ic_2\sin\alpha)^s$ is a homogeneous spherical harmonic of degree s. Then $m\rho C(C^s)=-\sum_{j=0}^{s}C_{s(j)}S_jS_{s-j}$, where $2^{s-1}C_{s(j)}=\pi\sum_{l=0}^{j}\sum_{k=j-l}^{s-l}(-1)^{j+l}\binom{j}{l}\binom{s-k}{l}\binom{k}{j-l}B_k$, with $B_k=2\pi\int_0^\infty\{1-P_k(\cos\phi)\}vb\,db$. From this, the value of $C(Y_s)$ for any homogeneous spherical harmonic $Y_s(o)$ of degree s is obtainable: $C(Y_s)=\int_0^{2\pi}C(C^s)f_s(\alpha)d\alpha$, where $f_s(\alpha)$ is a uniquely determined trigonometric polynomial of degree s such that $Y_s(o)=\int_0^{2\pi}C^sf_s(\alpha)d\alpha$. (Received November 10, 1955.)

232. J. R. Isbell (p) and W. H. Marlow: *Programming games.*

A linear fractional function in n-space can be factored into a linear transformation upon the plane and a plane linear fractional function. Thus a simple computational method for linear fractional programming is found. Further, certain tactical games involving strategic choice of distributions of fire of weapons over several targets, assuming attrition is governed by certain generalized Lanchester equations (due to R. N. Snow, Rand publication RA-15078), have payoff functions linear fractional in both players' strategies together (not merely bilinear fractional). Thus a simple computational method for solving these games is found. Both authors supported in part by the Office of Naval Research; the first is a private in the Army Ordnance Corps. (Received November 9, 1955.)

233. H. A. Osborn: *A new technique for eigenvalue problems.*

If $\phi(t)$ is a sufficiently smooth positive function of t in $[0, 1]$ the following method furnishes an efficient technique for computing the lowest eigenvalue λ of $u''+\lambda\phi(t)u=0$, $u(0)=u(1)=0$. For any $s\in[0, 1]$ and any $v\geq 0$ set $f(s, v)=\min\int_0^s w'^2(t)dt$ subject to $w(0)=0$, $w(s)=v$, and the normalization $\int_0^s\phi(t)w^2(t)dt=s^3$, assuming that the minimum is attained for some twice-differentiable function $w(t)$. Then for any $\sigma\in(0, 1)$ we have the functional inequality $f(s, v)\geq\min\{\int_\sigma^s w'^2(t)dt+r^{-2}f(\sigma, rv^*)\}$, where the minimum is taken over $v^*>0$ and $w(t)$ for $t\in[\sigma, s]$ subject to $w(\sigma)=v^*$, $w(s)=v$, and where r is a factor depending on $w(t)$ which accounts for the renormalization to the interval $[0, \sigma]$. The two sides of the inequality actually differ only by $o(s-\sigma)$. Replacing $\int_\sigma^s w'^2(t)dt$ and r by approximations correct to $o(s-\sigma)$ one finds a difference equation which permits the computation of $\lambda=f(1, 0)$ to within $O(h)$, where h is the difference interval. The initial values of the difference scheme do not depend on a trial value of λ. (Received November 10, 1955.)

234. H. H. Rachford (p) and Jim Douglas, Jr.: *On the numerical solution of heat condition problems in two and three space variables.*

The objective of this paper is to introduce a new implicit finite difference method for obtaining approximate solutions of both transient and permanent heat conduction problems in two or three space variables. Let $f_{i,j,k,n} = f(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ and $\Delta_x^2 f_{i,j,k,n} = (f_{i+1,j,k,n} - 2f_{i,j,k,n} + f_{i-1,j,k,n})/(\Delta x)^2$. The difference analogue of the parabolic case with three space variables is the following three step procedure: $\Delta_x^2 w^{*}_{i,j,k,n+1} + \Delta_y^2 w_{i,j,k,n} + \Delta_z^2 w_{i,j,k,n} = (w^{*}_{i,j,k,n+1} - w_{i,j,k,n})/\Delta t$, $\Delta_y^2 w^{**}_{i,j,k,n+1} = \Delta_y^2 w_{i,j,k,n} + (w^{**}_{i,j,k,n+1} - w^{*}_{i,j,k,n+1})/\Delta t$, $\Delta_z^2 w_{i,j,k,n+1} = \Delta_z^2 w_{i,j,k,n} + (w_{i,j,k,n+1} - w^{**}_{i,j,k,n+1})/\Delta t$. It is proved that this method is stable and that its solution for the initial- and boundary-value problem converges in the mean to that of the heat flow equation with the error being $O((\Delta x)^2 + (\Delta t)^2 + (\Delta x)^4(\Delta t)^{-1})$. The method is then applied to the elliptic case by considering the time step as an iteration parameter. A sequence of parameters is constructed so that the number of calculations necessary to reduce the norm of the difference between the iterated solution and the solution of $\Delta_x^2 w_{i,j,k} + \Delta_y^2 w_{i,j,k} + \Delta_z^2 w_{i,j,k} = 0$ by the factor $\exp(-Q)$ is $O(QN \log N)$, where N is the number of interior points in the lattice. The two space variable case is treated by the analogous two step procedure. (Received November 7, 1955.)

235t. H. E. Salzer: *Osculatory extrapolation and a new method for the numerical integration of differential equations.*

One may define "confluent binomial coefficients," or multipliers of the function and its derivative, which play a role in both extrapolation and checking similar to that of binomial coefficient multipliers for the function alone. The coefficients C_k and D_k in the n-point osculatory extrapolation formula of $(2n-1)$th degree accuracy. viz. (1) $f_{[n/2]+1} = \sum_{k=-[(n-1)/2]}^{k=[n/2]} (C_k f_k + hD_k f'_k) + R_{2n}$, are tabulated exactly for $n = 2(1)11$, with the factor F_n where $R_{2n} \approx F_n \cdot \Delta^{2n} f$. Comparison with ordinary extrapolation and checking by differencing shows advantages in (1) due to the smallness of F_n as n increases. The most important use of (1) is in providing, for the lower values of n, a new and more accurate set of formulas for the stepwise numerical integration of $y' = \phi(x, y)$. Because of the error in $f_{[n/2]+1}$ due to the magnitude of C_k and D_k, (1) is used in conjunction with an entirely new type of refining formula (2) $f_{[n/2]+1} = \sum_{k=-[(n-1)/2]}^{k=[n/2]} A_k f_k + h \sum_{k=-[(n-1)/2]}^{k=[n/2]+1} B_k f'_k$, i.e., "$n$ 1/2-point" osculatory type based upon f and f' at n points and f' at the $(n+1)$th. The coefficients A_k and B_k in (2) were found up to $n = 5$ (10th degree accuracy). For complex integration of differential equations of the form $w' = \phi(z, w)$, over a Cartesian grid, osculatory extrapolation and refining formulas analogous to (1) and (2) were derived, of 7th and 8th degree accuracy respectively, based upon f and f' at the corners of a square and the extra f' at each of the four closest points, which is sufficient to continue the integration until all the desired grid points are included. (Received October 7, 1955.)

236. G. L. Spencer, II: *A method for solving mixed initial problems for quasi-linear hyperbolic systems.*

The method presented solves a subclass of the mixed initial problems discussed by Courant and Lax (*On nonlinear partial differential equations with two independent variables*, Courant and Lax, "Communications on Pure and Applied Mathematics," vol. II, 1949). It also solves a class of singular problems not covered by Courant and Lax. (Received November 14, 1955.)

237t. G. L. Thompson: *On the solution of a game-theoretic problem.*

During the consideration of a generalization of von Neumann's model of an expanding economy the following, essentially game-theoretic, problem arose: Given real-valued matrices A and B with non-negative entries and such that $v(-A) < 0$ and $v(B) > 0$; set $M(\alpha) = B + \alpha(-A)$ and find an α so that $v(M(\alpha)) = 0$; then find a pair of probability vectors (x, y) such that $xBy>0$ and x is optimal for the maximizing player and y is optimal for the minimizing player in the game $M(\alpha)$. It is shown that at least one solution to this problem exists. Elsewhere it has been shown that there are at most a finite number of α's for which solutions exist. Our proof utilizes the minimax theorem from the theory of games in the strong form proved by A. W. Tucker. Specifically, the concept of a central solution to a matrix game is used. Examples of solutions plus possible extensions of the theorem are also considered. (Received November 14, 1955.)

238. Deonisie Trifan: *Stress theory of plastic flow.*

General stress-strain relations for the stress theory of plastic flow of an incompressible, isotropic material exhibiting a gradual transition from the elastic to the plastic state are developed by using the stress-strain relation for simple tension as a model, resulting in $ds_{ij}=2G_0de_{ij}-G\{\partial g/\partial e_{ij}-(4E_2E_3/3)\partial g/\partial E_2^2\delta_{ij}\}dg$ for loading and $ds_{ij}=2G_0de_{ij}$ for unloading. The quantities ds_{ij} and de_{ij} are differentials of stress and strain deviations respectively, while the functions G and g are positive definite functions of the strain invariants E_3 and E_2^2. The loading-unloading criterion is given by the differential of g. Conditions on G and g are obtained and a minimum principle for the differential of strain deviation introduced, namely: For a suitable set of differentials of strain de'_{ij} (called admissible) and corresponding differentials of stress deviation ds'_{ij}, the integral $(1/2)\int_V de'_{ij}ds'_{ij}dV-\int_{S_2}du'_i\,dT_idS$ is an absolute minimum for the actual differentials of stress and strain occurring in the body; the boundary condition dT_i being given over S_2. Inverted stress-strain relations to the above are calculated, and a corresponding minimum principle for the differential of stress deviation given. (Received November 1, 1955.)

GEOMETRY

239. L. M. Blumenthal: *Remarks on the fundamental theorem concerning metric arcs with zero curvature at each point.*

The existing literature does not contain a proof of the theorem that a metric, ptolemaic arc with zero Menger curvature at each point is a metric segment, that is both complete-in-itself (in an obvious sense) and which is free from error and troublesome lacunae. The purpose of the present investigation is to supply such a proof. (Received November 14, 1955.)

240. S. S. Cairns: *On the partition of the vertices of an n-cube by an $(n-1)$-plane.*

The work is motivated by the problem of characterizing the convex of the vertices of the n-cube $0\leqq x_i\leqq 1$ satisfying an inequality $\sum a_ix_i<c$ $(a_i>0)$. A vertex p is identified with the subset J of $(1, \cdots, n)$ such that, at p, $x_i=1$ if $i\in J$, and $x_i=0$ otherwise. Given the a's, the plane $\pi(c):\sum a_ix_i=c$ induces a permutation $\Pi(a)$ of the set $\{J\}$ of all 2^n subsets of $(1, \cdots, n)$, defined by the order in which $\pi(c)$ passes through the vertices, regarded as elements of $\{J\}$, when c increases from a negative value through the value $\sum a_i$. By minor adjustments of the a's it can be arranged that $\pi(c)$ never passes simultaneously through two vertices. Various general properties are established for the permutations thus generated, and a recurrent method with respect to n is developed. Some questions of consistency of linear inequalities make the general step of the recurrency difficult. The step from $n=3$ to $n=4$ reveals that there are 336 possible permutations $\Pi(a)$ for the vertices of a 4-cube, 148 possible sets of vertices defined by $\sum a_i x_i < c$ and only 27 types of convex hulls of such sets, two convexes being of the same type if one can be carried into the other by a permutation of axes. (Received November 14, 1955.)

241. N. A. Court: *On four real mutually orthogonal spheres.*

Considering the sphere of similitude, the radical sphere, and the two spheres of antisimilitude of the given spheres taken two at a time, twenty four spheres are obtained which may be grouped into six tetrads such that the spheres in each tetrad are mutually orthogonal. If the given spheres are taken three at a time, four pairs of real points of intersection are obtained. Taking one and only one point in each pair, sixteen tetrahedrons are formed. All these tetrahedrons are isodynamic, and may be grouped into eight pairs of homological tetrahedrons, their planes of homology coinciding with the eight planes of similitude of the four given spheres, etc. It follows from the latter proposition that the problem: "Through four given points, taken three at a time, to pass four mutually orthogonal spheres" admits of no solution, unless the four given points form an isodynamic tetrahedron. If this condition is satisfied, the problem may, in general, have two solutions. (Received November 14, 1955.)

242. M. K. Fort, Jr.: *A geometric problem of S. Stein.*

S. Stein has proposed the following problem (Bull. Amer. Math. Soc. Research Problem 61-5-25): Let $J \subset R_2$ be a rectifiable Jordan curve, with the property that for each rotation R, there is a translation T, depending on R, such that $(TRJ) \cap J$ has a nonzero length. Must J contain the arc of a circle? If "length" is interpreted to mean the one-dimensional Lebesque measure which is induced on J by arclength, then an example shows that J need not contain the arc of a circle. The following theorem is then proved: If $J \subset R_2$ is a (not necessarily rectifiable) Jordan curve, with the property that for each of an uncountable number of rotations R about some point there is a translation T, depending on R, such that $(TRJ) \cap J$ has a nondegenerate component, then J must contain the arc of a circle. An example shows that it is not possible to replace "uncountable" by "infinite" in the above theorem. The proof of the theorem is based on the following lemma: If A and B are topological arcs in R_2 and A contains an infinite number of subarcs which are congruent to B, then B is either an arc of a circle or a straight line segment. (Received October 24, 1955.)

243. Michael Goldberg: *Trammel rotors in regular polygons.*

A rotor in a regular polygon is a closed convex curve which remains tangent to all the sides of a fixed n-gon during a complete rotation of the curve. Among these, the rotors bounded entirely by arcs of circles have a special interest for several reasons. They are called trammel rotors because of the method of their generation. The rotors of minimal area in the equilateral triangle and in the square have been shown to be trammel rotors and it has been conjectured by the author and others that minimal rotors for other polygons are also trammel rotors [*Circular-arc rotors in regular polygons*, Amer. Math. Monthly vol. 55 (1948) pp. 393–402]. The author, in the cited paper, gave a method for generating trammel rotors in all regular polygons. These rotors have only one or two axes of symmetry. He now shows how to construct regular trammel rotors of the same circular arcs but with $n-1$ and $n+1$ axes of symmetry. Some of these new rotors have less area than the earlier ones. Fujiwara had already discovered the one for the pentagon [Tôhoku Science Reports vol. 8 (1919) pp. 245–246]. More general trammel rotors are obtained by linear combinations of the regular rotors. Also, the corresponding rotors in spherical polygons are derived. (Received November 9, 1955.)

244. J. S. Griffin, Jr.: *Affine connections in terms of the tangent bundle.*

Let $\mathfrak{B}(B, M, \pi, E^m)$ be a differentiable bundle of linear spaces, and let $\mathfrak{R}=(R, M, \mu, E^n)$ and $\mathfrak{T}=(T, B, \nu, E^{m+n})$ be the tangent bundles. Now $d\pi: T\rightarrow R$, and in fact $\mathfrak{Q}=(T, R, d\pi, E^{2m})$ is a bundle of linear spaces, say with operations $\oplus$ and $\circ$. In the particular case that $\mathfrak{B}=\mathfrak{R}$, a differentiable retraction $\rho: T\rightarrow T$ which is linear on each fibre $\nu^{-1}(b)$ and whose kernel is that of $d\pi$ defines an affine connection for M if and only if ρ is invariant under these operations: $\rho(x\oplus y)=\rho(x)\oplus\rho(y)$ and $\rho(r\circ x)=r\circ\rho(x)$ for any real number r. As to the operations $\oplus$ and $\circ$, they may be defined in a natural manner from coordinate functions for $\mathfrak{Q}$, or in the following intrinsic fashion. Since $\mathfrak{B}$ is a bundle of linear spaces for any r there is $\tau_r: B\rightarrow B$ defined by $\tau_r(x)=rx$, and thus there is $d\tau_r: T\rightarrow T$; let $r\circ x=d\tau_r(x)$. Let (C, M, η, E^{2m}) be the Whitney product of $\mathfrak{B}$ with itself and let (S, C, ζ, E^{2m+n}) be the tangent bundle; then there are ξ, ι, and σ on C to B defined by $\xi(x, y)=x+y$, $\iota(x, y)=x$, and $\sigma(x, y)=y$; these induce $d\xi$, $d\iota$, and $d\sigma$ on S to T; one may define $x\oplus y=z$ if there is $v\in S$ such that $d\iota(v)=x$, $d\sigma(v)=y$, and $d\xi(v)=z$. (Received December 16, 1955.)

245*t*. H. G. Helfenstein and Max Wyman: *Geodesic mapping of minimal surfaces.*

In general the only mappings preserving the geodesic lines of a minimal surface onto some other surface are isometries and similarities. There are, however, six classes of exceptional surfaces admitting nontrivial geodesic mappings. They are determined in this paper by means of the theorems of Dini and Lie. Every such surface shares of course this property with all its associates as well as with their similar images; they form an equivalence class. Choosing a simple representative in each equivalence class we obtain a complete system of normal types. Among our surfaces there is a complex one-parameter family of different normal types; in addition there are five other, discrete types. Four of the six cases lead to real minimal surfaces. (Received October 19, 1955.)

246. L. M. Kelly and Leo Lapidus (p): *Metric operations and Brouwerian geometries*. Preliminary report.

If with each two points of a collection, S, is associated as the distance between them an element, u, of a lattice L with a 0 (the latter serving as the distance between coincident points only), the resulting structure is called a lattice-metrized $(L-M)$ space if, for each three points of the space, the associated distances u, v, w, satisfy the lattice "triangle inequality," $u\cup v\supseteq w$. When $S=L$, such an association is called a metric operation, and L is said to be autometrized. When $S=L$ is a Boolean algebra, it is known that symmetric difference is such a metric, yielding a "Boolean geometry." (D. O. Ellis, *Autometrized Boolean algebras* I and II, Canadian Journal of Mathematics vol. 3 (1951) pp. 87–93 and 145–147.) An analogous symmetric difference function in the more general Brouwerian algebra (logic), gives rise to a "Brouwerian geometry." (E. A. Nordhaus and L. Lapidus, *Brouwerian geometry*, Canadian Journal of Mathematics vol. 6 (1954) pp. 217–229.) It is shown in the latter paper that the congruence order (L. M. Blumenthal, *Distance geometry*, Oxford, 1953, pp. 91–92) of a Brouwerian geometry relative to the class of $L-M$ spaces is three iff it is a Boolean geometry. This paper continues the study of Brouwerian geometries. It is shown that the congruence order of an arbitrary Brouwerian geometry is four. The proof is based on the representation theorem for a distributive lattice as a subdirect union of chains. The implications of the coincidence of metric and lattice betweenness in autometrized lattices is also studied. (Received November 14, 1955.)

247t. E. R. Lorch: *Riemannian geometry and smooth convex bodies in Hilbert space.*

There is associated with each smooth convex body in Hilbert space a Riemannian geometry defined over that space and which completely characterizes the body. Let $\mathfrak{K}$ be a convex body defined by $2G(x) \leqq 1$ where $G(x)$ is positively homogeneous of degree 2 and sufficiently differentiable; furthermore, for the second derivatives we assume $L\|y\|^2 \leqq G_{yy}(x) \leqq M\|y\|^2$. The quadratic form $G_{yy}(x)$ then defines the Riemannian metric. Making use of the fundamental homeomorphism $x \rightarrow \mathfrak{x} = G.(x)$ relating $\mathfrak{K}$ to its polar body $\overline{\mathfrak{K}}$ (see the author's paper *On the volume of smooth convex bodies in Hilbert space*, Math. Zeit. vol. 61 (1955) pp. 391–407), the Riemann curvature tensor is calculated. The existence of the Ricci tensor cannot be ascertained due to excessive contraction. The development uses the recent results of Detlef Laugwitz (*Differentialgeometrie ohne Dimensionsaxiom*, Math. Zeit. vol. 61 (1954) pp. 100–118). In particular his result that the classical (n-dimensional) i, j, k notation can be extended intact to Banach spaces forms the basis of all tensor computations. (Received November 18, 1955.)

248t. T. G. Ostrom: *Double transitivity in finite projective planes.*

In this paper, the author proves the following two theorems: Theorem 1. If a finite plane π, in which n is odd and not a square, is doubly transitive, then π is Desarguesian. Theorem 2. If a finite plane π, in which n is odd and not a square, is doubly transitive for points not on a certain line, then π is a Veblen-Wedderburn plane with the line as its special line. Under the restrictions placed on n collineations of order two are perspectivities. A doubly transitive plane contains enough collineations of order two to generate other groups of perspectivities which imply (see R. Baer, *Homogeneity of projective planes*, Amer. J. Math. vol. 64 (1942) Theorem 6.2) the minor theorem of Desargues. This in turn implies Theorems 1 and 2. (Received November 7, 1955.)

249. T. K. Pan: *Indicatric torsion in a subspace of a Riemannian space.*

A generalization of geodesic torsion of a curve in a surface of an ordinary space has been given and studied by the author in a former paper [Bull. Amer. Math. Soc. Abstract 61-2-335; Proc. Amer. Math. Soc. to appear]. Further generalization of this concept leads to indicatric torsion of a vector field in a subspace V_n of a Riemannian space V_m, which includes as a special case the geodesic torsion of a curve in a V_n in a V_m. Results corresponding to those in the preceding paper are obtained, some new properties are found. Let a vector field v along a curve C in V_n in V_m consist of unit tangent vectors to the orthogonal trajectories of C. The geodesic torsion of C in V_n in V_m, which corresponds to a vector N normal to V_n, is discovered to be identical with the normal curvature of v along C in V_n in V_m corresponding to N in a sense as introduced by the author [Amer. J. Math. vol. 74 (1952) pp. 955–966]. (Received November 7, 1955.)

250. C. E. Springer: *Union parallel displacement.*

It is known that if a vector with components λ^α is transported by Levi-Civita parallelism around an infinitesimal circuit in a Riemannian manifold, the change $\Delta\lambda^\alpha$ in the components is zero if, and only if, all of the components of the Riemann tensor vanish. In this paper, union parallelism is introduced. The usual components of connection represented by the Christoffel symbols are replaced by components of connection which appear in the differential equations of union curves with respect to a given rectilinear congruence. It is shown that a congruence may be chosen such that if the vector λ^α is displaced around an infinitesimal circuit in a union parallel manner with respect to this congruence, the vector λ^α returns to its original position. (Received November 7, 1955.)

251. Saly Ruth Struik: *Affine geometry. III. On the affine axiomatic theory of equal area and volume, defined by affine two-space and three-space reflection respectively.*

To the 5 groups of axioms conventionally accepted for affine geometry another axiom is added, the two-space affine reflection, to define equal area. Transitivity is included. The three-space affine reflection defines equality of volume, transitivity included. Using reflections addition of triangles and tetrahedrons is defined. Scales of measuring area and volume are reduced to the ratio of parallel vectors. Dehn's proof that compass and ruler fails to produce a regular tetrahedron half the size of a regular one makes it obvious that Hilbert's axiomatic theory of area has no three-space analogon. He (and Hilbert later in the second edition of the *Grundlagen*) indicates the necessity of an entirely other approach, e.g. the Cavalieri principle. The reflection does it, though by a definition narrower than the Cavalieri principle. Besides, to demand rigidity when comparing volumes is *not in keeping* with the spirit of the Erlanger program and would have *forestalled*, if adhered to, in history, the whole development of *integral calculus* which so fittingly was initiated by the dire necessity to accurately compare curved barrels of different shape. (Received November 14, 1955.)

252*t*. R. S. Underwood: *Algebraic results obtained by use of extended analytic geometry.*

Consider, by way of illustration, the simultaneous equations (1) $x^2+y^2+z^2+u+v=D$ and (2) $x^2+y^2/2+z^2/3+u^2/4+v^2/5=1$. Let $X=x^2+z^2+v$ and $Y=y^2+u$, so that the locus of (1) on the XY plane is the family of lines $X+Y=D$. By this plotting rule and the author's method (Amer. Math. Monthly vol. 61, pp. 525–542) the locus of (2) is found to be a convex area bounded by an ellipse, two parabolas, and a straight line. The slopes of adjacent curves coincide at the junction points. Simple calculation then shows that (1) and (2) have common real solutions only when $-3\leq D\leq 15/4$, the sole solutions at the respective end points being $(0, 0, 0, -4/3, -5/3)$ and $(0, 0, \pm 3/2, 2/3, 5/6)$. The method is applicable to similar equations in n variables when the first equation has some linear terms, though if it has only one such term families of parabolas instead of lines are used. (Received November 9, 1955.)

253. R. S. Underwood: *Some generalisations of theorems about quadric surfaces.*

It is known that if $P(x_0, y_0, z_0)$ is a point on a quadric surface the equation of the plane tangent to the surface at P is obtained by replacing x^2 by x_0x, xy by $(x_0y+xy_0)/2$, etc. If the surface is a hyperboloid of two sheets the tangent plane touches it at one point only, and the two equations have exactly one common real solution. On the other hand, a plane tangent to a hyperboloid of one sheet cuts through the surface. The question arises: Does the equation $\sum_1^m x_i^2/a_i^2 - \sum_{m+1}^n x_i^2/a_i^2 = 1$ have the algebraic properties associated with a hyperboloid of one or two sheets? The answer is: Two sheets when $m = 1$ and $n \geqq 3$, and one sheet when $1 < m < n$. Moreover, when the first degree equation is plotted as a straight line by the author's method (Amer. Math. Monthly vol. 61, pp. 525–542), the locus of the "quadric surface" becomes the silhouette of a hyperboloid of one or two sheets as the case may be. The one-sheet area-locus may appear as seen from any one of various positions, but in the other case the line clearly touches the locus of the "quadric" at exactly one point. (Received November 9, 1955.)

254*t*. G. A. Baker, Jr.: *A formulation of quantum mechanics based on the quasi-probability distribution induced on phase space.*

In a previous paper (*Equivalence theorem for quantum mechanics*, Bull. Amer. Math. Soc. Abstract 61-3-423) the author proved that a formulation of quantum mechanics based on a quasi-probability distribution is equivalent to the standard Schrödinger one. In this paper the formulation is extended to include measurement. A method is developed whereby, given the quasi-probability distribution representing an ensemble, one may compute the joint distribution of any group of physical quantities. This is a true distribution (no regions of "negative probability") if and only if the physical quantities are simultaneously measurable. In the course of this development, a method is obtained by which, in principle, any quantum mechanical eigenvalue problem may be solved exactly. The great flexibility of this approach is pointed out and the canonical invariance of quantum theory is investigated. Some useful mathematical properties are developed and they are used to introduce some very convenient notation. The framework herein postulated is believed to be complete for purely dynamical systems and to form a clear, logically consistent, self-contained basis for quantum mechanics. In addition, it provides one with an intuitive picture of what the system does in phase space. (Received October 31, 1955.)

255*t*. G. A. Baker, Jr.: *The relation of the quasi-probability distributional representation of quantum mechanics to classical mechanics and certain allied results.*

In the limit as Planck's constant, h, tends to zero, the formalism of the quasi-probability representation tends to the hamiltonian formalism of classical mechanics. Also approximating differential expressions are developed for key quantities of the theory. From these approximate expressions the conditions under which the interpretation suggested by Bohm (David Bohm, *A suggested interpretation of the quantum theory in terms of "hidden" variables*. I and II, Phys. Rev. vol. 85 (1952) p. 166) gives results equivalent to those of quantum theory are developed. (Received October 31, 1955.)

256*t*. G. A. Baker, Jr.: *A note on the physical principles underlying the quasi-probability distribution for quantum mechanics.*

The point distribution in phase space of Newtonian mechanics satisfies the equation $f = (0^+)ff$. In quantum mechanics one seeks to smear out the classical point. This alteration is accomplished by replacing (0^+) by Planck's constant, h, and ff by the product of f with itself with respect to a particular non-associative multiplication. With the aid of this change of the classical concept of a representation in terms of a single moving phase point, and Bohr's correspondence principle, the derivation of an equation equivalent to Schrödinger's wave equation follows. The effect of a measurement on a quantum mechanical system is seen to be the placement of the system in a state such that the expected value is an extremum in the sense of the calculus of variations with respect to the class of quasi-probability functions. The measured quantity may assume only these extreme values. These considerations suffice to establish the postulates of the quasi-probability distributional representation described in the preceding papers. (Received October 31, 1955.)

257*t*. Hugo Ribeiro: *Universal completeness for relational systems.*

For terminology see A. Tarski, *Contributions to the theory of models*, I and II, Proc. Dutch Acad. Sci. Ser. A, vol. 57, no. 5, and vol. 58, no 1. Equational completeness for abstract algebras has been defined by Kalicki and Scott and studied also by Tarski, Bull. Amer. Math. Soc. Abstracts 58-6-582, 59-1-116, 59-1-119, 60-2-202. A set, Σ, of sentences of the formalized theory $T(R)$ constructed for a given similarity class, R, of relational systems is said to be universally complete if and only if for any universal sentence ϕ, of $T(R)$, either every model of Σ is a model of ϕ or there is a universal sentence, α, involving only the identity symbol, together with the other logical constants, such that every model of Σ is a model of ϕ if and only if it is a model of α. The set of sentences defining the class of all systems $\langle A, R\rangle$ where A is a nonvoid set and R is a binary relation which orders A linearly is universally complete. (Received November 14, 1955.)

258*t*. H. D. Sprinkle: *A primitive recursive foundation for randomness.* Preliminary report.

Recursive function theory is used to define the notion of an element being random with respect to certain sequences (probability collectives). The main theorem proves the existence of a probability collective (of any finite cardinal) such that each element of it is random with respect to the collective. The proof is constructive if there is an effective well-ordering of a certain subclass of primitive recursive functions. (Received November 7, 1955.)

STATISTICS AND PROBABILITY

259. G. E. Baxter and M. D. Donsker (p): *On the zeros of the sample functions of certain gaussian processes.*

Let $\{y(s),\ 0\leq s<\infty\}$ be the Wiener process. For each fixed $t>0$, the integral $x(t)=\int_0^\infty e^{-ts}dy(s)$ exists with probability one, and the process, $\{x(t),\ 0<t<\infty\}$, is Gaussian with zero mean function and covariance function $(t_1+t_2)^{-1}$. The sample functions $x(t)$ are analytic with probability one. This paper is principally concerned with the question: what is the expected number of crossings that the sample functions $x(t)$ make with a curve $r(t)$ on a given interval (a, b)? For analytic $r(t)$ an explicit answer is obtained. The corresponding result is derived for Gaussian processes obtained from the Wiener process by the transformation $z(t)=\int_0^\infty e^{-g(t)s}dy(s)$ where $g(t)$ is analytic. The following theorem is also obtained: let $f(t)$ be an increasing continuous function on (a, ∞) vanishing at $a>0$. The Gaussian process, $\{z(t),\ a\leq t<\infty\}$, whose mean function is zero and whose covariance function is $(e^{2\pi f(t_1)}+e^{2\pi f(t_2)})^{-1}$ has exactly $f(T)$ as its expected number of zeros on the interval (a, T) for every T. The preceding results are used to obtain certain strong limit theorems on the asymptotic behavior of the sample functions of Gaussian processes obtained from the Wiener process by the transformations above. (Received November 14, 1955.)

260t. S. S. Cairns: *Balance scale sorting.*

Given (1) a set W of n objects, indistinguishable save that the members of a subset H of k objects are slightly heavier than the rest, (2) a balance scale, one seeks weighing programs minimizing either [Problem $M(n, k)$] the maximum number of weighings which may be required to cull out H or [Problem $E(n, k)$] the expected number of such weighings. Problem $M(n, 1)$ is a familiar puzzle. Problem $E(n, 1)$ is here solved, under various hypotheses. Problem $M(n, 2)$ is partially solved. The methods and results may serve as a basis for more extended work on Problems $M(n, k)$ and $E(n, k)$. Some of the techniques, moreover, involve manipulations of series which may prove useful in other combinatorial problems. (Received November 14, 1955.)

261t. K. L. Chung: *Some new developments in Markov chains.*

The new concept of a post-exit process imbedded in a continuous parameter Markov chain is introduced. It is shown that this process has the same transition probabilities as the original chain but its initial distribution is of a novel kind. An imbedded renewal process is also studied. Together they yield various relations implying the differentiability of the transition probability functions and certain generalized Kolmogorov differential equations. (Received November 15, 1955.)

262t. J. C. Kiefer and Jacob Wolfowitz: *Consistency of the maximum likelihood estimator in the presence of infinitely many incidental parameters.*

The authors show, under usual regularity conditions, that the maximum likelihood estimator of a structural parameter is strongly consistent, when the (infinitely many) incidental parameters are independently distributed chance variables with a common unknown distribution function. The latter is also consistently estimated although it is not assumed to belong to a parametric class. Application is made to several problems, in particular to the problem of estimating a straight line with both variables subject to error, which thus after all has a maximum likelihood solution. (Received September 12, 1955.)

263. Klaus Krickeberg: *Convergence of nondenumerable martingales.* Preliminary report.

Since a Boolean σ-algebra $\mathfrak{B}$ with unity E bearing a finite strictly positive measure μ and the vector lattice $\mathfrak{F}$ of all $\mathfrak{B}$-measurable real functions on E are complete, every Moore-Smith sequence $(f_\sigma; \sigma \in \Theta)$ in $\mathfrak{F}$, with an index set Θ directed by $\ll$, has an upper and lower limit in $\mathfrak{F}$. In a previous paper of the author the usual theorems on interchanging integral and limits have been generalized to the present case, and criteria for $\int_E (\limsup f_\sigma) d\mu \leqq \limsup \int_E f_\sigma d\mu$ derived. An application yields convergence theorems on martingales with index set Θ. The sequence (f_σ) is called a martingale relative to a monotone increasing sequence $(\mathfrak{B}_\sigma; \sigma \in \Theta)$ of Boolean σ-subalgebras of $\mathfrak{B}$, if f_σ is $\mathfrak{B}_\sigma$-measurable and summable over E, and $\int_A f_\sigma d\mu = \int_A f_\tau d\mu$ for $A \in \mathfrak{B}_\tau$ and $\tau \ll \sigma$. For fixed $(\mathfrak{B}_\sigma)$, the space of all martingales relative to $(\mathfrak{B}_\sigma)$ is a complete vector lattice. $(\mathfrak{B}_\sigma)$ is said to enjoy Vitali's property, if for any $\epsilon > 0$, any $A \in \mathfrak{B}$, and any sequence $(K_\sigma; \sigma \in \Theta)$ with $K_\sigma \in \mathfrak{B}_\sigma$ and $A \leqq \bigvee \{K_\sigma; \tau \ll \sigma\}$ for every τ, there exist a finite subset $\{\xi_1, \cdots, \xi_r\}$ of Θ and disjoint $L_i \in \mathfrak{B}_{\xi_i}$, $i = 1, \cdots, r$, such that $L_i \leq K_{\xi_i}$ and $\mu(A - (A \wedge \vee L_i)) < \epsilon$. Under this assumption many theorems known before only in the case $\Theta = (1, 2, \cdots)$ still hold, e.g. convergence of (f_σ), if $\sup \int_E |f_\sigma| d\mu < +\infty$. Vitali's condition is indispensable here, as shown by examples, but is satisfied, if $\ll$ is a linear order. (Received November 10, 1955.)

264*t*. J. M. Shapiro: *Convergence of moments of sums of truncated random variables.*

For $n = 1, 2, \cdots$ let $S_n = x_{n1} + \cdots + x_{nk_n}$ be the sum of independent, infinitesimal, random variables, and let $F_n(x)$ be the distribution function of S_n. Assume (A): there exists an $a > 0$ such that $F_{nk}(x) = 0$ for $x \leq -a$ and $F_{nk}(x) = 1$ for $x \geq a$ where $F_{nk}(x)$ is the distribution function of x_{nk}. In a previous paper (Bull. Amer. Math. Soc. Abstract 61-3-435) the author has shown that if $F_n(x)$ converges to a limiting distribution $F(x)$ then $F(x)$ has finite moments of all orders. Assuming $F_n(x) \rightarrow F(x)$ it is shown that for each positive integer k the kth moment of $F_n(x)$ approaches the kth moment of $F(x)$ as $n \rightarrow \infty$. If the assumption (A) is not made let $x^*_{nk} = x_{nk}$ if $|x_{nk}| < a$, $x^*_{nk} = 0$ otherwise, and let $F^*_n(x)$ be the distribution function of $S^*_n = x^*_{n1} + \cdots + x^*_{nk_n}$. Using the results of the above paper conditions are given for the existence of a function $F^*(x)$ such that the kth moment of $F^*_n(x)$ approaches the kth moment of $F^*(x)$ and the form of $F^*(x)$ is also discussed. (Received November 14, 1955.)

TOPOLOGY

265. L. W. Anderson: *Cutpoints in topological lattices.*

A topological lattice is a triple $(L, \wedge, \vee)$ where L is a Hausdorff space and $\wedge: L \times L \rightarrow L$ and $\vee: L \times L \rightarrow L$ are continuous functions satisfying the usual conditions stipulated for a lattice. It is shown that if L is a connected topological lattice and if $a \in L$ then a is related to every element of L (i.e. $a \leq x$ or $x \leq a$ for all $x \in L$) if, and only if, either $a = 0$ or $a = 1$ or a is a cutpoint of L. It follows from this result that (1) a connected topological lattice has at most two end points, (2) a connected topological chain with 0 and 1 is irreducibly connected about 0 and 1. (Received November 15, 1955.)

266. R. D. Anderson: *Atomic decompositions of continua.*

An upper semi-continuous decomposition G of a continuum M into continua is said to be atomic provided that if K is any subcontinuum of M intersecting two elements of G, then K contains each of the elements of G which it intersects. The author shows that if S is any continuum (compact, metric, connected set) and n is any positive integer, then there exist a continuum M and a monotone open mapping f of M onto S such that the collection of inverses of the points of S under f is atomic, and each of M and the various inverse sets under f contains an n-cell and is of dimension n. The conclusion may be altered to assert that each of M and the inverse sets contains a homeomorphic image of the Hilbert cube. The argument used to establish this result is a modification of the author's proof of the existence of monotone interior dimension raising mappings (Duke Math. J. vol. 19 (1952) pp. 359–366). (Received November 14, 1955.)

267*t*. R. D. Anderson and Gustave Choquet: *Plane continua, no two nondegenerate subcontinua of which are homeomorphic.*

The authors give a construction of a plane continuum such that no two subcontinua of it are homeomorphic to each other. The continuum cited has the additional properties that (1) No subcontinuum of it separates the plane, (2) It does not contain uncountably many mutually exclusive non-degenerate subcontinua, and (3) Each non-degenerate subcontinuum admits an irreducible monotone mapping onto a continuum which is the sum of finitely many arcs, each pair having exactly an end point of all in common. A somewhat similar alternative construction is suggested yielding such a continuum having the property, in lieu of (1) and (3) above, that every non-degenerate subcontinuum of it separates the plane. (Received November 14, 1955.)

268. R. W. Bagley: *Transfinite sequential order topology.*

It is shown that transfinite sequences are not adequate to describe the topology of pointwise convergence on $2^{\mathfrak{R}}$ (taking the discrete topology on $\{0, 1\}$) when $\mathfrak{R}$ is uncountable. Here, "transfinite sequence" means a function on a well ordered set into a set. This result holds for the Tychonoff cube as well as any direct product of lattices which contains $2^{\mathfrak{R}}$ as a sublattice. In each case it is assumed that there are uncountably many coordinate spaces. The transfinite sequential order topology on a complete lattice is defined in a manner analogous to the sequential order topology and the order topology (G. Birkhoff, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloquium Publications, 1948). The above result follows from the fact that, when $\mathfrak{R}$ is uncountable, the transfinite sequential order topology is properly less than the order topology and properly greater than the sequential order topology. The order topology coincides with the topology of pointwise convergence (O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 569–582 and J. W. Tukey, *Convergence and uniformity in topology*, Princeton, 1940). (Received November 14, 1955.)

269. Lida K. Barrett: *On a question raised by R. H. Bing.*

R. H. Bing in *Partitioning continuous curves* (Bull. Amer. Math. Soc. vol. 58 (1952) p. 541) raises the following question, which if answered in the affirmative would give an alternate method of partitioning continuous curves. *Question*: Does there exist a positive integer n such that the following holds for each continuous curve M, each positive number ϵ, and each pair of mutually exclusive closed subsets H and K of M? If R is a finite subset of M such that each point of R belongs to an arc in M of diameter less than ϵ that intersects $H+K$, there are two collections A_H and A_K of arcs satisfying the following conditions: (a) Each element of A_H intersects H but not K and each element of A_K intersects K but not H nor any element of A_H. (b) Each element of R belongs to an element of A_H+A_K. (c) Each element of A_H+A_K is of diameter less than $n\epsilon$. Bing gives an example composed of two arcs to show that the question cannot be answered in the affirmative for $n=1$. The present paper gives an example composed of twenty-one arcs to show that it cannot be answered in the affirmative for $n=2$. (Received November 4, 1955.)

270. R. H. Bing: *An alternate proof that 3-manifolds can be triangulated.*

The author's approximation theorem for surfaces is extended to apply to topological 2-complexes. The approximation theorem for 2-complexes may be stated as follows. If in E^3, C is a topological 2-complex and f is a nonnegative continuous function defined on C, then there is a topological 2-complex C' and a homeomorphism h of C onto C' such that h moves no point x of C by more than $f(x)$ and C' is locally polyhedral at $h(x)$ if $f(x)>0$. This theorem has interesting applications when applied to the 2-skeleton of a triangulation of an open subset U of E^3. It shows that if f is a continuous nonnegative function on U and h is a homeomorphism of U into E^3, then there is a homeomorphism h' of U into E^3 such that $\rho(h(x), h'(x)) \leq f(x)$ and h' is piecewise linear at x if $f(x) > 0$. It may likewise be used to show that each 3-manifold can be triangulated. These last two results are due to Moise who proved them by other methods. (Received November 14, 1955.)

271. A. H. Copeland, Jr.: *The Whitehead bracket product on CW-complexes.*

Suppose X, N, M are CW-complexes with X being 1-connected, N, n-connected, and M, m-connected $(1 < n \leq m)$, $\pi_n(N) \approx \pi_n(X)$, $\pi_m(M) \approx \pi_m(X)$. Let $Y \supset X$ be a realization of the $(n+m-1)$ Postnikov complex associated with X, and $k \in H^{n+m}(Y; \pi_{n+m-1}(X))$ the Postnikov invarient. In addition, suppose that there are maps $\xi: N \rightarrow Y$, $\eta: M \rightarrow Y$ such that the induced homomorphisms $\xi_\#: \pi_n(N) \rightarrow \pi_n(Y)$, $\eta_\#: \pi_m(M) \rightarrow \pi_m(Y)$ are isomorphisms. The maps ξ, η give rise to a map $\theta: N \times M \rightarrow Y$. The element $\theta^* k \in H^{n+m}(N \times M; \pi_{n+m-1}(X))$ defines an element $W \in \mathrm{Hom}\,(\pi_n(X) \otimes \pi_m(X); \pi_{n+m-1}(X))$ such that $W(\alpha \otimes \beta) = [\alpha, \beta]$ for $\alpha \in \pi_n(X)$, $\beta \in \pi_m(X)$. In some cases this construction may be used to give an effective procedure for finding bracket products on X. (Received November 14, 1955.)

272. M. L. Curtis: *A spectral sequence associated with a topological space.*

The singular chain group $C(X)$ of a topological space X can be given a decreasing filtration by letting $C^p(X)$ be the subgroup of chains of $C(X)$ which are obtainable from the p-iterated loops by a transgression on chains. This filtration gives rise to a certain exact couple which, in turn, gives rise to a spectral sequence. The term E_∞ consists of images of $\pi_k(X)$ in $H_k(X)$. The term E_1 can be interpreted as follows. Let K be the total singular complex of X. There exists a sequence of subcomplexes $K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \cdots$ such that E_1 is the direct sum of the relative homology groups $H_r(K_m, K_{m+1})$. Instead of the total singular complex one may use a minimal subcomplex of K. (Received November 14, 1955.)

273. M. L. Curtis and R. L. Wilder (p): *The existence of certain types of manifolds.*

The first part of the paper gives a class of polyhedral 3-manifolds M, each separating the 4-sphere S^4 and with homology groups $H_1(M) (= H_2(M)) = 0$, but with $\pi_1(M) \neq 0$; thus some "Poincaré spaces" are imbeddable in S^4. Also, given any finitely-generated abelian group G, there exists a 3-manifold M in S^4 with $H_1(M) = G$. In the second part of the paper a negative answer is found for the question: Since the metric cases of the 2-gcms reduce to the classical types, and the 3-gcms generally do not, can one state that a sphere-like 3-gcm in euclidean space E^4 must be a classical manifold? 3-gcms are constructed in E^4 which have the same homology and homotopy character as the 3-sphere (globally and locally), but which are not locally euclidean. In the third part of the paper a negative answer is found for a question proposed by Griffiths (Michigan Mathematical Journal vol. 2 (1953) pp. 61–89, Section 6.19) by showing the existence of a 3-dimensional "homotopy manifold" which is not locally euclidean (this is precisely a space previously obtained by R. H. Bing by identifying the points on a certain set of tame arcs in S^3). (Received November 14, 1955.)

274. W. F. Davison: *Convergent sequences and mosaics.*

Certain topological spaces are investigated, wherein sequences are the principal associated convergence objects. A *mosaic* of compact metric spaces is a collection $\{(X_a, T_a): a\in A\}$ such that each (X_a, T_a) is compact metric, and E being T_a-closed implies $E\cap X_b$ is T_b-closed for $a, b\in A$. The *mosaic topology* T on $X=\bigcup\{X_a: a\in A\}$ is obtained by E being T-closed provided $E\cap X_a$ is T_a-closed for all $a\in A$. In mosaic spaces the following hold: (i) limits of T-convergent sequences are unique; (ii) the topology derived from the T-convergent sequences is T itself; (iii) countable compactness is equivalent to sequential compactness; and (iv) countably compact sets are closed. A space satisfying (i) and (ii) is the mosaic space of some mosaic of compact metric spaces. In mosaic spaces open sets and closed sets are mosaic spaces in their relative topology, and the following are equivalent: (1) every subspace is a mosaic space; (2) a limit point of a set is the limit of some sequence in the set; (3) $\{x_{nm}\}$ converging to x_n and $\{x_n\}$ converging to x imply a sequence in the set of $\{x_{nm}\}$ converges to x; and (4) sequence closure is topological closure. Finally the construction of mosaic spaces is generalized, and a non-Hausdorff mosaic space is exhibited. (Received November 14, 1955.)

275. Eldon Dyer: *Open mappings and regular convergence.*

If G is a collection of compact subspaces of a compact metric space X, $\mathfrak{G}$ denotes the metric space in which the points are the elements of G and the metric is the Hausdorff metric between elements of G. J. H. Roberts has shown (Duke Math. J. vol. 2 (1936) pp. 10–19) that if G is a continuous collection of mutually exclusive arcs filling X, there is a subcollection G_1 of G such that $\mathfrak{G}_1$ is a dense G_δ subset of $\mathfrak{G}$ and every element g of G_1 is approached 0-regularly by every sequence of elements of G converging to g. In the present paper it is shown that if n is a non-negative integer and G is a collection of compact lc^n subspaces of X such that $\mathfrak{G}$ is a topologically complete space, then there is a sub-collection G_1 of G such that $\mathfrak{G}_1$ is a dense G_δ subset of $\mathfrak{G}$ and every element g of G_1 is approached n-regularly by every sequence of elements of G converging to g. It is also shown that if f is an open mapping of an open subset of an n-sphere into a metric space M such that for each point x of M, $f^{-1}(x)$ is compact, lc^n, and homologically trivial, then f is a homeomorphism. The coefficient group is any field or elementary compact group. (Received November 14, 1955.)

276. J. G. Horne, Jr.: *On o-ideals in* $C(X)$.

We study o-ideals of Milgram in a class of sub-semigroups of $C(X)$ ($=$semigroup under multiplication of all continuous complex functions on the topological space X) which Civin and Yood have called *Silov* semigroups. In a commutative ring $(R, +, \cdot)$ with unit, a semigroup $S\subset(R, \cdot)$ is Silov if and only if three elements e_1, e_2, e_3, in R, related by the equations $e_2e_1=e_1$ and $e_3e_2=e_2$, yield an $e_2'\in S$ such that $e_2'e_1=e_1$ and $e_2'e_3=e_2'$ hold. For such semigroups S we have the equivalence of (1) I is an o-ideal in S, and (2) there is an o-ideal I' of $(R, \cdot)$ such that $I=I'\cap S$. Further, the correspondence $I'\rightarrow I'\cap S$ is one-to-one. In the case $(R, \cdot)=(C(X), \cdot)$ then (1) and (2) are equivalent to (3) there exists a completely regular filter $g(I)$ of open sets in X such that $I=\{f\in S;\ Z_f\supset G,$ for some $G\in g(I)\}$. I is a maximal if and only if $g(I)$ is a completely regular end. A completely regular filter $\mathcal{G}$ of open sets is an ideal in the lattice of open sets such that for each $G_1\in\mathcal{G}$, there is a $G\in\mathcal{G}$ and a continuous real function e which vanishes on G_0 and is one outside G_1. (Received November 14, 1955.)

277. I. S. Krule: *Structs on the* 1*-sphere.*

Let X be the 1-sphere and let L be a reflexive monotone continuous struct on X. (For terminology see A. D. Wallace, *Struct ideals*, Proc. Amer. Math. Soc. vol. 6

(1955) pp. 634–638.) For $A \subset X$ let $L \cap (A \times A)$ be called the *L-induced relation on A*. Let $G_L = \{x \in X: L(x) = X\}$. The following result is proved: *If $x \in K_L$ then $L_x = x$ or $L_x = K_L$. If $X \neq K_L$ then* (1) *G_L consists of exactly one point*; (2) *if C_1 and C_2 are the components of $X - (K_L \cup G_L)$ then $L(C_1) \cap C_2 \neq \square \neq C_1 \cap L(C_2)$*; (3) *if $x \in X - K_L$ then $L_x = x$ or L_x consists of exactly two points, one in C_1 and one in C_2*; (4) *if R_i is the L-induced relation on $\overline{C}_i$ then $\overline{C}_i$ is an ordered space with respect to $R_i (i = 1, 2)$*; (5) *if $x \in \overline{(X - K_L)}$ then $\sigma L(x)$ is connected*; (6) *$G_L \cup \{x \in X: L_x \neq x\}$ is closed*; (7) *the quotient space $X/(L \cap \sigma L)$ is a one-dimensional cyclic chain which is topologically embeddable in the plane*. (Received November 10, 1955.)

278. P. S. Mostert (p) and A. L. Shields: *On semigroups on a manifold.*

Let S be a topological semigroup with identity 1. *Theorem.* If there is a neighborhood of 1 homeomorphic to a Euclidean space, then there is an open, connected Lie subgroup of S containing 1. *Corollary.* Let (S, B) be a compact relative manifold with boundary (i.e., B is a closed subset of S and $S - B$ is locally Euclidean). If S is a topological semigroup with identity, then, if $H(1)$ denotes the maximal subgroup of 1, either $H(1) = S$, or $H(1) \subset B$. (Received November 14, 1955.)

279*t*. C. D. Papakyriakopoulos: *On a lemma of Kneser.*

The following theorem is proved: *Let M be a 3-manifold, compact or not, with boundary N formed by a number* (>0, $\leq \infty$) *of surfaces closed or not. Let L be a loop belonging to an open set U of an orientable component N' of N such that $L \simeq 0$ in M, and not $\simeq 0$ on N (the symbol $\simeq$ means homotopic to). Then there exists a simple (= without multiple points) loop L_0 in U such that $L_0 \simeq 0$ in M, and not $\simeq 0$ on N. The first to study* problems of this sort was H. Kneser (*Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jber. Deutschen Math. Verein vol. 38 (1929) pp. 248–260, Hilfsatz of p. 248). J. H. C. Whitehead proved an important special case of the theorem (*On doubled knots*, J. London Math. Soc. vol. 12 (1937) pp. 63–71, lemma of p. 65). The proof of the theorem makes use of Whitehead's result. A consequence of the theorem is a negative answer to a problem of Bing (*Summary of Lectures and Seminars*, Summer Institute on Set Theoretic Topology, Madison, Wisconsin (1955) p. 57, problem 12). (Received November 4, 1955.)

280. B. J. Pearson: *A connected point set in the plane which spirals down on each of its points.*

Let ABC be a triangle in the plane and G the collection of all straight line intervals with one end point at A and the other end point on the interval BC. If G' is a collection of arcs in the plane such that (1) each arc of G is a proper subarc of some arc of G', (2) each arc of G' has one end point at A and contains some arc of G, and (3) no two arcs of G' have any point in common except A, then G' is said to be an extension of G. It is shown that there exist an extension G' of G and a compact, connected inner limiting set M not containing A which is connected im kleinen at all but a countable number of its points such that (1) each point of M is an end point of some arc of G', (2) each arc of G' has one end point in M, and (3) each arc of G' spirals down on the point of M belonging to it but does not spiral down on any other point. (Received November 14, 1955.)

281*t*. J. P. Roth: *A combinatorial topological method for the synthesis of switching systems in n variables.*

The switching circuit problem, arising e.g. in the design of telephone switching systems, computing machines, and control systems, has previously been treated from the viewpoint of symbolic logic, notably by Shannon and Quine. There exists no effective mechanical algorithm for finding circuits performing prescribed functions and having a minimum number of switches. Here the problem is considered to define a cubical complex; a routine, "fast" mechanical process is given which automatically gives a switching system in n variables which has a minimum number of switches over a very restricted class of series-parallel circuits. Other operations are defined which permit further reduction of the system, allowing for bridge circuits, etc. But no general procedure has yet been found to obtain the absolute minimum. (Received November 15, 1955.)

282. R. F. Williams: *Concerning dimension raising mappings of cells.*

It is shown that if n and m are integers, $m>n>1$, and f is a mapping of an n-cell I^n into an m-cell I^m, then there exists a sub-continuum of I^n whose image under f is of dimension $\leq m-2$. The proof is based on a category argument in the space of sub-continua of I^n and the existence of certain (familiar) $m-2$ dimensional continua in I^m. (Received November 14, 1955.)

J. W. T. Youngs,
Associate Secretary

BOOK REVIEWS

Analytische Fortsetzung. By L. Bieberbach. (Ergebnisse der Mathematik und ihrer Grenzgebiete, N.S., no. 3.) Berlin, Springer, 1955. 4+168 pp. 24.80 D.M.

The general scope of this monograph is approximately the same as the Chapter of the same title in the author's *Lehrbuch der Funktionentheorie* (Chap. 7, vol. II, Teubner, 1931) brought up to date by the developments of the last 25 years. Much of this was initiated by Pólya and those who have been influenced by him. The central problem can be stated somewhat as follows: how are properties of the coefficient sequence $\{a_n\}$ reflected in the behavior of $\sum a_n z^n$, and conversely? Three chief tools form the basis for the development, and are discussed in Chapter 1; these are the method of associated functions, linked with the Laplace-Borel transform, the Hadamard multiplication theorem, and the Euler series transformation. If $A(z) = \sum_0^\infty c_n z^n/n!$ is an entire function of exponential type, its Borel transform is $a(z) = \sum_0^\infty c_n/z^{n+1}$. The rate of growth of A along rays is closely connected with the region of regularity of $a(z)$; in turn, one may immediately obtain information about the location of singularities of the associated function $g(z) = \sum_0^\infty A(n)z^n$. The classical multiplication theorem of Hadamard deals with the three power series, $a(z) = \sum a_n z^n$, $b(z) = \sum b_n z^n$, and $c(z) = \sum a_n b_n z^n$. An open set S is a star (at 0) if $\lambda S < S$ for all λ, $0 \leqq \lambda \leqq 1$. If A and B are star sets, their star-product is $A \odot B = (A' \cdot B')'$. If $a(z)$ is regular for $z \in A$, and $b(z)$ for $z \in B$, then $c(z)$ is regular at least in $A \odot B$. It is not true [as other books have not always pointed out] that the singularities of $c(z)$ necessarily have the form $\alpha\beta$ where α and β are, respectively, singularities of $a(z)$ and $b(z)$. However, every suitably restricted boundary point of $A \odot B$ which is a singularity for $c(z)$ has this form. The Euler transformation provides a simple necessary and sufficient condition that a power series $\sum a_n z^n$ have 1 as a singularity.

It is difficult to give more than an indication of the skill and elegance with which the author combines these simple and familiar tools to obtain a unified treatment of a selected portion of the theory of Taylor series. In Chapters 2 and 3, one finds a discussion of the effect of coefficient gaps upon the presence and location of singularities of a power series, starting with the general theorems of Fabry and Pólya, and ending with more detailed and specialized theorems of Wilson, MacIntyre, Mandelbrojt and many others. In particular, §2.1 is devoted to a clarification [with enlightening comments] of Fabry's original proof. In Chapter 4, the subject of study is the class of power series with unit radius of convergence, and various justifications of the general remark that most of these have the unit circle for a cut; for example [Boerner], the set of $\beta=(\beta_0, \beta_1, \cdots)$ with $|\beta_j| \leq \pi$ for which $\sum a_n \exp(i\beta_n)z^n$ is continuable has measure zero, while [Pólya-Hausdorff] the class of non-continuable power series is open and dense in the space of power series.

Another general remark of somewhat similar nature is that whenever $\{a_n\}$ is a sufficiently "nice" sequence, $\sum a_n z^n$ is either rational, or has the circle of convergence for a cut. This is illustrated by the material of Chapter 6 which revolves around the classical theorems of Eisenstein, Szegö, Pólya, and Carlson. As an instance of the method of associated functions, the author digresses in §6.3 to include a sketch of some known results dealing with integral valued entire functions. If $A(z)$ is an entire function of exponential type such that $A(n) \in D$ for $n=0, 1, \cdots$ where D is a domain of algebraic integers, then one may consider the associated function $g(z) = \sum_0^\infty A(n)z^n$. If A is of sufficiently slow growth, $g(z)$ is regular in a set of mapping radius greater than 1; if D is either the rational integers, or a quadratic complex domain, then the Pólya-Carlson theorem may be applied to show that $g(z)$ is a rational function, from which the form of $A(z)$ may be obtained. Finally, Chapter 5 deals with certain consequences of the Hadamard multiplication theorem, and Chapter 7 discusses the connection between a power series $\sum a_n z^n$ and the related series $\sum \phi(a_n)z^n$ where $\phi(z)$ is a preassigned analytic function.

The author has not entered upon the general coefficient problems for schlicht functions and for bounded functions, nor has he discussed analytic continuation of power series by means of summability. Within his chosen framework, he has produced a remarkably interesting and coherent summary of recent work and literature.

R. C. BUCK

Theory of differential equations. By E. A. Coddington and N. Levinson. New York, McGraw-Hill, 1955. 14+429 pp. $8.50.

It has become fashionable of late, in various mathematical centers, to present the fundamental tools of analysis, real and complex variable theory, in an increasingly abstract manner to those most defenseless, namely fledgling graduate students. In the process, motivation for the introduction of new concepts has been on the whole by-passed as an atrophied relic of those early pioneer days when mathematicians were forced to consort with astronomers and physicists, and indeed, in some cases, were indistinguishable from them. It may be true, howsoever improbable, that mutation or inbreeding has produced a new type of mathematical mind that can absorb winged words before pedestrian ones. It is doubtful, however, whether this species can perpetuate itself. Most likely, mathematical education will continue along the same simple logical principles that have guided the great scientists of the past, from the simple to the complex, from the concrete to the abstract. If these premises be accepted, a very good case can be made for choosing the subject of differential equations, with its ramifications, as a basic course of study for first-year graduate students.

To begin with, existence and uniqueness theorems furnish an excellent testing ground for introducing new concepts and techniques and for training the skeptical mind. Three fundamental methods of analysis, successive approximations, finite differences, and fixed-point techniques, all enter in very natural fashion. The last method may serve, if one wishes, as an introduction to the general study of differential and integral operators and functionals. Alternately, in connection with finite difference and successive approximations, one enters the new domain of the study of numerical methods in relation to the optimal use of high-speed computers.

Turning to the study of specific classes of equations, matrix theory enters as the elegant approach to the treatment of linear equations. Those with constant or linear coefficients afford an excellent showcase for the Laplace transform. The consideration of basic physical problems leads to Sturm-Liouville problems. By way of the basic principle of superposition, we encounter Fourier series and Fourier integrals, these last occasionally requiring Stieltjes integrals. Alternate approaches lead to the theory of integral equations, this time of Fredholm type, and to Rayleigh-Ritz principles, a domain of great richness.

The concept of generalized solution and summability methods enter as soon as Fourier expansions are used to solve initial value and boundary problems. Furthermore, complex variable methods are an essential tool, following Cauchy, in discussing the expansion problem for general Sturm-Liouville problems. Expansion of the solution of a linear equation with polynomial coefficients as a power series introduces the concept of asymptotic series, and expansion in terms of a parameter appearing in the equation may be treated by means of the same ideas.

Finally, we enter the nonlinear domain, where the principle of superposition is a wistful memory. Immediately, we encounter the stability problem, a paramount concept in the study of functional equations. The study of periodic solutions of differential equations, of particular significance in the nonlinear case, leads, following the footsteps of Poincaré, to the study of families of curves and surfaces, and so into the field of topology. Systems with more than two degrees of freedom introduce almost-periodicity as the suitable substitute for the more restrictive periodicity.

With differential equations as a springboard, practically the whole sea of analysis lies before one.

If the above program sounds interesting, the book by Coddington and Levinson on differential equations will serve as an admirable text as far as both range and virtuosity are concerned. In any case, the book stands as an excellent and comprehensive survey of much of the modern theory of differential equations.

Let us now summarize the contents of the volume. The first two chapters are devoted to the question of the existence and uniqueness of solutions, and the dependence of the solution upon parameters and initial values. The next four chapters contain a study of linear systems, with constant, periodic and rational coefficients. A number of results due to the authors in connection with the asymptotic behavior of solutions is contained in these chapters. In particular, there is an interesting application of Phragmén-Lindelöf theorems to the study of asymptotic series in the complex plane. Those interested in parts of the theory of linear equations with periodic coefficients not covered in the present text may refer to V. M. Starzinskii, *A survey of works on the conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients*, American Mathematical Society Translations Project, series 2, vol. 1.

The following five chapters contain a discussion of Sturm-Liouville problems, for both second order and general nth order equations, including a chapter devoted to non-self-adjoint problems, an area of great current interest. The method used goes back to Cauchy, depending upon Green's function and the method of residues. The details, however, are far from trivial, and much of the content of these chapters was developed by the authors. The next chapter is devoted to an exposition of the stability theory of Poincaré and Liapounoff, as developed and perfected by Perron. There is, however, no reference to the great volume of work done by the Russian school since Liapounoff, contained in the papers of Malkin, Persidiski, and others. In two succeeding chapters, there is a detailed study of the perturbation theory of systems with periodic solutions. The penultimate chapter presents the Poincaré-Bendixson theory of two-dimensional orbits, and the last chapter treats the difficult topic of periodic solutions on a torus.

An important feature of the book is the inclusion of approximately one hundred and seventy-seven problems of varying degrees of difficulty, with hints as to the solution. This greatly increases the scope of the book. Finally, let us note that the book is printed in the attractive easy-on-the-eyes style which we have grown to expect from McGraw-Hill.

RICHARD BELLMAN

Elemente der Funktionalanalysis. By L. A. Ljusternik and W. I. Sobolew. (Mathematische Lehrbücher und Monographien, vol. 8.) Berlin, Akademie-Verlag, 1955. 10+253 pp. 25.00 DM.

This book is a translation of *Elementy funkcional'nogo analiza* [Gostehizdat, Moscow-Leningrad, 1951], which was reviewed in Mathematical Reviews in Vol. 14 (1953) p. 54. It is an introduction to the theory of normed linear spaces and operations on them, and contains few surprises. It is more complete and more sophisticated than *Elementy teorii funkcii i funkcional'nogo analiza* by Kolmogorov and Fomin [Izdatel'stvo Mosk. Universiteta, 1954], and is totally different from *Leçons d'analyse fonctionnelle* by Riesz and Sz.-Nagy [Akadémiai Kiadó, Budapest, 1952]. There are many points of contact with Banach's classical monograph *Théorie des opérations linéaires* [Monografje Matematyczne, Warszawa, 1932]. So far as the reviewer knows, there is no single treatise in English covering the same material as the book under review.

The reader is tacitly expected to know the elementary theory of functions of a real variable: continuity; differentiation; functions of finite variation; Lebesgue integration on $[0, 1]$. For students with this background, the book is highly recommended as an introduction to functional analysis.

Chapter I deals with metric spaces. The only nonstandard topic here is Banach's theorem on contracting mappings: a mapping A of a space X (with metric ρ) into itself such that $\rho(Ax, Ay) \leq \alpha\rho(x, y)$ for all $x, y \in X$ and some α, $0 < \alpha < 1$, admits exactly one fixed point. Several applications of this theorem are given. Chapters II and III deal with linear spaces, linear operators, and linear functionals; this part bears a strong family resemblance to Banach's book. Several interesting and not universally known facts are given: for example, if E is a Banach space with conjugate space $\bar{E}$, and if E is not reflexive, then

$$E, \bar{E}, \bar{\bar{E}}, \bar{\bar{\bar{E}}}, \ldots$$

are all different (Plessner). Banach's theorem that $C([0, 1])$ is universal for separable Banach spaces is stated and proved. Chapter IV, dealing with completely continuous operators, is standard and also good. Chapter V contains a standard discussion of the spectral theorem for self-adjoint bounded operators on a separable Hilbert space. Chapter VI, the last in the book, is credited mainly to Ljusternik. It deals with nonlinear functional analysis. The principal topics are: derivatives and integrals for functions on $[0, 1]$ with values in a Banach space; Fréchet's differential; implicit function theorems for Banach-space valued functions; extreme values and their calculation.

Limitations of the book in subject matter, perhaps justified in an introductory textbook, are the following. The L_p-spaces dealt with are all on $[0, 1]$. The only space of continuous functions considered is $C([0, 1])$. Compactness for nonmetric spaces is ignored: in discussing the unit ball in $\bar{E}$, for example, the authors show only that it is weakly sequentially compact if E is separable—an assertion much weaker than the theorem of Alaoglu-Bourbaki. Countability arguments are used wherever possible, and the necessary appeal to transfinite induction in proving the Hahn-Banach theorem is slurred over.

The book is fairly discursive, and should be easy reading. It is beautifully printed. The translation is on the whole good, although it is misleading in a few places. A couple of errors in the Russian original have been quietly corrected. The book is marred, however, by a ridiculous exaggeration of the rôle played by Russian, and in particular by Soviet, mathematicians in the development of functional analysis. The rule adopted seems to be: if some Russian had anything to do with it, mention him and no one else; if not, mention no one if you can help it.

EDWIN HEWITT

BRIEF MENTION

Einführung in die Verbandstheorie. By H. Hermes. Berlin, Springer, 1955. 8+160 pp. 22.80 DM.

The quickest way to describe the book under review is to say that if there were a Bourbaki treatment of lattice theory, it would be pretty much like that of Hermes. (It is unlikely that there ever will be a Bourbaki treatment of the subject; cf. Bull. Amer. Math. Soc. vol. 59 (1953) p. 483.) The book is an introduction to the elements of lattice theory; the exposition is handled with painstaking care and thoroughness. Once the author decides to discuss a subject, he discusses it systematically; his treatment, for instance, of the various kinds of homomorphisms (join, meet, lattice, order) is remarkable for its completeness and clarity. There are a few exercises, but they are not very numerous or exciting. The book is divided into five chapters, as follows: (I) Foundations; (II) The simplest classes of lattices; (III) Modular lattices; (IV) Distributive and Boolean lattices; (V) Miscellany. The fifth chapter discusses Zorn's lemma, congruence relations, and some connections between Boolean algebra and logic. There is an appendix that treats in great generality the universal algebraic concepts (e.g., homomorphism) of which special cases are treated in the body of the book. There is nothing new here for the expert, but a beginning student might find the treatment a pleasant illustration of some of the easier techniques of algebra.

PAUL R. HALMOS

RESEARCH PROBLEMS

7. H. E. Salzer: *Numerical analysis.*

It is known that I: an nth degree polynomial is determined by $(n+1)$ values, considering its value and the *first consecutive* r derivatives at a point as $(r+1)$ values. The writer found for $(2n-1)$th degree formulas of the type (1) $f_{[n/2]+1} = \sum_{i=-[(n-1)/2]}^{[n/2]} \cdot (A_i f_i + h^2 B_i f_i'')$ the existence of solutions A_i, B_i for $n=2$ and 4, but not for $n=1$ (trivial), 3, or 5, suggesting the following:

(a) In general is a polynomial of $(2n-1)$th degree determined by its value and its *second* derivative at n equally-spaced points if and only if n is even?

(b) What generalizations of I are there, involving $(r+1)$ *non-consecutive* derivatives (value considered as 0th derivative)? Thus the writer found $(2n)$th degree formulas of the type (2) $f_{[n/2]+1} = \sum_{i=-[(n-1)/2]}^{[n/2]} A_i f_i + h \sum_{i=-[(n-1)/2]}^{[n/2]+1} B_i f_i'$ for $n=1(1)5$, indicating that f can be determined from n values of f and f' at n points and f' at an $(n+1)$th point. (Received December 5, 1955.)

8. H. E. Salzer: *Numerical analysis.*

The integration formula (1) $\int_{-1}^{1} (f(x)/(1-x^2)^{1/2})dx = (\pi/n)[f(\cos \pi/2n) + f(\cos 3\pi/2n) + \cdots + f(\cos (2n-1)\pi/2n)]$ is exact for $f(x)$ any $(2n-1)$th degree polynomial, so that the weight function $1/(1-x^2)^{1/2}$ gives a Gaussian-type quadrature formula for a finite interval, which happens to be equally-weighted. Can one find weight functions $w(x)$ and $W(x)$ yielding Gaussian-type quadrature formulas of the form (2) $\int_0^\infty w(x)f(x)dx = (a/n)\sum_{i=1}^{n} f(x_i)$ and (3) $\int_{-\infty}^{\infty} W(x)f(x)dx = (b/n)\sum_{i=1}^{n} f(x_i)$, combining the advantage of orthogonality or algebraic degree of precision equal to $2n-1$ with equal weights? Unlike (1), do solutions to (2) and (3) exist only when $w(x)$ and $W(x)$ depend upon n? (Received December 5, 1955.)

THE NATIONAL SCIENCE FOUNDATION PROGRAM FOR MATHEMATICAL SCIENCES

LEON W. COHEN[1]

The purpose of this note is to inform mathematicians of the variety of support for research and education available through the Program for Mathematical Sciences in the National Science Foundation. The mathematical content of the program is determined by the interests of the mathematicians themselves. The program has no specific mission but endeavors to respond to the needs of basic research, to the demand for an increase in the number of competent students, both undergraduate and graduate, and to the requirements of new mathematical methods in the sciences. Some of the ways in which the program has aided in the advance of mathematics are listed below. Suggestions for other methods are most welcome.

Support is normally provided by a grant to the college, university or other institution through which a proposal is submitted but grants directly to individuals are possible. Information as to the usual content of a proposal, which is to be formulated by the one or more mathematicians seeking support, is set out in a booklet, *Grants for Scientific Research*. This will be supplied upon request to the Program Director for Mathematical Sciences, National Science Foundation, Washington 25, D. C. There are no printed forms to be filled out. Proposals may be submitted at any time during the year.

The following list gives some of the items which have been covered by grants for the support of research.

1. For individual mathematicians as principal investigators
    1.1 Supplemental salary during sabbatical leave
    1.2 Salary during special leave for research
    1.3 Salary up to 50% to reduce excessive teaching or administrative load during the academic year
    1.4 Salary for summer research
    1.5 Salary for research by professors emeriti

2. For collaboration with principal investigators
    2.1 Salary for graduate students as research assistants
    2.2 Salary for mathematicians with the doctorate as research associates

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[1] Program Director for Mathematical Sciences. This note is not to be interpreted as a formal expression of Foundation policy.

3. For the development of research in departments of mathematics
    3.1 Salary for visiting research professors and lecturers
    3.2 Support for seminars in one or more broad fields of mathematics

This may involve several principal investigators and combine several items in 1 and 2.

4. For auxiliary purposes
    4.1 Travel related to research
    4.2 Secretarial assistance, particularly for the preparation of manuscripts
    4.3 Publication costs
    4.4 Computation costs
    4.5 Permanent equipment

5. For the applications of mathematics
    5.1 Numerical methods for high speed computation
    5.2 New mathematical models for application to the sciences and operations research

6. For research and education in undergraduate colleges
    6.1 Experimental programs for accelerated courses leading to faculty-student research groups in the junior and senior years.

In formulating a proposal one or more of the items listed may enter according to the requirements of the principal investigators. In some cases, proposals budget no funds for salaries of principal investigators who are full professors. The National Science Board has adopted the policy of encouraging proposals for a period of at least two and not more than five years. It is clear, however, that proposals dealing with sabbatical leave will have a term of one year.

Item 6, Experimental Programs for Research and Education in Undergraduate Colleges, is represented by only a few grants in the history of the program. Proposals embodying ideas pertinent to the local institution will be considered with great interest.

In addition to the direct support of research, the program has sponsored a three-day Conference on the Theory of Numbers, a Conference on Mathematical Tables and a Survey of Research and Training in Applied Mathematics. There is currently under way a Survey of Research Potential and Training in the Mathematical Sciences. The results of Survey visits to about 60 departments of mathematics are now being analyzed and shortly a questionnaire will be distributed to about 3,000 mathematicians having the doctorate. It is hoped that a picture of the existing capabilities and future needs of mathematics in the graduate schools will emerge. Other directions which the Survey has planned to investigate are the undergraduate situation, nonacademic careers in mathematics and the publication of mathematical material.

An indication of the growth of the program is given in the following annual totals of research grants:

Year (ending June 30)	Number of Grants	Total Dollars
1952	1	$ 19,300
1953	19	85,200
1954	21	173,950
1955	48	562,400

DUALITY AND S-THEORY

E. H. SPANIER

This is an account of a systematization of certain parts of homotopy theory by means of the suspension category (also called the S-category). One of the most important applications is that a formal analogy becomes a rigorous duality in the S-category. We begin by summarizing some of the background material and results leading to the S-theory and duality.

1. Given topological spaces X and Y and two continuous mappings f_0 and f_1 from X to Y (denoted by $f_0, f_1\colon X\to Y$) we say that f_0 is *homotopic* to f_1 (denoted by $f_0\simeq f_1$) if there exists a continuous family of continuous mappings $h_t\colon X\to Y$ for $0\leq t\leq 1$ such that $h_0=f_0$ and $h_1=f_1$. Intuitively $f_0\simeq f_1$ if the map f_0 can be continuously deformed into the map f_1. It is easily seen [4; 7] that the relation of homotopy thus defined is reflexive, symmetric, and transitive and, therefore, partitions the set of continuous maps from X to Y into disjoint equivalence classes called *homotopy classes*. We denote the set of homotopy classes of mappings from X to Y by $[X, Y]$, and if $f\colon X\to Y$, then $[f]$ will denote the homotopy class of f. It is of fundamental importance in present day topology to determine the structure of $[X, Y]$. Specifically, we would like to have a method of determining whether two given mappings are homotopic, and also we would like to determine how many elements there are in $[X, Y]$. In many cases our information is so limited that we do not even know whether $[X, Y]$ contains more than one element. (In the above when X is contractible we assume Y is arcwise connected.)

It is clear that if either X or Y is a contractible space then there is exactly one homotopy class of maps from X to Y. In particular if either X or Y is a cell of any dimension, the structure of $[X, Y]$ is completely known as this set consists of a single element.

Perhaps the most natural spaces to consider after the cells are the spheres. We denote by S^n the unit sphere in a euclidean space of $(n+1)$ dimensions. We want to discuss the homotopy classes $S^n\to Y$ and $X\to S^n$.

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2. Let us consider first the case where we map S^n into a space Y. We choose a base point $p \in S^n$ and a base point $y \in Y$ and restrict our attention to maps $f\colon S^n \to Y$ such that $f(p)=y$ and to homotopies h_t such that $h_t(p)=y$ for all $0 \leq t \leq 1$. Given two maps $f, g\colon S^n \to Y$ with $f(p)=g(p)=y$ we define a map

$$(f+g)\colon S^n \to Y$$

with $(f+g)(p)=y$ to be the composite

$$S^n \xrightarrow{\Lambda} S^n \vee S^n \xrightarrow{f \vee g} Y$$

where $S^n \vee S^n$ denotes the subset $S^n \times p \cup p \times S^n$ of $S^n \times S^n$, Λ pinches an equator of S^n through p to a point and maps one hemisphere onto $S^n \times p$ and the other hemisphere onto $p \times S^n$, and $f \vee g$ maps $S^n \times p$ by f and $p \times S^n$ by g [4; 7; 9; 10]. It turns out that this "addition" of maps defines a group structure on the set of homotopy classes of maps of S^n into Y which take p into y, and this group is abelian if $n>1$. The resulting group is denoted by $\pi_n(Y, y)$ and is called the ***nth homotopy group*** of Y with ***base point*** y. For $n=1$ the group is also known as the ***fundamental group*** or ***Poincaré group*** of Y.

The group structure helps greatly in studying the structure of the set of homotopy classes from S^n to Y; not that the solution is known in general but that many natural mappings between sets of homotopy classes are homomorphic in terms of this group structure, and these homomorphisms have provided a solution in some cases.

For example, a continuous map $f\colon X \to Y$ induces homomorphisms of the integral homology groups $f_*\colon H_p(X) \to H_p(Y)$ for every p and $f \simeq g$ implies $f_* = g_*$ [5]. Since $H_n(S^n)$ is infinite cyclic, choice of a generator $s \in H_n(S^n)$ determines a map $\phi\colon \pi_n(Y, y) \to H_n(Y)$ defined by $\phi[f] = f_* s$, and this map is a homomorphism between these two groups. *If Y is arcwise connected and simply connected and if $H_i(Y)=0$ for $0<i<n$ (where $n>1$), then $\pi_i(Y, y)=0$ for $0<i<n$ and ϕ is an isomorphism of $\pi_n(Y, y)$ onto $H_n(Y)$.* This result, known as the ***Hurewicz isomorphism theorem***, shows that the first nontrivial homotopy group of a simply connected space can easily be determined from its homology structure. It also follows that for such a space Y two maps $f, g\colon S^n \to Y$ are homotopic if and only if $f_* s = g_* s$, which can be regarded as a generalization of the theorem of Brouwer to the effect that two maps of S^n into itself are homotopic if and only if they have the same degree [3].

3. Consider next the case where we map a space X into S^n. If we assume dimension $X \leq 2n-2$, it turns out that we can make

$[X, S^n]$ into an abelian group. Given two maps $f, g\colon X\to S^n$ we define a map $(f, g)\colon X\to S^n\times S^n$ by $(f, g)(x)=(f(x), g(x))$. Let $S^n\vee S^n$ denote the subset $S^n\times p\cup p\times S^n$ of $S^n\times S^n$ as above and define

$$\Omega\colon S^n\vee S^n\to S^n$$

by $\Omega(z, p)=\Omega(p, z)=z$. Because of the dimension restriction on X the map (f, g) is homotopic to a map $h\colon X\to S^n\times S^n$ such that $h(X)\subset S^n\vee S^n$. Then the composite Ωh is a map of X into S^n representing the sum of $[f]$ and $[g]$. The definition of addition in this case is due to Borsuk [2; 12], and the resulting group is called the nth *cohomotopy group* of X and denoted by $\pi^n(X)$.

A continuous map $f\colon X\to Y$ induces homomorphisms

$$f^*\colon H^p(Y)\to H^p(X)$$

of the integral cohomology groups for every p and $f\simeq g$ implies $f^*=g^*$. Since $H^n(S^n)$ is infinite cyclic, a generator $s^*\in H^n(S^n)$ determines a homomorphism

$$\phi^*\colon \pi^n(X)\to H^n(X)$$

defined by $\phi^*[f]=f^*s^*$. *If X is a complex and $H^i(X)=0$ for $i>n$, then $\pi^i(X)=0$ for $i>n$ and ϕ^* is an isomorphism of $\pi^n(X)$ onto $H^n(X)$.* This theorem is known as the *Hopf classification theorem* [4; 8; 22]. As in the case of the Hurewicz isomorphism theorem, it can also be regarded as a generalization of Brouwer's theorem on the degree.

The Hurewicz isomorphism theorem and the Hopf classification theorem are examples of dual theorems. That is, if we interchange homotopy group with cohomotopy group, homology group with cohomology group, the map ϕ with the map ϕ^*, and lowest nontrivial dimension for homology with highest nontrivial dimension for cohomology, then one of them becomes the other (assuming simple connectedness for the Hurewicz theorem and a suitable dimension restriction for the Hopf theorem). The duality we seek is such as to make these two theorems dual. There are, of course, other theorems dual in the same sense. For example, the second nontrivial homotopy group from below is related to the homology groups [21] in a way dual to the way that the second nontrivial cohomotopy group from above is related to the cohomology groups [12; 18] (again if suitable dimension restrictions are imposed on the first nontrivial group in each theorem).

4. When we consider maps $S^m\to S^n$ the resulting set $[S^m, S^n]$ can be regarded as a homotopy group and, if $m\leqq 2n-2$, also as a cohomotopy group (the base point condition being unnecessary for the

homotopy groups of spheres because a sphere is simple in all dimensions [4; 7; 9]). The two group structures on $[S^m, S^n]$ defined when $m \leq 2n-2$ turn out to be the same, and the groups $[S^m, S^n]$ are particularly important as the other groups of homotopy classes of mappings between complexes are built from them. A complete knowledge of the homotopy groups of spheres would be the major step in obtaining information about the homotopy classes of maps from one complex to another.

To study the homotopy groups of spheres Freudenthal [6] introduced the ***suspension homomorphism*** (*Einhängung*) $S\colon \pi_k(S^n) \to \pi_{k+1}(S^{n+1})$ defined as follows. We regard S^{k+1} as the join of S^k with a pair of points (a, b); that is, S^{k+1} is the set of line segments joining a point of an equator S^k to the two poles a and b. Similarly S^{n+1} is the join of S^n with a pair of points (a', b'). Given a map $f\colon S^k \to S^n$ its suspension $Sf\colon S^{k+1} \to S^{n+1}$ is defined to map S^k into S^n (regarded as equators of S^{k+1}, S^{n+1}, respectively) as f does, to map a into a', b into b', and a line segment from a point x of S^k to a (or b) linearly onto the line segment from $f(x)$ to a' (or b'). The map $f \to Sf$ induces the suspension homomorphism

$$S\colon \pi_k(S^n) \to \pi_{k+1}(S^{n+1}).$$

This homomorphism is of great importance in the study of the homotopy groups of spheres. For $k \leq 2n-2$ it is an isomorphism onto [6; 7; 19]. It follows that for fixed d in the sequence

$$\pi_{n+d}(S^n) \xrightarrow{S} \pi_{n+1+d}(S^{n+1}) \xrightarrow{S} \cdots \xrightarrow{S} \pi_{m+d}(S^m) \xrightarrow{S} \cdots$$

all the maps S except for a finite number at the beginning are isomorphisms onto. Note that S is an isomorphism onto for the same range of values for which the cohomotopy group $\pi^n(S^k)$ is defined.

5. The suspension map can be extended to more general spaces. If X denotes an arbitrary space, SX will denote the join of X with an ordered pair of points. Then by a natural generalization of the map S defined above we define a map

$$S\colon [X, Y] \to [SX, SY]$$

called the ***suspension map***. The isomorphism theorem for the suspension homomorphism for spheres has an extension to the more general case [13; 16] and states that *if $\pi_i(Y) = 0$ for $i < n$ and if X is a complex with dimension $\leq 2n-2$, then S is a* 1-1 *correspondence between* $[X, Y]$ *and* $[SX, SY]$. In particular, it follows that for the range of values for which the cohomotopy groups are defined S is an isomorphism of

$\pi^k(X)$ onto $\pi^{k+1}(SX)$. On the other hand, it is not true for homotopy groups that the map S: $\pi_k(Y) \to \pi_{k+1}(SY)$ is always an isomorphism onto.

Hence, there are two difficulties to the immediate establishment of a duality interchanging homotopy groups with cohomotopy groups. First, the cohomotopy groups are only defined when certain dimension restrictions are satisfied, and secondly, the suspension map is not always an isomorphism onto for the homotopy groups. Since the dimension restriction in order that the cohomotopy group be defined is the same as the one for which the suspension map is a 1-1 correspondence, we might expect both difficulties to disappear if we could construct a system in which the suspension map is always a 1-1 correspondence.

6. In order to obtain a system in which the suspension map is always a 1-1 correspondence we pass to the limit under the suspension map. This leads us to the *suspension category* or *S-category* which we now describe. For arbitrary spaces X, Y consider the sequence

$$[X, Y] \xrightarrow{S} [SX, SY] \xrightarrow{S} \cdots \xrightarrow{S} [S^kX, S^kY] \xrightarrow{S} \cdots$$

and define $\{X, Y\} = \lim_k [S^kX, S^kY]$. The elements of $\{X, Y\}$ are called *S-maps*, and the S-category is the category [5] whose objects are topological spaces and whose mappings are S-maps. If f: $S^kX \to S^kY$ for some $k \geq 0$, then $\{f\}$ will denote the S-map determined by f. It is clear from the definition that $\{X, Y\} = \{SX, SY\}$ so if we use S-maps instead of the usual homotopy classes we have a system in which the suspension map is always a 1-1 correspondence. It is also clear from the theorem stated in §5 that if $\pi_i(Y) = 0$ for $i < n$ and X is a complex with dimension $X \leq 2n-2$, then $[X, Y]$ is in 1-1 correspondence with $\{X, Y\}$. For k sufficiently large it is known [1] that $[S^kX, S^kY]$ can be given a law of composition so that it becomes an abelian group and S: $[S^kX, S^kY] \to [S^{k+1}X, S^{k+1}Y]$ is a homomorphism. Therefore, $\{X, Y\}$ is an abelian group for every pair of spaces X, Y. Furthermore, composition defines a bilinear pairing of $\{X, Y\}$ and $\{Y, Z\}$ to $\{X, Z\}$.

7. Having defined the S-category we are ready to formulate the duality for S-maps. The S-homotopy groups will be dual to the S-cohomotopy groups. The examples of dual theorems quoted above will be consequences of the duality because in the range of values for which they hold the set of S-maps is in 1-1 correspondence with the set of usual homotopy classes. It is also clear that dual theorems in

the S-category give rise to dual theorems about ordinary homotopy classes for the range of values of dimension and connectedness assumptions so that $[X, Y]$ is in 1-1 correspondence with $\{X, Y\}$.

The duality is an extension of the Alexander duality which equates the pth homology group of a subset of S^n with the $n-p-1$st cohomology group of its complement. We summarize its main properties below [15].

Given a subpolyhedron X of a triangulation of S^n we define a polyhedron $D_nX \subset S^n - X$ as *n-dual* to X if D_nX is an S-deformation retract of $S^n - X$; that is, if the inclusion map $D_nX \subset S^n - X$ represents an S-map which is an S-equivalence. Any subpolyhedron of an n-sphere has an n-dual because if we triangulate S^n so that the subpolyhedron X is a complete subcomplex (every simplex of S^n having all its vertices in X belongs to X) then the set of simplexes of S^n with no vertex in X is an n-dual of X because every simplex of S^n is the join of a unique simplex of X and of a unique simplex with no vertex in X. It can be shown that if D_nX is an n-dual of X, then X is an n-dual of D_nX and SD_nX is an $(n+1)$-dual of X (where SD_nX denotes that subset of S^{n+1} obtained by regarding S^n as an equator of S^{n+1} and taking the join of D_nX, which lies in this equator, with the poles of S^{n+1}).

The main theorem on duality follows. Given *subpolyhedra* X, $Y \subset S^n$ *and n-duals* D_nX, $D_nY \subset S^n$ *there exists a unique map*

$$D_n\{X, Y\} \to \{D_nY, D_nX\}$$

with the properties:

(1) *If* $i: X \subset Y$ *and* $i': D_nY \subset D_nX$, *then* $D_n\{i\} = \{i'\}$.

(2) *Given* $\{f\} \in \{X, Y\}$, $\{g\} \in \{Y, Z\}$ *and n-duals* D_nX, D_nY, D_nZ, *then* $D_n(\{g\}\{f\}) = D_n\{f\}D_n\{g\}$.

(3) *Using* X, Y *as n-duals of* D_nX, D_nY, *respectively, we have maps*

$$\{X, Y\} \xrightarrow{D_n} \{D_nY, D_nX\} \xrightarrow{D_n} \{X, Y\},$$

and this composition is the identity.

(4) D_n *is a homomorphism.*

(5) *Taking* SD_nX, SD_nY *as* $(n+1)$*-dual to* X, Y, *then*

$$SD_n = D_{n+1}.$$

(6) *Taking* D_nX, D_nY *as* $(n+1)$*-dual to* SX, SY, *then*

$$D_{n+1}S = D_n.$$

(7) *Commutativity holds in the diagram*

$$\begin{tikzcd}
H_p(X) \arrow[r, "f_*"] \arrow[d, "\mathfrak{D}_n"'] & H_p(Y) \arrow[d, "\mathfrak{D}_n"] \\
H^{n-p-1}(D_nX) \arrow[r, "(D_nf)^*"] & H^{n-p-1}(D_nY)
\end{tikzcd}$$

where $\mathfrak{D}_n$ denotes the Alexander duality isomorphism.

The first four properties assert that D_n is an isomorphism from $\{X, Y\}$ onto $\{D_nY, D_nX\}$ with certain functorial properties. Properties (5) and (6) assert that this isomorphism doesn't depend on the integer n and is stable under suspension in the sense that complete commutativity holds in the diagram

$$\begin{tikzcd}
\{X, Y\} \arrow[r, "D_n"] \arrow[d, "S"'] \arrow[dr] & \{D_nY, D_nX\} \arrow[d, "S"] \\
\{SX, SY\} \arrow[r, "D_{n+2}"] \arrow[ur] & \{SD_nY, SD_nX\}
\end{tikzcd}$$

where each of the diagonal maps is D_{n+1}.

As an example of this duality let us take $S^p \subset S^{p+q+1}$ and as $(p+q+1)$-dual to S^p we can take a sphere $S_1^q \subset S^{p+q+1}$. Assuming $q \leq 2p-2$ we know from the suspension theorem that

$$\{S_1^q, S^p\} \approx [S_1^q, S^p] = \pi_q(S^p).$$

Taking $D_{p+q+1}(S^p) = S_1^q$, $D_{p+q+1}(S_1^q) = S^p$, we get an automorphism of period two,

$$D_{p+q+1}\colon [S_1^q, S^p] \approx \{D_{p+q+1}S^p, D_{p+q+1}S_1^q\} \approx [S_1^q, S^p].$$

Hence, the duality defines an automorphism of period two in the stable (under suspension) homotopy groups of spheres, and this automorphism is itself stable. It need not be the identity. In fact, for $p=q$ it can be shown that $D_{2p+1}\colon [S_1^p, S^p] \approx [S_1^p, S^p]$ is given by $D_{2p+1}[f] = -[f]$. On the other hand, if $q=p+1$ and $p \geq 3$, then $[S_1^{p+1}, S^p]$ is a group of order two so D_{2p+2} is necessarily the identity on this group. Similarly for $q=p+2$ and $p \geq 3$, D_{2p+3} is the identity.

We define the S-homotopy and S-cohomotopy groups by

$$\Sigma_p(X) = \{S^p, X\}, \qquad \Sigma^p(Y) = \{Y, S^p\}.$$

Since S^p and S^{n-p-1} are n-dual, it follows from the above properties that D_n is an isomorphism onto

$$D_n\colon \Sigma_p(X) \approx \Sigma^{n-p-1}(D_nX).$$

Analogous to the homomorphisms defined for the ordinary homotopy and cohomotopy groups there are homomorphisms

$$\phi: \Sigma_p(X) \to H_p(X), \qquad \phi^*: \Sigma^p(Y) \to H^p(Y).$$

It is a consequence of (7) and the other properties of the duality that commutativity holds in the diagram

$$\begin{tikzcd}
\Sigma_p(X) \arrow[r, "\phi"] \arrow[d, "D_n"'] & H_p(X) \arrow[d, "\mathfrak{D}_n"] \\
\Sigma^{n-p-1}(D_nX) \arrow[r, "\phi^*"] & H^{n-p-1}(D_nX).
\end{tikzcd}$$

In the S-category the Hurewicz isomorphism theorem asserts that if $H_i(X)=0$ for $i<m$ (where $H_0(X)$ is taken to be the reduced group [5]), then $\Sigma_i(X)=0$ for $i<m$ and $\phi: \Sigma_m(X) \approx H_m(X)$. Similarly in the S-category the Hopf classification theorem asserts that if Y is a polyhedron with $H^i(Y)=0$ for $i>m$, then $\Sigma^i(Y)=0$ for $i>m$ $\phi^*: \Sigma^m(Y) \approx H^m(Y)$. Both theorems as thus stated are stable, and we note that from the commutativity of the last diagram above D_n takes one theorem into the other, which was one of the features we desired in trying to find the duality.

To define the map $D_n: \{X, Y\} \to \{D_nY, D_nX\}$ note that (1) defines it in the case that we are considering an inclusion map (and the duals satisfy a reverse inclusion). (1) and (2) show that if $Y \subset X$ and the inclusion map $i: Y \subset X$ is an S-equivalence and if $i': D_nX \subset D_nY$, then i' is an S-equivalence and

$$D_n(\{i\}^{-1}) = (D_n\{i\})^{-1} = \{i'\}^{-1}.$$

It can be shown directly that i' is an S-equivalence when i is and then the above equation defines D_n on $\{i\}^{-1}$. By using the concept of a mapping cylinder any map can be seen to be homotopic to a map which is a composition of inclusion maps and S-retractions by deformation (i.e. S-inverses of inclusion maps). The remarks above and (2) show how to define D_n for such a composition. D_n is then defined for an arbitrary map by noting that if it is defined for a homotopic map which is a composition of inclusion maps and S-retractions by deformation then the resulting map doesn't depend on the choice of the representation as a composition of such maps. The properties (1)–(7) are then established in straightforward fashion.

8. If we consider spaces which are not subpolyhedra of spheres, there is a duality for them if they are of the same S-homotopy type as subpolyhedra of spheres. We restrict attention to CW complexes [20], and we shall consider maps $\xi: X \to X'$ which are S-equivalences

of the CW complex X onto some subpolyhedron X' of S^n. We say that X, ξ and X^*, ξ^* are *weakly n-dual* if ξ, ξ^* are S-equivalences of X, X^* onto n-dual subpolyhedra of S^n. The duality D_n can be extended to weakly n-dual complexes in an obvious manner. The importance of this generalization lies in the fact that it enables us to dualize the concept of adjoining a cell to a complex by a given map. The basic result in this direction is that *if* $f\colon X\to Y$ *is weakly n-dual to* $f^*\colon Y^*\to X^*$ *and if* $Z=CX\cup Y$ *is the space obtained by taking a cone over* X *disjoint from* Y *and identifying* $x\in X$ *with* $fx\in Y$ *and* Z^* *is similarly defined, then* Z *and* Z^* *are weakly* $(n+1)$*-dual in a natural way* [15].

This last result can be used to extend the duality to relative theory [17]. To any CW complex X there corresponds, for sufficiently large n, a CW complex X^* and an anti-isomorphism $\sigma\leftrightarrow\sigma^*$ of the cells of X with those of X^* such that if A is a subcomplex of X and A^* denotes the subcomplex of X^* of cells which correspond to the cells of $X-A$, then the complexes A and $X^*\cup CA^*$ are weakly n-dual in such a way that the natural maps $A\to X\to X\cup CA$ are weakly n-dual to the natural maps $X^*\cup CA^*\leftarrow X^*\leftarrow A^*$. It follows that the S-homotopy exact couple [11] of X is isomorphic to the S-cohomotopy exact couple of X^* so that theorems about these exact couples can also be dualized.

If $\phi\colon X\to Y$ is a cellular carrier [13; 16] and X^*, Y^* are weakly n-dual complexes as in the above, there is defined in a natural way a dual carrier $\phi^*\colon Y^*\to X^*$. Then the relative theorem [17] for D_n asserts that D_n maps the S-homotopy classes carried by ϕ (denoted by $\{\phi\}$) isomorphically onto $\{\phi^*\}$. This duality converts an obstruction to extending a continuous map [4] into an obstruction to compressing a continuous map into a subset [14; 16].

BIBLIOGRAPHY

1. M. G. Barratt, *Track groups* (1), Proc. London Math. Soc. (3) vol. 5 (1955) pp. 71–106.

2. K. Borsuk, *Sur les groupes des classes de transformations continues*, C. R. Acad. Sci. Paris vol. 202 (1936) pp. 1400–1403.

3. L. E. J. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Math. Ann. vol. 71 (1912) pp. 97–115.

4. S. Eilenberg, *Cohomology and continuous mappings*, Ann. of Math. vol. 41 (1940) pp. 231–251. See also, *Lectures in topology*, University of Michigan Press, 1941, pp. 57–100.

5. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952.

6. H. Freudenthal, *Über die Klassen der Sphärenabbildungen* I, Compositio Math. vol. 5 (1937) pp. 299–314.

7. P. J. Hilton, *An introduction to homotopy theory*, Cambridge University Press, 1953.

8. H. Hopf, *Die Klassen der Abbildungen der n-dimensionalen Polyeder auf der n-dimensionale Sphäre*, Comment. Math. Helv. vol. 5 (1933) pp. 39–54.

9. S. T. Hu, *An exposition of the relative homotopy theory*, Duke Math. J. vol. 14 (1947) pp. 991–1033.

10. W. Hurewicz, *Beitrage zur Topologie der Deformationen* I–IV, Proc. Akad. Amsterdam vol. 38 (1935) pp. 112–119, 521–528 and vol. 39 (1936) pp. 117–126, 215–224.

11. W. S. Massey, *Exact couples in algebraic topology*, Ann. of Math. vol. 56 (1952) pp. 353–396 and vol. 57 (1953) pp. 248–256.

12. E. H. Spanier, *Borsuk's cohomotopy groups*, Ann. of Math. vol. 50 (1949) pp. 203–245.

13. E. H. Spanier and J. H. C. Whitehead, *A first approximation to homotopy theory*, Proc. Nat. Acad. Sci. vol. 39 (1953) pp. 655–660.

14. ———, *Obstructions to compression*, Quarterly J. of Math. vol. 6 (1955) pp. 91–100.

15. ———, *Duality in homotopy theory*, Mathematika vol. 2 (1955) pp. 56–80.

16. ———, *The theory of carriers and S-theory* (to appear).

17. ———, *Duality in relative homotopy theory*, Algebraic geometry and topology (A symposium in honor of S. Lefschetz), Princeton University Press, 1956.

18. N. E. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. vol. 48 (1947) pp. 290–320.

19. G. W. Whitehead, *On the Freudenthal theorems*, Ann. of Math. vol. 57 (1953) pp. 209–228.

20. J. H. C. Whitehead, *Combinatorial homotopy* I, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 213–245.

21. ———, *A certain exact sequence*, Ann. of Math. vol. 52 (1950) pp. 51–110.

22. H. Whitney, *The maps of an n-complex into an n-sphere*, Duke Math. J. vol. 3 (1937) pp. 51–55.

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TOPOLOGICAL ANALYSIS

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1. Introduction. Apart from the really spectacular development of the field of topology during the last 50 years, one of the most interesting and satisfying related phenomena has been the widespread interaction between topology and other branches of mathematics. This has occurred in nearly all fields of mathematics. It has been notable in algebra, algebraic geometry, differential geometry and in various types of analysis. However it is the connection with the theory of functions, and particularly functions of a complex variable, which has developed and is being actively developed at present that I should like to discuss in some detail today. As I have used the term, *topological analysis* refers to those results of the analysis type, theorems about functions or mappings from one space onto another or about real or complex valued functions in particular, which are topological or pseudo-topological in character and which are obtainable largely by topological methods. Thus in a word we have analysis theorems and topological proofs. As just indicated however, what I shall say today will be confined largely to results closely related to analytic functions of a complex variable. Since one of the main roots—if indeed not the taproot— of topology rests firmly in the recognition by Riemann and Poincaré of the fundamental and inescapable topological nature of such functions, much of the work to be described represents a return of topology to some of the original situations and problems which motivated its beginnings and to which it owes much for its early development.

Contributions of fundamental concepts and results in this type of work have been made during the past 25 years by a large number of mathematicians. Among these should be mentioned (1) Stoïlow [1], the originator of the interior or open mapping, who early recognized lightness and openness as the two fundamental topological properties of the class of all nonconstant analytic functions; (2) Eilenberg [2] and Kuratowski [3] who introduced and used an exponential representation for a mapping and related it to properties of sets in a plane, (3) Morse and Heins [4], whose studies on invariance of topological indices of a function under admissible deformations of curves in the

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complex plane greatly clarified the action of the mapping in the region and near the boundary and opened the way toward admissible simplifying assumptions, (4) the Nevanlinna [5] brothers and L. Ahlfors [6] whose oustanding work on exceptional values of analytic functions lead to conclusions partly topological in character which contain the suggestion of new connections with topology still awaiting development, and (5) Ursell and Eggleston [7] as well as Titus and Young [8] who have contributed elementary proofs for the lightness and openness of analytic mappings using novel methods which have stimulated considerable further effort using these methods in the same area as well as for mappings in a more general topological setting. My own work on this subject began around 1936 as a result of reading some of Stoïlow's early papers, and has been published in an extended sequence of papers spanning the interval of nearly 20 years to the present. On two occasions, however, summaries of some of my results have been given and these are to be found in Memoirs No. 1 of the American Mathematical Society series, entitled *Open mappings on locally compact spaces* and as Lecture No. 1 in the University of Michigan's recently published *Lectures on functions of a complex variable*. In what will be said today an attempt will be made to minimize the overlapping with these earlier lectures—although some will be unavoidable—and concentrate mostly on recent results not previously announced and on some promising areas for new development.

2. **Openness and closedness of mappings.** We begin with a brief discussion of openness and closedness of mappings in a general setting, together with some closely related mapping types. Spaces mentioned will always be assumed to have an open set topology in which the weak separation axiom is satisfied that for any two distinct points x and y there is an open set containing x but not y. Thus we have T_1-spaces. Actually the points to be considered are of primary interest in much more restricted spaces, such as separable metric ones or even Euclidean spaces so that little will be lost in following the discussion if one thinks in terms of these spaces. A mapping $f(X) = Y$, that is, a single valued continuous transformation, of a space X onto a space Y is *open* or *closed* provided the image of every open or closed set in X is open or closed respectively in the space Y.

For example, the mapping of the complex z-plane Z onto the w-plane W generated by the function $w = z^2$ is open since the image of any open set in the z-plane such as a sector is an open set, a larger sector in the w-plane. On the other hand, if the function is modified making it $w = z^2$ on the left-hand half-plane and keeping it $w = z^2$ on the right half-plane, the mapping is no longer open. For the effect now is to fold the left half-plane onto the right half-plane across the y-axis and then stretch the right half-plane so as to cover the whole w-plane by drawing the positive and negative y-axis together along the negative u-axis. Thus, a circular region with center on the positive y-axis would have as its image a semicircular region containing points of the negative u-axis but no points below this axis so that the image is not open.

Similarly, for closedness, any mapping on an ordinary compact metric space is necessarily closed, since all closed sets in such a space are compact and compactness is an invariant property for all mappings. On the other hand the mapping $w=e^z$ of Z into W is not closed because, for example, the set of points on the right arm of the hyperbola $y=1/x$ is closed in Z whereas its image under the exponential mapping approaches the circle $|w|=1$ asymptotically but contains no point of this circle. For an even simpler example it is of interest to note that the mapping $x=\cos t$, $y=\sin t$ of the half-open interval $0\leq t<2\pi$ onto the unit circle C, although a 1-1 mapping, is neither open nor closed. A sequence of values of t converging to 2π is closed in our original space whereas its image converges to the point $(1, 0)$ of C and thus is not closed. For 1-1 mappings such as this openness and closedness are equivalent properties and each implies that the mapping is topological. In a general way it may be said that a mapping is open provided the vicinity of a point is not violated or intruded on under the mapping—nothing comes from outside in and approaches the image of the point without actually doing it through the image of the given vicinity. Similarly a mapping is closed provided that a set which wanders off to infinity (i.e., has no limit point) in the original space, has as image a set which behaves similarly—it cannot come in and approach a limit in the image space.

Openness and closedness of a mapping are relative properties in case the image Y of X under f is embedded in a larger space Y_0. In such a situation we say that the mapping is strongly open (or closed) provided the image of each open (closed) set in X is open (closed) in the whole containing space Y_0. This is particularly significant for analytic functions when we usually are considering mappings of regions in the z-plane *into* but not *onto* the w-plane. However, we note that the distinction disappears whenever the image set Y of X is itself open (or closed) in the larger space Y_0.

3. The Brouwer theorem. A problem. Perhaps the earliest theorem of note concerned with openness of a mapping was the celebrated one due to Brouwer to the effect that *Every homeomorphism of a Euclidean space into itself is necessarily strongly open.* This basic proposition seems to have been a powerful influence in the studies of Stöllow and probably suggested to him the concept of openness for a mapping in general when he had noted that many other mappings, among them the ones generated by analytic functions, have this property. Otherwise stated, the Brouwer theorem asserts that *In a given Euclidean space any set homeomorphic with an open set is itself necessarily an open set.* In this connection it seems worthwhile to suggest consideration of the converse question, namely, *To what extent are the Euclidean manifolds characterized by this property?* In other words, if a suitably restricted space X has the property that each subset of X which is homeomorphic with some open set in X is necessarily open, what, if any, additional conditions are needed to insure that X is a manifold of some dimension? Phrased still another way: *How does one characterize those spaces having the property that any subset homeomorphic with an open set is open?* The answer here is entirely unknown to me, but it probably could be had by some concentrated effort using already available tools in topology and it might well be worth the effort required.

4. **Quasi-compactness.** If instead of requiring that *all* open (or closed) sets in X have open (closed) images, this restriction is limited to open (closed) *inverse* sets, we obtain the notion of a *quasi-compact* mapping. An *inverse* set in X is a set X_0 satisfying $X_0=f^{-1}f(X_0)$, i.e., a set which is the inverse of its transform under f. Thus a mapping is quasi-compact provided the image of every open inverse set is open—or provided the image of every closed inverse set is closed. Clearly the same concept is obtained whichever way we state the condition because for inverse sets the image of the complement is the complement of the image. Thus quasi-compactness is a weaker condition than either openness or closedness for a mapping and in many respects represents the common ground shared by these two properties. Quasi-compactness is of particular interest in connection with mappings generated by decompositions of a topological space and it was in this connection that the concept was initially formulated by Alexandroff and Hopf [9], who referred to it under the name of *strong continuity*. The natural mapping for any such decomposition of a space X onto the hyperspace X' in which a set is open provided the union of its elements is open in X is always quasi-compact. Further, a given mapping $f(X)=Y$ will be quasi-compact if and only if the decomposition of X into point inverses $f^{-1}(y)$, $y\in Y$, has a

natural mapping $\phi(X)=X'$ which is topologically equivalent to X. Indeed *any mapping* $f(X)=Y$ *has a topologically unique representation in the form*

$$f(x) = h\phi(x)$$

where ϕ *is a quasi-compact mapping and* h *is a* 1-1 *mapping.* In case f is quasi-compact, h is then a homeomorphism, and f and ϕ are topologically equivalent.

Quasi-compactness is of interest also in connection with the invariance of local connectedness [10]. It has long been known that local connectedness is invariant under all mappings provided the original space X is compact but not in general otherwise. Also it was observed that local connectedness is invariant under open mappings for all types of spaces X. However, it turns out much more generally that this property is invariant under all quasi-compact mappings for all spaces X. Stated in this way this result includes the ones previously mentioned and in addition the invariance of the same property under all closed mappings and under all retractions, because all such mappings are quasi-compact.

5. **Relation to semi-continuity of decompositions.** The exact relationship between quasi-compactness, closedness and openness of a mapping is best exhibited in terms of semi-continuity properties of the decomposition into point inverses generated in the original space by the mapping. In general a decomposition of a space X into a collection G of disjoint closed sets is *upper semi-continuous* (u.s.c.) provided the union of all elements of G intersecting any closed set in X is closed (or, equivalently, the union of all elements containined in any open set is open). Dually, the decomposition is *lower semi-continuous* (l.s.c.) provided the union of all elements of G intersecting an open set in X is open (or, the union of all elements contained in any closed set is closed). We can now state [11] the

THEOREM. *In order that a mapping* $f(X)=Y$ *be open* {*closed*} *it is necessary and sufficient that it be quasi-compact and generate an l.s.c.* {*a u.s.c.*} *decomposition of* X *into point inverses.*

This readily follows. For if f is open or closed it is quasi-compact as already noted. Also if U is any open set in X, $f(U)$ is open if f is open and $f^{-1}f(U)$ is open by continuity and this is exactly the union of all elements of the decomposition intersecting U. Similarly, if f is closed and K is a closed set in X, $f(K)$ and $f^{-1}f(K)$ are closed, giving u.s.c. of the decomposition. The converse follows by an equally simple argument. Thus we see that semi-continuity of the point-inverse decomposition represents precisely the difference between quasi-compactness and openness or closedness of a mapping.

6. Quasi-openness. A word should be added in this connection about another closely related property, namely, quasi-openness. A mapping $f(X)=Y$ is *quasi-open* provided that for any $y\in B$ and any open set U in X which contains a compact component of $f^{-1}(y)$, y is interior to $f(U)$. Similarly f is *strongly quasi-open* if y is interior to $f(U)$ relative to the whole space Y_0 in which Y may be embedded. In case the mapping is light, that is $f^{-1}(y)$ is totally disconnected for each $y\in Y$, quasi-openness is the same as openness but in general not, of course, since openness clearly implies quasi-openness but not conversely. In particular, every monotone mapping on a compact set is quasi-open. This property of quasi-openness is of interest and importance in connection with the topological properties of analytic functions to which we now turn our attention.

7. Plane region mapping theorem. Before restricting our attention to the mapping generated by an analytic function we mention an especially useful characterization of quasi-openness for mappings from a plane region into another or the same plane. It reads as follows:

THEOREM. *In order that the mapping $f(X)=Y$ be strongly quasi-open, where X is a plane region and Y lies in a plane π, it is necessary and sufficient that for every elementary region R in X with boundary C in X we have*

$$(*)\qquad f(R+C)=f(C)+\textit{the union of a set of bounded components of } \pi-f(C).$$

An elementary region is a bounded connected open set whose boundary consists of a finite number of disjoint simple closed curves. To see that quasi-openness implies (*), we suppose some complementary region Q of $f(C)$ in π intersects but does not lie wholly in $f(R)$. Compactness of $f(\bar{R})$ and connectedness of Q then assures the existence of a point $y\in Q\cdot f(R)$ which is a limit point of $Q-Q\cdot f(R)$. Now R contains a component K of $f^{-1}(y)$ and K is compact because $f^{-1}(y)$ cannot intersect C. Then y must be interior to $f(R)$ relative to π by quasi-openness of f, contrary to the fact that y is a limit point of $Q-Q\cdot f(R)$. The converse implication is established by showing first that if U is any open set in the plane P containing X such that U contains a compact component K of $f^{-1}(y)$ for $y\in Y$, then there exists an elementary region R with boundary C such that

$$K\subset R\subset R+C\subset U \quad\text{and}\quad C\cdot f^{-1}(y)=0.$$

Once this is done, our conclusion readily follows. For y lies in a component Q of $\pi - f(C)$ since it is not in $f(C)$. By (*) we have $y \in Q \subset f(R) \subset f(U)$ so that y is interior to $f(U)$.

8. **Differentiable functions: lightness and openness.** We now consider in some detail the type of mapping from the complex z-plane Z to the w-plane W generated by a function $w = f(z)$ which is nonconstant and differentiable in a region S of Z. It was first noted by Stoïlow that all such mappings are light and strongly open. In other words they are not constant on any continuum, since the set on which the function takes a given value must be totally disconnected (this is lightness) and open sets in S go into open sets in W. Furthermore, these two properties of lightness and openness of all such mappings are the fundamental topological properties of analytic functions in the sense that any topological property of all nonconstant analytic functions necessarily is a consequence of these two. For not only does every nonconstant analytic function have these properties but, conversely, any light open mapping from an orientable triangulable 2-dimensional manifold to the complex plane or sphere is topologically equivalent to an analytic function. This is the primary conclusion on topological properties of analytic functions as originally formulated by Stoïlow.

From the viewpoint of topological analysis the best known method at present for exhibiting lightness and openness for differentiable functions is by means of the circulation index of a mapping about a point, or the winding number about a point. By this method it is possible to prove the properties directly from the assumption of differentiability, making no use of further developments in analysis such as integrals, continuity or zeros of the derivative and the like. This is of importance because many of the results of classical analytic function theory are to be deduced from lightness and openness rather than conversely. The circulation index $\mu_C(f, p)$ of our mapping f about a point $p \in W - f(C)$ taken over a simple closed curve C in the region S of definition of f is defined by taking a mapping $\zeta(x)$ of an interval (a, b) onto C with $\zeta(a) = \zeta(b)$ but which otherwise is 1-1—i.e., a *traversal* of C, and representing the mapping $f\zeta(x) - p$ in the exponential form $e^{u(x)}$ where $u(x)$ is continuous on $(0, 1)$. Then $\mu_C(f, p)$ is defined to be $u(b) - u(a)$. Usually ζ is chosen as a positive traversal of C relative to orientation in the plane Z. For convenience we may define the winding number $\omega_C(f, p)$ of f about p as the number obtained by dividing $\mu_C(f, p)$ by $2\pi i$. Then ω is an integer and measures the net number of times $f(z)$ goes around p when z traverses C once in the positive sense.

Aside from the usual clarifying results on the nature of the circulation index, it can be proven, significantly, that as a function of p, this index is continuous and thus is constant on each complementary domain of $f(C)$ in W and that it vanishes on the unbounded one of these domains. Further, if R is an elementary region lying with its boundary C wholly in S, this index computed over the whole boundary C of R vanishes at a point p of W if and only if p does not belong to $f(R+C)$. In proving this latter statement use is made of the differentiability of f in order to find a point q in the same component of $W-f(C)$ with p such that $f'(z)\neq 0$ for all $z\in f^{-1}(q)$ and to show that for any such z with $f'(z)\neq 0$, $\mu_J(f, q)$ has the value $2\pi i$ for any sufficiently small circle J about z. Once this is established, however, in view of the characterization discussed earlier we then have proven quasi-openness of our mapping f. For if R is any elementary region in S with boundary C in S we must have (*) satisfied because the circulation index vanishes throughout the unbounded component U of $W-f(C)$ so that f takes no value in U; and when f takes on R one value p in a bounded component Q of $W-f(C)$, $\mu_C(f,p)\neq 0$ and hence f takes all other values in Q on R as the index is constant on Q.

Thus we have strong quasi-openness of f; and this means that we will have established strong openness and lightness as soon as we prove lightness. This latter is accomplished by means of the property (*) just noted. Assuming, contrary to lightness, that our function is constant on some continuum in S we are able to construct, with the aid of properties of the circulation index, a function $g(z)$ which will be differentiable inside and on a simple polygon P in S and such that $|g(z)| < \delta = |g(z_0)|/2$ for all z on P where z_0 is inside P. Clearly this contradicts (*). The function $g(z)$ here has the form

$$(\dagger)\qquad g(z) = \prod_{r=0}^{q-1} \{f[(z - z_0)e^{2\pi i r p/q} + z_0] - a\}.$$

For details of this argument as well as other proofs mentioned in this connection reference is made to the previously mentioned summaries of mine together with articles referred to therein—in particular here to a paper of Ursell and Eggleston [7] which contains some of the basic ideas which make the proofs successful.

9. **Local and global analysis of light open mappings.** The important converse conclusion: that every light (strongly) open mapping from a 2-manifold to the complex plane or sphere is topologically equivalent to an analytic function is best treated in the more general setting of light open mappings acting on 2-manifolds. We only out-

line the treatment in brief since it has been discussed in detail in the sources quoted earlier. In the first place it can be shown that the property of being a 2-manifold is invariant under light open mappings. Thus it is not necessary to assume such a highly simplified structure of the image space. In the next place, although we assume only that the inverse of each point is totally disconnected, it follows from results of plane topology closely related to the Jordan Curve Theorem that each point inverse is a completely scattered set, that is, if our mapping f is constant on a set X, then no point of X is a limit point of X. Thus the situation is greatly simplified and we are in position to complete a local analysis of the action of the mapping. This consists in showing that if $f(A)=B$ is our mapping where for simplicity we suppose A and B are 2-manifolds without edges, then for any $y\in B$ and any $x\in f^{-1}(y)$ there exists 2-cell neighborhoods V of y and U of x such that $f(U)=V$ and the mapping of U onto V is topologically equivalent to a power mapping $w=z^k$ on $|z|\leq 1$ for some integer k.

Since it also follows in this situation that orientability of B implies that of A, it is now possible, in case B is a region on the complex plane or sphere, to construct a Riemann surface Σ onto which A can be mapped by a homeomorphism $h(A)=\Sigma$; and, using standard procedures of analysis, an analytic function $\phi(z)$ can be constructed on Σ in such a way that the relation

$$f(x) = \phi h(x)$$

is satisfied for all $x\in A$. Thus f is topologically equivalent to the analytic function ϕ.

10. Maximum modulus results. Rouché's Theorem. Degree and zeros. Among the numerous well known fundamental results of analytic function theory which are direct consequences of lightness and openness of the associated mapping brief mention will be made of only a few. These include the Maximum Modulus Theorem in its full strength together with a group of related results on existence of zeros such as the Fundamental Theorem of Algebra. Also, Rouché's Theorem in its full generality is readily obtainable using easily developed properties of the circulation index, as are also other classical results concerning the number of zeros and poles of a meromorphic function inside a simple closed curve on which it is analytic and $\neq 0$. Homotopy classification of analytic and meromorphic functions can be clarified by similar methods and greatly illuminates the topological character and action of the mappings generated by such functions.

Studies along these latter lines have been carried out in recent years by Kuratowski with striking success.

11. **Sequence results.** In connection with sequences of functions and of mappings, two theorems fit together in a peculiarly interesting way. They read as follows:

1. *If A and B are locally compact connected and locally connected separable metric spaces, and if* $f_n: A \to B$, $n=1, 2, \cdots$, *is any sequence of strongly quasi-open mappings of A into B which converges uniformly to the mapping* $f: A \to B$, *then f is strongly quasi-open.*

2. *If the sequence of functions* $w_n = f_n(z)$, *each differentiable in a region S of Z, converges uniformly in S to a nonconstant function* $w=f(z)$, *then f is light.*

Thus by No. 1 quasi-openness carries over to the limit function from members of the sequence. In general lightness does not carry over for sequences of mappings—indeed not even for sequences of homeomorphisms. Remarkably, however, by No. 2 it does carry over for uniformly convergent sequences of differentiable functions. Now applying No. 1 to the case covered by No. 2 we see at once that, in No. 2, f is also strongly quasi-open because this holds for each f_n. Thus, in No. 2, f is both light and strongly open and hence has the characteristic topological character of an analytic function. We have then a theorem giving the topological content of the Weierstrass Double Series Theorem.

Now using the lightness of f as given in No. 2, which enables us to apply the circulation index to f, the theorem of Hurwitz is readily obtained. This theorem asserts that, under conditions in No. 2, if ζ is an m-fold zero of $f(z)$, then every sufficiently small neighborhood D of ζ contains exactly m zeros of $f_n(z)$ for $n > N(D)$. This important conclusion is thus obtained purely from the topology of the situation with no need for knowledge of the differentiability of the limit function f.

12. **Dimension and nondensity preservation of mappings. General setting.** The remainder of this lecture will be devoted to a consideration of results and questions concerned with dimension and nondensity preservation of mappings and their applications to differentiable functions. The results to be mentioned are largely new ones, announced here for the first time. We begin by recalling a simple example of a monotone mapping of a square S onto a square Σ which alters dimension of a compact nondense subset. This example is not new, though perhaps it is put to a new use. This mapping can be effected by dividing S into 9 equal squares and mapping the middle one onto

the center of Σ; then divide each of the 8 remaining squares into 9 equal squares and map the 8 middle ones onto 8 points symmetrically distributed in Σ and so on. This procedure can be continued so as to obtain a uniformly continuous mapping of a set of squares in S with union dense in S onto a set of points dense in Σ; and the continuous extension of this mapping is our required mapping ϕ of S onto Σ. When carefully described, ϕ will be monotone and if K denotes the complement in S of the union of the interiors of the center squares selected in the definition, then K is nondense and thus 1-dimensional whereas $\phi(K)=\Sigma$ and thus is 2-dimensional. Also, it should be noted that on the set $D_\phi = S-U$, where U is the union of all center squares, ϕ is 1-1 and D_ϕ is dense in K and $\phi(D_\phi)$ is dense in Σ. However, D_ϕ is not dense in any open set in S.

Thus it is possible to alter dimension of a compact subset under a mapping of a quite restricted type even when the dimension of the whole set is not changed. This fact is not new by any means as I believe it was mentioned by Menger [12] in his early papers on dimension theory in connection with this same example. In this connection we recall next a theorem of Alexandroff's [13] to the effect that under open mappings having totally imperfect point inverses, dimensionality of the whole space is not altered provided the original space X is separable, metric and locally compact and the image space is metric. This result does not apply to show that the dimension of subsets of X are not altered under these conditions, because the openness assumed on X does not necessarily hold on the subset of X in question. However, using the method of Alexandroff, it is readily shown that *if X and Y are locally compact, separable and metric and $f(X)=Y$ is an open mapping having scattered point inverses, then X is the union of a countable sequence of compact sets X_n such that $f|X_n$ is topological.* It follows from this, of course, that for any closed set $K\subset X$, $\dim f(K)=\dim K$. This result in this form is of importance and will be used later in connection with our discussion of differentiable functions.

Now for a mapping in general, $f(X)=Y$, we let L_f denote the set of all $x\in X$ such that $f^{-1}f(x)$ is totally disconnected and D_f the set of all $x\in X$ such that x is a component of $f^{-1}f(x)$. Obviously $L_f\subset D_f$ for any f. We shall say that f *preserves nondensity* for compact sets provided that if K is a compact nondense set in X, i.e., K contains no open set in X, then $f(K)$ is nondense in Y. If X and Y are locally compact separable and metric and the mapping $f(X)=Y$ preserves nondensity of compact sets, it turns out that $L_f\neq 0$. Indeed it can be proven that *$f(L_f)$ is dense in B and L_f itself is semi-dense X*, i.e., dense in some open subset of every open set U in X with an open image. The proof here presents little difficulty and follows readily using countable coverings by small open sets and taking unions of boundaries. It may be remarked also that the same conclusion about L_f can be obtained under the assumption that f preserves nondensity for compact sets K with $\dim K < k = \dim X < \infty$. It is of interest to note again that in the example of the mapping ϕ just given, $f(L_\phi)$ is dense in Y but L_ϕ is not dense in any open set whatever of S. Here L_ϕ and D_ϕ are the same, of course, since ϕ is monotone.

13. **Quasi-open mappings on 2-manifolds.** We now limit ourselves to the specific setting in which the original space X is a 2-manifold and where the mapping is quasi-open. We continue using the notion of *semi-density* in the sense defined in §12. First we have the

THEOREM. *If X is a region on a sphere, $f(X) = Y$ is compact and quasi-open and no component of a point inverse separates X, then* $\dim f(K) \leq \dim K$ *for all compact* 1*-dimensional sets K in X if and only if D_f is semi-dense in X.*

Compactness of the mapping f here means that the inverse of every compact set is compact or, equivalently, that f is closed and point inverses are compact. The "only if" part of this theorem is an immediate consequence of the result just quoted for mappings in general, because $D_f \supset L_f$. The proof of the reverse implication, that density of D_f is an open subset of every open set in X with an open image implies the asserted dimension invariance is effected by an interesting combination of factorization of the mapping along with the invariance of dimension under open mappings with scattered point inverses quoted earlier. For compactness of f enables us to factor f into monotone and light factors:

$$f = lm, \qquad m(X) = X', \qquad l(X') = Y.$$

Then the monotone mapping m can be "extended" to the whole sphere S on which X lies by decomposing S into the sets $m^{-1}(x')$, $x' \in X'$, together with components of $S - X$. The natural mapping $\phi(S) = S'$ of this decomposition maps S monotonically onto another sphere S' by a classic result of R. L. Moore [14]. Then $l(X') = Y$ is a light, open, compact mapping of X', a region on S', onto a set Y which is necessarily a 2-manifold. Thus l necessarily has finite point inverses. Our density assumption for D_f insures that $\dim m(K) \leq 1$ when $K \subset X$ is compact and of dimension ≤ 1 because m is 1-1 on D_f, of course, and $\dim lm(K) \leq 1$ since l is open and has finite point inverses.

The general case of a quasi-open mapping $f(X)=Y$ of an arbitrary 2-manifold X without edge points onto a locally connected generalized continuum Y can be reduced essentially to the case just discussed where X is a region on a sphere by means of the following:

LEMMA. *Suppose $f(L_f)$ is dense in Y and that for some compact nondense set K in X, $f(K)$ contains an open set in Y. Then there exists a region R in X contained in a 2-cell of X such that $Q=f(R)$ is open in Y, the mapping $f(R)=Q$ is compact and quasi-open and for some compact subset K_1 of $K\cdot R$, $f(K_1)$ contains an open set.*

This reduction is effected by taking a point $y\in f(L_f)\cdot V$ where V is an open set in $f(K)$ and covering each $x\in Kf^{-1}(y)$ by a 2-cell in X whose edge does not intersect $f^{-1}(y)$. After reducing this to a finite covering, it can then be shown that one of these 2-cells includes a region R meeting our conditions. Using this reduction we can then apply the previous theorem and thus handle the general case embodied in the

THEOREM. *Given a quasi-open mapping $f(X)=Y$ of a 2-manifold X without edge points onto a locally connected generalized continuum Y such that no component of a point inverse lying inside a closed 2-cell on X separates X. In order that f preserve nondensity for compact sets it is necessary and sufficient that the set L_f be semi-dense in X.*

In case the space Y is a 2-manifold, or if it has the property that every subset of dimension $\geqq 2$ contains an open set, the conclusion of *preserving nondensity* clearly is the same as *preserving the property of being of dimension* $\leqq 1$ for compact subsets. If $f^{-1}(y)$ has a degenerate component (or a component lying in a 2-cell of X which it does not separate) for every $y\in Y$, then Y will be a 2-manifold and the theorem could then be stated in terms of dimension preserving.

14. **Differentiability and dimension raising.** Returning now to the case of a function $w=f(z)$ continuous on a region X of Z we note at once that if $f(z)$ is differentiable everywhere in X, then the mapping is light and open so that $L_f=X$. Accordingly by the theorems just discussed, any compact nondense set K in X has a nondense image set or, in other words, $\dim f(K)\leqq\dim K$. This fact also is a consequence of the theorem stated earlier about open mappings with scattered point inverses, because in this situation f not only is light but has scattered point inverses. However, it is of interest and of some importance to know to what extent the differentiability assumption can be weakened. Making no use of the lightness and openness of differentiable functions it is still possible to prove the following

THEOREM. *Let $w=f(z)$ be continuous in a region X of Z and differentiable at all points of $f^{-1}(Y_0)$ for some open subset Y_0 of Y dense in Y. Then f is strongly quasi-open, no component of a point inverse lying inside a closed 2-cell in X separates X and the set $f(L_f)$ is dense in Y. Further, if f is not constant on any open set in X, L_f is semi-dense in X.*

Under the condition last stated, all conditions of the theorem for 2-manifolds are satisfied so that f preserves nondensity for compact sets, or does not raise dimensionality for any compact set of dimension ≤ 1. In this connection it is of interest to note that in the example given earlier of a monotone mapping from a square S to a square Σ, the mapping function ϕ there is differentiable and $\phi' \equiv 0$ in the open everywhere dense subset U of S consisting of the union of all interiors of center squares selected. However, U is not an inverse set nor is its image open in Σ. As already noted here, although $f(L_\phi)$ is dense in Σ, L_ϕ itself is not dense in any open set whatever of S.

15. **Applications. Concept of a pole.** We now apply these results to a situation in which dimension or nondensity preservation is important, as are also reduced differentiability assumptions. Suppose our function $w=f(z)$ is nonconstant and continuous in a region R of Z and differentiable on $R-H$ where H is a closed nondense subset of R on which f is constant. Can it still be asserted that f is light so that in particular H is totally disconnected? The answer here is: Yes, it can. However, the proof is far from easy and it involves these same issues we have been discussing. For in the proof of lightness of a differentiable function referred to earlier, the new function g which was mentioned is defined as a product of a finite number of factors each of which is a value of f at a point obtained by rotating the given z about a fixed center z_0 [see (†) above]. Thus we can be sure of differentiability of g only on the complement of the union of the images of H under the finite number of rotations involved. However, it can be shown that g will be differentiable on everywhere dense open inverse sets and hence our previous theorem applies to give us strong quasi-openness of g and this is exactly what is needed to show that H is totally disconnected. This yields the following results, listed A–D.

A. *If $w=f(z)$ is nonconstant and continuous in a region R and differentiable on $R-f^{-1}(a)$ where a is some value of f in R, then f is light and strongly open in R.*

It could be asserted, further, that f has completely scattered point inverses, so that $f^{-1}(a)$, in particular, is completely scattered. Also f is locally topologically equivalent to a power mapping even at points of $f^{-1}(a)$.

B. *Let* $w=f(z)$ *be differentiable in* $R-K$ *where* K *is a closed nondense subset of* R *such that for each* $z_0 \in K$ *we have* $\lim_{z\to z_0} f(z)=\infty$. *Then* K *is a completely scattered set and the mapping of* R *into the complex sphere is light and strongly open.*

Thus all points of K are topologically precisely like poles of f. Hence the concept of a pole is approachable topologically. No reference to expansions in series need be made. A pole is simply a point where $f(x)\to\infty$ and such that in some neighborhood of this point there are only points of this same type and points where f is finite and differentiable.

C. *Let* $w=f(z)$ *be nonconstant and differentiable in a region* R. *Then if* f *is continuous at all points of a continuum* K *of* $\mathrm{Fr}(R)$ *and constant on* K, K *is a continuum of condensation of* $\mathrm{Fr}(R)$.

D. *In particular, in* C, *if* R *is an elementary region, there can exist no arc in* $\mathrm{Fr}(R)$ *on which* f *is continuous and constant. Thus if* f *is continuous on* $\mathrm{Fr}(R)$, *it must be light on* $\mathrm{Fr}(R)$.

BIBLIOGRAPHY

1. S. Stoïlow, *Principes topologiques de la théorie des fonctions analytiques*, Paris, 1938.

2. S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. vol. 26 (1936) pp. 61–112.

3. C. Kuratowski, *Homotopie et fonctions analytiques*, Fund. Math. vol. 33 (1945) pp. 316–367.

4. M. Morse and M. H. Heins, *Deformation classes of meromorphic functions and their extensions to interior transformations*, Acta Math. vol. 79 (1947) pp. 51–103.

5. R. Nevanlinna, *Eindeutige Analytische Funktionen*, Berlin, Springer, 1936.

6. L. V. Ahlfors, *Development of the theory of conformal mapping*, . . . , "Contributions to the theory of Riemann surfaces," Annals of Mathematics Studies, No. 30, 1953.

7. H. D. Ursell and H. G. Eggleston, *On the lightness and strong interiority of analytic functions*, J. London Math. Soc. vol. 27 (1952) pp. 260–271.

8, C. J. Titus and G. S. Young, *A Jacobian condition for interiority*, Michigan Mathematical Journal vol. 1 (1952) pp. 89–94.

9. P. Alexandroff and H. Hopf, *Topologie*, Berlin, Springer, 1935, p. 65.

10. G. T. Whyburn, *On quasi-compact mappings*, Duke Math. J. vol. 3 (1952) pp. 445–446. See also R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications vol. 32 (1949) p. 70.

11. ———, *Open and closed mappings*, Duke Math. J. vol. 17 (1950) pp. 69–74.

12. K. Menger, *Zur allgemeinen Kurventheorie*, Fund. Math. vol. 10 (1927) pp. 106–107.

13. P. Alexandroff, *Über abzählbar-fache offene Abbildungen*, C. R. (Doklady) Acad. Sci. URSS. N.S. vol. 4 (1936) pp. 295–299.

14. R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 416–428.

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THE MATHEMATICAL THEORY OF THREE-DIMENSIONAL CAVITIES AND JETS

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1. Introduction. The theory of two-dimensional steady flows with free streamlines has been studied intensively for over a century and a thorough knowledge has been obtained both of methods for solving special problems in closed form and of techniques for establishing the existence of solutions for more general models. In contrast, the principal investigations concerning three-dimensional free surface flow have only appeared during the past decade and progress in this direction has been confined almost entirely to problems possessing axial symmetry. Even in the case of axial symmetry, most of the discussions have centered about existence theorems [2] or qualitative properties of the flow [3; 4], and very little has been learned about explicit treatment of the specific models which arise in engineering applications. We shall describe in the present article a systematic procedure for the numerical solution of the free boundary problem which was developed in an effort to fill this gap.

The chief significance of the method which we shall outline lies in the wide variety of free boundary problems to which it applies. However, for the sake of clear, concise presentation we prefer to restrict our attention to one quite special, albeit important, example, with the belief that in this manner we illustrate the basic ideas better and still indicate their full scope. Thus we wish to place as much emphasis on the general approach to be described as we give to the particular numerical results mentioned, some of which might even be obtained more simply by a slightly different attack [1].

We shall study in detail the axially symmetric Riabouchinsky flow past a circular disk. This model consists of a flow parallel to the x-axis past an obstacle composed of two identical disks, perpendicular to the x-axis, which are joined by a free surface attached at their perimeters. The motion is governed in the meridian (x, y)-plane by a Stokes stream function ψ satisfying the partial differential equation

$$\psi_{xx} + \psi_{yy} - \frac{1}{y}\psi_y = 0. \tag{1}$$

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Along the x-axis, or axis of symmetry, along the two circular disks $x=\pm X$, $0\leq y\leq Y$ comprising the nose and tail of the obstacle, and along the free streamline joining these disks in the meridian plane, the Stokes stream function ψ vanishes,

$$\psi = 0. \tag{2}$$

In addition, along the free streamline the normal derivative $\partial\psi/\partial n$ satisfies the condition

$$\frac{1}{y}\frac{\partial\psi}{\partial n} = 1, \tag{3}$$

which states that the pressure is constant there. At infinity ψ has the expansion

$$\psi = \frac{y^2}{2(1+\sigma)^{1/2}} - \frac{ay^2}{(1+\sigma)^{1/2}r^3} + \cdots, \qquad r^2 = x^2 + y^2, \tag{4}$$

where σ is the so-called cavitation parameter of the flow and where the coefficient a is connected with the virtual mass of the obstacle [1; 2].

In the physical interpretation of the Riabouchinsky model, the free surface is to be thought of as a surface of discontinuity separating the moving liquid from a vapor cavity which is considered to be at rest. The principal difficulty encountered in the mathematical analysis of the model lies in finding the shape of the free streamline, which is presumably determined by the extra boundary condition (3). The cavitation parameter σ, which gives a dimensionless measure of the difference between the pressure at infinity and the pressure in the vapor cavity, varies monotonically with the distance $2X$ between the two disks generating the obstacle, for a fixed choice of their radius Y, and hence its determines the size of the cavity.

We shall be interested in developing a systematic, and at the same time practical, procedure for calculating the stream function ψ characterized by the conditions (1), (2), (3) and (4). The first possibility which suggests itself is to attempt to derive a construction of ψ from the proof of the existence of the solution of the axially symmetric free surface flow problem [2]. The existence proof is based on a variational principle which states that the flow with a free boundary fulfilling (3) is the one which solves the extremal problem

$$4\pi a - (1+\sigma)V = \text{minimum} \tag{5}$$

among all flows ψ of the above type satisfying (1), (2) and (4) alone, where a and σ are defined by (4) and where V denotes the volume of the varying obstacle. This minimum principle does not actually lead to a useful construction of the flow pattern, but merely serves to indicate that the stationary functional on the left in (5), which represents the energy of the motion, can be estimated from above with exceptional accuracy by computing the corresponding quantity for the flow past an obstacle whose boundary approximates the true free surface.

Although the energy is in general a quantity of secondary physical interest in hydrodynamics, it turns out that for our special example of the steady state cavity flow past a circular disk we can express the drag in terms of $4\pi a-(1+\sigma)V$ and thus can apply the result (5) effectively. We define the drag coefficient $C_D=C_D(\sigma)$ by the formula

$$\frac{C_D}{1+\sigma} = 1 - \frac{2}{Y^2}\int_0^Y \left(\frac{\partial\psi}{\partial n}\right)^2 \frac{dy}{y}, \tag{6}$$

where the integral is evaluated over only one of the circular disks forming the obstacle. The dimensionless coefficient C_D is a measure of the pressure force acting on this disk, which is to be thought of as the actual body placed in the fluid, whereas the remaining disk should be interpreted as an idealization of the turbulent wake at the opposite end of the cavity. The relation between the drag and the energy for flow past a circular disk can be written in the form

$$\frac{2\pi XY^2}{3} C_D = (1+\sigma)V - 4\pi a, \tag{7}$$

and it obviously permits us to obtain from the variational principle (5) good lower bounds on the drag coefficient C_D whenever the cavitation parameter σ is known in terms of X and Y.

Contour integration yields an easy proof of the identity (7). We note that, together with ψ, the function $\phi=x\psi_x+y\psi_y$ is also a solution of (1). Hence, letting s denote arc length, we have

$$\int \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)\frac{ds}{y} = 0 \tag{8}$$

whenever the path of integration is a simple closed curve surrounding a portion of the flow region. If we choose for this path a large semicircle and two segments of the x-axis, plus the intersection of the meridian plane with the two disks and the free surface bounding our obstacle, and if we allow the radius of the large semicircle to become infinite, we find readily that (8) leads to (7).

The relation (7) provides only one step toward a description of the cavity flow past a disk. A disadvantage of the formula is that when

X and Y are given, not only the coefficient a, but also the cavitation parameter σ must be calculated before an application can be made. Even after this has been done, a loss of significant figures can result from the cancellation on the right between the relatively large terms $(1+\sigma)V$ and $4\pi a$. Thus we turn in the next section to a completely independent, and in practice more effective, attack on the free boundary problem which, however, has not yet produced a rigorous proof of the existence of the solution.

2. **Stationary combination of the boundary conditions.** As we pointed out earlier, the main trouble in solving the boundary value problem (1), (2), (3), (4) is our ignorance of the position of the free surface. In order to overcome this difficulty, we find a linear combination of the two boundary conditions (2) and (3) which is stationary with respect to normal displacements of the free streamline. This enables us to derive a decisively improved approximation to the free boundary from any reasonably chosen first guess and thus to set up a rapidly convergent iterative scheme for the solution of the problem.

On the free streamline we can rewrite the differential equation (1) for ψ in terms of normal and tangential coordinates in the form

$$\frac{\partial}{\partial n}\frac{1}{y}\frac{\partial\psi}{\partial n}+\frac{\kappa}{y}\frac{\partial\psi}{\partial n}=0 \tag{9}$$

by virtue of the boundary condition (2), where κ denotes the curvature of the streamline. Through another application of the boundary condition (2), we derive from (9) the result

$$\frac{\partial}{\partial n}\left(\frac{1}{y}\frac{\partial\psi}{\partial n}+\frac{\kappa}{y}\psi\right)=0. \tag{10}$$

We can combine (2) and (3) on the free streamline to obtain

$$\frac{1}{y}\frac{\partial\psi}{\partial n}+\frac{\kappa}{y}\psi=1 \tag{11}$$

there, and (10) shows that this particular boundary condition is stationary in the sense that a shift of the free boundary by an infinitesimal amount δn along its inner normal results in an error of the order of magnitude $(\delta n)^2$ in the relation (11), with the relevant quantities evaluated along the displaced curve. Thus the solution ψ of the boundary value problem defined by the partial differential equation (1), by the boundary condition (2) imposed along the x-axis and on the circular disks bounding our obstacle, by the asymptotic expansion (4) at infinity, and by the boundary condition (11) imposed along an approximation to the free streamline corresponding to a normal shift of the true curve by the amount δn, yields an approximation for the stream function of the actual free surface flow with an error of the order of magnitude $(\delta n)^2$.

On our first approximation to the free boundary, the stream function ψ found in the above manner will not in general vanish, and, indeed, the quantity

$$\delta n = -\psi/y \tag{12}$$

is easily seen to represent a normal shift that carries this first guess into a second approximate free streamline which is accurate up to terms of the order $(\delta n)^2$. We can repeat the construction of ψ, using this improved form of the free surface, to obtain a third approximation with an error of the order $(\delta n)^4$, and so forth. Thus an iterative procedure, based essentially on linearization, is established for the solution of the free boundary problem. The convergence of the method is quite rapid, since the error is virtually squared at each successive step.

It should be pointed out that in mathematical examples the geometric parameters X and Y will be given rather than the cavitation parameter σ, which appears in the expansion (4). No real difficulty is involved here, since it is known [1] that the stream function ψ has at the point of separation (X, Y) of the free streamline from the fixed boundary a regular series expansion in terms of the real and imaginary parts of the complex variable $(x+iy-X-iY)^{1/2}$, and since the boundedness of the velocity of the flow thus implies the nontrivial requirement

$$\lim_{x\to X, y\to Y} (x + iy - X - iY)^{-1/2}\psi(x, y) = 0 \tag{13}$$

there, which suffices to determine σ uniquely. In practice it is best to sharpen (13) by adding the more precise condition

$$\frac{1}{y}\frac{\partial\psi}{\partial n} = 1 \tag{14}$$

at the point of separation.

The boundary value problem defined by (1), (4), (11), (13) and (14), plus (2) along the fixed boundary, has a unique solution, and we have found in specific applications that it overcomes quite effectively the difficulty arising because the shape of the free boundary is not given in advance. Thus there only remains to discuss a linear mixed boundary value problem in a specific domain. The following section will be devoted to a numerical treatment of this in principle simpler question.

3. Numerical analysis of the flow pattern. The method we shall use to solve numerically the boundary value problem (1), (4), (11), (13), (14), with (2) prescribed along the fixed boundary, consists in expressing the stream function ψ as a linear combination

$$\psi = \sum_{j=1}^{N} \lambda_j p_j \qquad (15)$$

of suitable known solutions p_j of (1) with coefficients λ_j determined by the requirement that the boundary conditions be satisfied in the sense of a least squares approximation at $M > N$ properly selected points. We thus choose the simplest, most direct type of interpolation to fit the boundary conditions in preference, for example, over that based on the Dirichlet integral associated with the partial differential equation (1), since we can thus avoid numerical integration and the resulting difficulties with accuracy.

We reject two other less promising approaches for the solution of our boundary value problem, namely, the method of integral equations and the method of finite differences. It would be quite tedious to calculate with sufficient accuracy the integrals involved in the first of these two methods because of the logarithmic singularity of the kernel associated with the differential equation (1), and thus the computational work would considerably exceed what we are prepared to undertake here. The same criticism of prohibitively onerous calculation applies even more strongly to the method of finite differences, and it is questionable whether relaxation should be attempted at all in cases, such as the one here, where many types of explicit solutions of the differential equation are known. The principal disadvantage of the interpolation technique based on (15) which we intend to use is that its accuracy is worst at the boundary, but this drawback is offset by the rapid over-all convergence of the procedure. It should be noted, furthermore, that the parameters a and σ appearing in the formula (7) for the drag coefficient C_D can be computed exceptionally well by our method.

For a first guess of the shape of the free boundary we choose an affine transformation of the free streamline of the relevant plane Riabouchinsky flow past a flat plate. This curve can be described completely in terms of the Jacobian elliptic functions sn t, cn t, dn t and zn t, whose modulus we denote by k. Thus our first approximation of the free boundary has the form

$$x = k^{-1}\operatorname{zn} t + tk^{-1}(EK^{-1} - k^{*2}), \tag{16}$$

$$y = k^{-1}(k^{*2} + E^{*} - k^{2}K^{*}) + \beta(\operatorname{dn} t - k^{*}), \tag{17}$$

where k^{*} is the complementary modulus such that $k^{*2}+k^{2}=1$, where K and E are the complete elliptic integrals of the first and second kinds with modulus k, where K^{*} and E^{*} are the complete elliptic integrals of the first and second kinds with modulus k^{*}, and where β is a parameter associated with the above affine mapping. The variable t ranges over the interval $-K \leqq t \leqq K$, and the coordinates X and Y of the point of separation are given by

$$X = k^{-1}(E - k^{*2}K), \qquad Y = k^{-1}(k^{*2} + E^{*} - k^{2}K^{*}). \tag{18}$$

For our numerical example we assign to X and Y specific values by setting $k=.96$. Furthermore, we take $\beta=.25629688$ on the basis of earlier calculations [1] which are equivalent, in effect, to one iteration of the scheme outlined in the previous section. The second iteration, to be discussed here, will yield an adequate description of the physical phenomena connected with the free surface flow problem.

To define solutions of (1) for substitution into (15) which are appropriate for a representation of the flow near the separation point, we shall need the two pairs of elliptic coordinates μ_m and ν_m given by

$$\begin{aligned}\mu_m = [\{X + (-1)^m x\}^2 + y^2 - Y^2 + ([\{X + (-1)^m x\}^2 \\ + y^2 - Y^2]^2 + 4Y^2\{X + (-1)^m x\}^2)^{1/2}]^{1/2}/(2^{1/2}Y),\end{aligned} \tag{19}$$

$$\nu_m = \frac{X + (-1)^m x}{Y\mu_m}, \tag{20}$$

for $m=1$, 2. The coordinate systems μ_1, ν_1 and μ_2, ν_2 have a square root behavior which is suitable for expansion of the stream function ψ in neighborhoods of the points (X, Y) and $(-X, Y)$. However, it is necessary to exercise a certain amount of caution if we wish to compute, for example, μ_1 and ν_1 in terms of t near $t=K$. The proper technique is to use the expressions

$$x - X = k^{-1}(t - K)(EK^{-1} - k^{*2}) + k^{-1}\operatorname{zn} t, \tag{21}$$

$$y^2 - Y^2 = 2Y\beta k^2\,\frac{\operatorname{cn}^2 t}{\operatorname{dn} t + k^{*}}\left[1 + \frac{\beta k^2}{2Y}\,\frac{\operatorname{cn}^2 t}{\operatorname{dn} t + k^{*}}\right] \tag{22}$$

in this range and to calculate the zeta-function $\operatorname{zn} t$ by means of its Taylor's series about $t=K$. The latter can be found easily through a power series solution of the ordinary differential equation

$$\frac{d^2}{dt^2}\operatorname{cn}^2 t = 2k^{*2} + (8k^2 - 4)\operatorname{cn}^2 t - 6k^2\operatorname{cn}^4 t \tag{23}$$

for $\operatorname{cn}^2 t = k^{-2}(EK^{-1} - k'^2) + k^{-2} d(\operatorname{zn} t)/dt$. Elsewhere, of course, it is better to represent the Jacobian elliptic functions in terms of theta-functions.

We shall take $N = 20$ as the number of explicit solutions p_j of (1) to be used in the present application of the approximation (15) for the stream function ψ. Let

$$x_1 = \xi - x, \quad x_2 = \xi + x, \quad r_1 = (x_1^2 + y^2)^{1/2}, \quad r_2 = (x_2^2 + y^2)^{1/2}. \tag{24}$$

We choose as our twenty solutions p_j the functions

$$p_1 = y^2, \tag{25}$$

$$p_j = y^2 \sum_{m=1}^{2} \left[P'_{2j-1}(\nu_m) q'_{2j-1}(\mu_m) + (-1)^j j(2j - 1) q'_1(\mu_m)\right], \qquad j = 2, \cdots, 6, \tag{26}$$

$$p_j = 4\xi y^2 \frac{r_1 + r_2}{r_1 r_2[(r_1 + r_2)^2 - 4\xi^2]}, \qquad \xi = .222(j - 6),\ j = 7, 8, 9, \tag{27}$$

$$p_j = y^2 \frac{P'_n(x_1/r_1)}{r_1^{n+2}} + y^2 \frac{P'_n(x_2/r_2)}{r_2^{n+2}}, \qquad \xi = .666,\ n = j - 9,\ j = 10, \cdots, 20, \tag{28}$$

where $P'_n(z)$ denotes the derivative of the nth Legendre polynomial and where $q'_n(z)$ is a Legendre function of the second kind defined by

$$\begin{aligned} q'_n(z) = &- \frac{2^{n+2}(n + 1)!}{1\cdot 3\cdot 5 \cdots (2n + 1)} \left[\frac{1}{z + (z^2 + 1)^{1/2}}\right]^{n-1} \\ &\cdot \left[\frac{1}{2(z^2 + 1)^{1/2}\{z + (z^2 + 1)^{1/2}\}}\right]^{3/2} \\ &\cdot F\left[\frac{3}{2}, -\frac{1}{2}; n + \frac{3}{2}; \frac{1}{2(z^2 + 1)^{1/2}\{z + (z^2 + 1)^{1/2}\}}\right] \end{aligned} \tag{29}$$

in terms of the classical hypergeometric series $F[a, b; c; w]$. The function (25) represents uniform flow; the ellipsoidal harmonics (26) represent flows with appropriate square root singularities at the point of separation; the source and sink functions (27) are familiar in the theory of flows past long, slender bodies of revolution; and, finally, the spherical harmonics (28) are designed to smooth out the more rapid oscillations in the error when we interpolate to fit the boundary conditions of our problem.

The functions $p_1, \cdots, p_{10}$ are all symmetric in the y-axis and all vanish on the x-axis. Thus it suffices to impose the boundary condition (11) along the arc $0 \leqq t \leqq K$ of the approximate free boundary curve (16), (17), and it suffices to impose the boundary condition (2) along the segment $x = X$, $0 \leqq y \leqq Y$. Our formulation of the problem will give special attention to the requirement, essential in practice, that the boundary conditions be weighted near the separation point in a manner adapted to the square root singularity of the stream function there. Thus we interpolate at the points $z_l = x_l + iy_l$ on the free boundary corresponding to the values $(l-1)K/36$ of the parameter t, with $l = 1, \cdots, 36$, and we interpolate at the points $z_l = x_l + iy_l$ on the fixed boundary given by the relations

$$x_l = X, \quad y_l = [1-(l-38)^2/24^2]Y, \qquad \text{with } l = 38, \cdots, 62.$$

We let $z_{37} = z_{38}$, since we shall impose the extra boundary condition (14) at that point, and we agree to define the curvature κ to be zero there. At each point z_l we set

$$(30) \qquad c_{lj} = \frac{p_j}{y^2}, \qquad d_{lj} = \frac{\partial p_j/\partial n + \kappa p_j}{y(1+\kappa y)},$$

and we set

$$(31) \quad b_l = \frac{1}{1+\kappa y} \text{ for } l = 1, \cdots, 37; \qquad b_l = 0 \text{ for } l = 38, \cdots, 62.$$

We define

$$(32) \quad a_{lj} = d_{lj} \text{ for } l = 1, \cdots, 37; \qquad a_{lj} = c_{lj} \text{ for } l = 38, \cdots, 62.$$

In terms of these quantities, our least squares approximation to the boundary conditions (2), (11), and (14) reduces to the extremal problem

$$(33) \qquad \sum_{l=1}^{62} e_l^2 = \text{minimum}, \qquad e_l = \sum_{j=1}^{N} a_{lj}\lambda_j - b_l,$$

which leads in turn to the symmetric system of simultaneous linear equations

$$(34) \qquad \sum_{j=1}^{N} \left(\sum_{l=1}^{62} a_{lm} a_{lj} \right) \lambda_j = \sum_{l=1}^{62} a_{lm} b_l, \qquad m = 1, \cdots, N,$$

for the determination of the coefficients λ_j in the representation (15) of the stream function ψ.

We computed the values of the coefficients b_l, c_{lj} and d_{lj} to six or seven significant figures on an IBM Card-Programmed Electronic Calculator, and various tricks of numerical analysis were used to avoid loss of accuracy in the results. The d_{lj} for $39 \leqq l \leqq 61$ and the c_{lj} for $1 \leqq l \leqq 36$ were computed in addition to the a_{lj} because they are needed for the calculation of the drag coefficient (6) and the normal shift (12) correcting the shape of the free boundary. The linear equations (34) were solved for the λ_j on an IBM Type 650 Magnetic Drum Data-Processing Machine, and thus the complete numerical solution of our boundary value problem was obtained.

A few comments should be made about the difficulties encountered in the numerical work. Although the conglomeration of solutions (25), (26), (27), (28) of (1) was selected on the basis of a good deal of experience and practical intuition concerning functions which would best represent an axially symmetric free surface flow, it was nevertheless impossible to avoid the occurrence of a certain amount of cancellation among the numerous terms on the right in our expression (15) for the stream function ψ. Such cancellations are always to be expected when we expand the solution of a boundary value problem defined in a peculiarly shaped domain in terms of functions which bear no direct relationship to that domain. In the present example, these cancellations contributed in the worst cases a loss of as many as two significant figures in our calculations, and this outcome was reflected in relative smallness of the determinant of the system (34). Thus it was necessary to compute the coefficients a_{lj} with exceptional accuracy in the first place in order to obtain meaningful final results.

4. **Interpretation of the data.** For the method sketched in the previous section, it is desirable to check numerically the convergence of the representation (15) as the number N of interpolating functions increases, and it is desirable to check that these functions have been evaluated at a sufficiently large number of points to describe all the oscillations of the error in fitting the boundary conditions. In our calculation we tested both these items by working out the solution of the problem using many different subsets of our basic system (25), (26), (27), (28) of twenty solutions of the differential equation (1). The results gave clear indication of geometric convergence of the procedure and established that the more reliable answers were those corresponding to the smaller values of the least squares error $62^{-1/2}(\epsilon_1^2+\cdots+\epsilon_{62}^2)^{1/2}$, as was, of course, to be expected. However, it turned out that the determinant of the system (34) was so small when all twenty interpolating functions were used that it was preferable to omit the nineteenth, which appeared to be the least significant one. In Tables I, II and III we give the complete numerical data compiled from the solution based on the first sixteen interpolating functions, from the solution based on the first eighteen interpolating func-

tions, and from the final solution based on all the interpolating functions except the nineteenth. In this section we shall discuss in detail the conclusions to be drawn from these results and our estimate of the errors involved.

The oscillating character of the distribution of values of the normal derivative $\partial\psi/y\partial n$ computed at the points $z_{39}, \cdots, z_{61}$ along the obstacle appears to be the largest source of error. Such oscillations are to be expected for the type of numerical approximation we have used in calculating the stream function ψ, and they should average out appreciably when we integrate to obtain the drag coefficient C_D from (6). However, since the pressure distribution even fails to be monotonic, we shall carry through an alternate computation of C_D based on the variational formula (7) and compare the answers found by the two different methods in order to establish their validity.

In applying (7), we shall use the value of the cavitation parameter σ calculated on the basis of the stationary boundary condition (11), whereas we use values of the volume V and the virtual mass coefficient a which are obtained from the classical potential flow past the body bounded by the two circular disks $x = \pm X$, $0 \leqq y \leqq Y$ and the approximate free streamline (16), (17). The stream function ψ^* of the latter flow is computed from a representation of the form

$$\psi^* = p_1 + \sum_{j=2}^{N} \lambda_j^* p_j \tag{35}$$

with coefficients λ_j^* determined by a least squares approximation to the boundary condition $\psi^* = 0$, which is imposed along the entire perimeter of the obstacle. Thus the boundary condition reduces in this case to the system of simultaneous linear equations

$$\sum_{j=2}^{N} \left(\sum_{l=1}^{38} c_{lm} c_{lj} + \sum_{l=39}^{62} c_{lm} c_{lj} \right) \lambda_j^* + \sum_{l=1}^{38} c_{lm} + \sum_{l=39}^{62} c_{lm} = 0 \tag{36}$$

for the λ_j^*. By means of (35) and (36) we transform (7) into the approximate relation

$$\frac{C_D}{1+\sigma} = 11.7737 - \frac{13.3469}{1+\sigma} \tag{37}$$

between the drag coefficient C_D and the cavitation parameter σ. For the actual computation here, we substituted into (35) all the explicit flows (25), (26), (27), (28) except the nineteenth. This gave the result $a = .0435088$, whereas substitution of only the first eighteen flows gave

$a = .0435114$ and substitution of only the first sixteen flows gave $a = .0435182$, which indicates that the numerical coefficients in (37) are exceptionally accurate.

From an earlier calculation [1] based on only ten solutions of (1) and based on a poorer choice of the stretching parameter β appearing in our approximation (16), (17) of the free boundary, we obtain the alternate numerical form

$$\frac{C_D}{1+\sigma} = 11.1910 - \frac{12.6498}{1+\sigma} \tag{38}$$

of the variational formula (7), which is useful as a check on the reliability of our results because it is derived from data independent of the computations discussed in this article.

Theoretically, both formula (37) and formula (38) should yield lower bounds on the drag coefficient C_D when σ is given, but the errors in the data presented in the tables overshadow this effect. It is consistent with the variational theory that in each table the value of $C_D/(1+\sigma)$ derived from (37) exceeds the corresponding value derived from (38), since (38) involves a less accurate choice of the free boundary. However, even in Table III the value of $C_D/(1+\sigma)$ found from (6) by direct application of Simpson's rule lies between the two values based on the variational formula (7), and the magnitude of the discrepancy here must serve as an estimate of the error occurring in our numerical evaluation of the cavitation parameter σ and of the integral in (6). Within the margin of accuracy necessary for a physically significant description of the flow, the general extent of agreement between the results based on (6) and the results based on (7) suffices, on the other hand, to establish the reliability of the data.

A study of Tables I, II and III shows that there is a quite definite trend in the numerical data as the number N of interpolating functions is increased. Indeed, as N increases we observe that the corresponding values of the cavitation parameter $\sigma = 1/4\lambda_1^2 - 1$ and of the scaled drag coefficient $C_D/(1+\sigma)$ decrease, as does also the cavity radius d, which is defined to be the value of y at the point on the free streamline where $x = 0$. A crude extrapolation based on the three examples presented in the tables indicates that the errors in the values of σ and d/Y listed in Table III do not exceed 1 per cent and that the true value of $C_D/(1+\sigma)$ must lie between .85 and .88. It is fair to say that the best estimates of these quantities which can be deduced from all the information compiled in our calculation are given by

$$\sigma = .2235, \qquad \frac{d}{Y} = 2.30, \qquad \frac{C_D}{1+\sigma} = .865, \tag{39}$$

and it is quite probable that the errors incurred here are less than 1/2 per cent.

We have presented in the tables values of x, y and $-\psi/y^2$ at the points $s_1, \cdots,$ $s_{[illegible]}$ along the curve (16), (17), and it is an easy matter to compute from this information the normal shift (12) which transforms the first guess (16), (17) of the shape of the free streamline into our final approximation to that curve. The quantity $-\psi/y^2$ can, indeed, be interpreted as a scaled normal shift. We note that it is everywhere positive, indicating that the true free boundary lies above our first guess, while at the same time the normal shift listed in Table III remains small enough so that the final curve lies well below the corresponding plane free streamline, in agreement with general theory [1]. The fact that when the index l increases the values of $-\psi/y^2$ increase significantly before they finally fall off toward zero near the separation point has the interesting interpretation that the axially symmetric free streamline is flatter at the wide section of the cavity than an affine map of the corresponding plane free streamline. This conclusion is consistent with Levinson's asymptotic formula [4] describing the shape of an infinite axially symmetric cavity. We remark that a quite accurate numerical solution of the mixed boundary value problem stated in §2 was required in order to provide a significant estimate of the normal shift.

Our calculation (39) of the scaled drag coefficient $C_D/(1+\sigma)$ combines with earlier investigations [1] to furnish a fairly complete picture of the dependence of this important physical quantity on the cavitation parameter σ. It was shown previously that $C_D/(1+\sigma)$ increases monotonically from the approximate value .827 to the exact value 1 as σ increases from zero to infinity, and that the derivative of $C_D/(1+\sigma)$ with respect to σ vanishes at $\sigma=0$. The result (39) now serves to describe more precisely the rate of growth of the drag coefficient for intermediate values of σ, and, indeed, it would be possible to base on the above data an empirical formula for $C_D/(1+\sigma)$ in terms of σ which would be quite accurate in the physically relevant range.

It is interesting at this stage to compare the plot of the drag coefficient obtained in the above manner with the corresponding graphs of experimental measurements. Such a comparison shows that our figures exceed the experimental values by roughly 2 per cent, an outcome which actually signifies substantial agreement of theory with experiment. Furthermore, the noticeable increase in $C_D/(1+\sigma)$ from .827 to .865 in the interval between $\sigma=0$ and $\sigma=.2235$, which is considerably bigger than the corresponding increment encountered in the two-dimensional flow theory, explains a relatively large slope in the experimentally observed values of C_D which has appeared heretofore to be rather puzzling.

The slight excess in the calculated values of $C_D/(1+\sigma)$ over the measured values may possibly be due to the effect of gravity, since experiments executed at high pressure indicate that an analogous discrepancy with theory (cf. [1]) in the determination of the contraction coefficient of a jet issuing from a circular orifice is caused by gravity. In fact, both the drag coefficient and the contraction coefficient can be expressed linearly in terms of an integral of the form

$$J = \int \left(\frac{\partial\psi}{\partial n}\right)^2 \frac{dy}{y} \tag{40}$$

involving the stream function ψ of the relevant flow, and while a minus sign appears before the integral in the case of the drag coefficient, the corresponding sign for the contraction coefficient is plus. Thus, since the accepted experimental value .61 of the contraction coefficient exceeds by a few per cent the calculated value .58, the apparent effect of introducing gravity is to increase the value assigned to J, and if J were correspondingly augmented in a measurement of the drag coefficient influenced by gravity, we would expect the result to be a value slightly smaller than that given by (39), an outcome consistent with the actual facts.

In closing, we should point out that the drag coefficient C_D and the cavitation parameter σ can be estimated from the variational formula (7) independently of the stationary boundary condition (11) formulated in §2. This is done simply by solving the two simultaneous equations (37) and (38), based on different approximations to the shape of the free surface, for σ and C_D. The answers obtained in this manner are $\sigma = .20$ and $C_D/(1+\sigma) = .62$, and the large errors involved are merely a consequence of the sizeable cancellations which prove to be unavoidable in such an approach. The only advantage which can be claimed for this method is that it depends only on calculation of the virtual mass coefficient a, which is probably the easiest and most accurate numerical step we have undertaken.

REFERENCES

1. P. R. Garabedian, *Calculation of axially symmetric cavities and jets*, Pacific Journal of Mathematics (to appear).
2. P. R. Garabedian, H. Lewy and M. Schiffer, *Axially symmetric cavitational flow*, Ann. of Math. vol. 56 (1952) pp. 560–602.
3. D. Gilbarg, *Uniqueness of axially symmetric flows with free boundaries*, Journal of Rational Mechanics and Analysis vol. 1 (1952) pp. 309–320.
4. N. Levinson, *On the asymptotic shape of the cavity behind an axially symmetric nose moving through an ideal fluid*, Ann. of Math. vol. 47 (1946) pp. 704–730.

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TABLE I

l	x_l	y_l	σ_l	$-\psi/y^2$	λ_l
1	.00000	.33099	−.00304	.0416	.451052
2	.07168	.33033	−.00140	.0421	$-.732372\ 10^{-1}$
3	.14257	.32837	.00255	.0433	$-.127971\ 10^{-1}$
4	.21190	.32516	.00555	.0441	$-.242056\ 10^{-2}$
5	.27898	.32079	.00306	.0437	$-.136522\ 10^{-3}$
6	.34319	.31536	−.00448	.0425	$-.154673\ 10^{-3}$
7	.40404	.30902	−.00594	.0428	$.399920\ 10^{-2}$
8	.46114	.30189	.00534	.0448	−.137642
9	.51423	.29414	.00707	.0455	.103587
10	.56315	.28591	−.00913	.0442	$-.309547\ 10^{-1}$
11	.60787	.27735	−.00081	.0455	$-.239546\ 10^{-2}$
12	.64841	.26860	.01219	.0472	$-.437344\ 10^{-3}$
13	.68491	.25977	−.00817	.0457	$-.416463\ 10^{-4}$
14	.71753	.25098	−.00848	.0460	$-.832451\ 10^{-5}$
15	.74650	.24231	.01274	.0483	$-.549745\ 10^{-6}$
16	.77206	.23385	.01091	.0481	$-.961096\ 10^{-7}$
17	.79447	.22567	−.00775	.0459	.000000
18	.81401	.21781	−.01673	.0445	.000000
19	.83094	.21031	−.01005	.0450	.000000
20	.84552	.20321	.00337	.0466	.000000
21	.85799	.19652	.01448	.0477	
22	.86858	.19025	.01941	.0477	
23	.87752	.18442	.01831	.0462	
24	.88500	.17903	.01314	.0434	
25	.89119	.17407	.00606	.0396	
26	.89626	.16955	−.00117	.0353	
27	.90036	.16545	−.00742	.0307	
28	.90363	.16177	−.01210	.0261	
29	.90617	.15850	−.01501	.0217	
30	.90811	.15565	−.01622	.0176	
31	.90953	.15318	−.01597	.0139	
32	.91054	.15111	−.01455	.0107	
33	.91120	.14943	−.01225	.0078	
34	.91159	.14813	−.00938	.0055	
35	.91179	.14720	−.00627	.0036	
36	.91187	.14664	−.00342	.0024	
37	.91188	.14646	−.01283		

l	σ_l	$\partial\psi/y\partial n$
38	−.00187	
39	−.00113	.7901
40	.00087	.6204
41	.00350	.4934
42	.00596	.4159
43	.00751	.3852
44	.00768	.3897
45	.00646	.4126
46	.00426	.4365
47	.00179	.4473
48	−.00022	.4379
49	−.00123	.4085
50	−.00112	.3657
51	−.00016	.3186
52	.00107	.2757
53	.00194	.2417
54	.00203	.2164
55	.00132	.1960
56	.00016	.1754
57	−.00089	.1509
58	−.00135	.1218
59	−.00105	.0901
60	−.00021	.0584
61	.00066	.0286
62	.00104	

$$\sigma = .22882, \quad d/Y = 2.354$$

$$\frac{C_D}{1+\sigma} = \begin{cases} .8881 & \text{by (6)} \\ .9121 & \text{by (37)} \\ .8967 & \text{by (38)} \end{cases}$$

$$62^{-1/2}(\sigma_1^2 + \cdots + \sigma_{62}^2)^{1/2} = .00832$$

TABLE II

l	x_l	y_l	σ_l	$-\psi/y^2$	λ_l
1	.00000	.33099	−.00159	.0271	.451680
2	.07168	.33033	−.00061	.0274	−.107688
3	.14257	.32837	.00158	.0280	$-.235624\ 10^{-1}$
4	.21190	.32516	.00268	.0284	$-.505784\ 10^{-2}$
5	.27898	.32079	.00032	.0281	$-.867247\ 10^{-2}$
6	.34319	.31536	−.00320	.0278	$-.271143\ 10^{-3}$
7	.40404	.30902	−.00054	.0286	$.221253\ 10^{-2}$
8	.46114	.30189	.00431	.0296	−.203479
9	.51423	.29414	−.00252	.0293	.171748
10	.56315	.28591	−.00153	.0298	$-.462914\ 10^{-1}$
11	.60787	.27735	.00506	.0307	$-.408437\ 10^{-2}$
12	.64841	.26860	−.00582	.0302	$-.712788\ 10^{-3}$
13	.68491	.25977	.00349	.0312	$-.836584\ 10^{-4}$
14	.71753	.25098	.00267	.0314	$-.169229\ 10^{-4}$
15	.74650	.24231	−.00723	.0307	$-.176830\ 10^{-5}$
16	.77206	.23385	.00194	.0315	$-.341559\ 10^{-6}$
17	.79447	.22567	.00798	.0320	$-.230466\ 10^{-7}$
18	.81401	.21781	.00048	.0310	$-.366784\ 10^{-8}$
19	.83094	.21031	−.00760	.0296	.000000
20	.84552	.20321	−.00777	.0291	.000000
21	.85799	.19652	−.00190	.0292	
22	.86858	.19025	.00488	.0294	
23	.87752	.18442	.00916	.0291	
24	.88500	.17903	.01000	.0280	
25	.89119	.17407	.00803	.0262	
26	.89626	.16955	.00443	.0238	
27	.90036	.16545	.00036	.0211	
28	.90363	.16177	−.00333	.0181	
29	.90617	.15850	−.00615	.0151	
30	.90811	.15565	−.00790	.0123	
31	.90953	.15318	−.00858	.0097	
32	.91054	.15111	−.00832	.0074	
33	.91120	.14943	−.00731	.0054	
34	.91159	.14813	−.00576	.0037	
35	.91179	.14720	−.00390	.0024	
36	.91187	.14664	−.00212	.0015	
37	.91188	.14646	−.00442		

l	σ_l	$\partial\psi/y\partial n$
38	−.00115	
39	−.00062	.8203
40	.00078	.6723
41	.00251	.5649
42	.00393	.5024
43	.00451	.4791
44	.00403	.4812
45	.00271	.4912
46	.00107	.4936
47	−.00027	.4786
48	−.00083	.4450
49	−.00051	.3990
50	.00038	.3504
51	.00123	.3085
52	.00152	.2776
53	.00102	.2557
54	.00001	.2363
55	−.00089	.2128
56	−.00105	.1824
57	−.00037	.1479
58	.00066	.1147
59	.00119	.0862
60	.00072	.0605
61	−.00034	.0326
62	−.00091	

$$\sigma = .22540, \qquad d/Y = 2.321$$

$$\frac{C_D}{1+\sigma} = \begin{cases} .8697 & \text{by (6)} \\ .8818 & \text{by (37)} \\ .8680 & \text{by (38)} \end{cases}$$

$$62^{-1/2}(\sigma_1^2 + \cdots + \sigma_{62}^2)^{1/2} = .00426$$

TABLE III

l	x_l	y_l	ϵ_l	$-\psi/y^2$	λ_l	l	ϵ_l	$\partial\psi/y\partial n$
1	.00000	.33099	.00035	.0230	.451839	38	−.00010	
2	.07168	.33033	.00022	.0230	−.138782	39	.00044	.8035
3	.14257	.32837	−.00020	.0230	$-.346617\ 10^{-1}$	40	.00180	.6451
4	.21190	.32516	−.00058	.0231	$-.822686\ 10^{-2}$	41	.00330	.5411
5	.27898	.32079	−.00013	.0235	$-.179683\ 10^{-3}$	42	.00414	.4957
6	.34319	.31536	.00086	.0240	$-.509280\ 10^{-3}$	43	.00379	.4974
7	.40404	.30902	.00032	.0242	$-.317849\ 10^{-2}$	44	.00223	.5237
8	.46114	.30189	−.00112	.0245	$.114032\ 10^{-1}$	45	.00002	.5480
9	.51423	.29414	.00081	.0251	$-.380732\ 10^{-1}$	46	−.00191	.5490
10	.56315	.28591	.00023	.0254	$-.743505\ 10^{-3}$	47	−.00268	.5173
11	.60787	.27735	−.00104	.0257	$.847660\ 10^{-3}$	48	−.00188	.4579
12	.64841	.26860	.00189	.0262	$-.433964\ 10^{-4}$	49	.00016	.3868
13	.68491	.25977	−.00237	.0262	$.205273\ 10^{-4}$	50	.00243	.3235
14	.71753	.25098	.00226	.0268	$-.227767\ 10^{-5}$	51	.00367	.2824
15	.74650	.24231	−.00013	.0267	$.489988\ 10^{-6}$	52	.00303	.2657
16	.77206	.23385	−.00312	.0265	$-.104352\ 10^{-6}$	53	.00066	.2628
17	.79447	.22567	.00251	.0268	$.686082\ 10^{-8}$	54	−.00224	.2568
18	.81401	.21781	.00374	.0266	$-.352631\ 10^{-8}$	55	−.00392	.2338
19	.83094	.21031	−.00133	.0257	.000000	56	−.00309	.1920
20	.84552	.20321	−.00504	.0248	$-.601794\ 10^{-10}$	57	.00001	.1427
21	.85799	.19652	−.00389	.0242		58	.00340	.1021
22	.86858	.19025	.00024	.0240		59	.00454	.0778
23	.87752	.18442	.00428	.0236		60	.00227	.0619
24	.88500	.17903	.00639	.0228		61	−.00172	.0380
25	.89119	.17407	.00621	.0214		62	−.00378	
26	.89626	.16955	.00431	.0196				
27	.90036	.16545	.00154	.0174				
28	.90363	.16177	−.00130	.0150				
29	.90617	.15850	−.00368	.0125				
30	.90811	.15565	−.00531	.0101				
31	.90953	.15318	−.00610	.0079				
32	.91054	.15111	−.00610	.0058				
33	.91120	.14943	−.00542	.0041				
34	.91159	.14813	−.00420	.0025				
35	.91179	.14720	−.00262	.0013				
36	.91187	.14664	−.00101	.0005				
37	.91188	.14646	−.00201					

$\sigma = .22454, \quad d/Y = 2.312$

$$\frac{C_D}{1+\sigma} = \begin{cases} .8624 & \text{by } (6) \\ .8742 & \text{by } (37) \\ .8608 & \text{by } (38) \end{cases}$$

$$62^{-1/2}(\epsilon_1^2 + \cdots + \epsilon_{62}^2)^{1/2} = .00302$$

THE FEBRUARY MEETING IN NEW YORK

The five hundred twenty-second meeting of the American Mathematical Society was held at Hunter College in New York City on Saturday, February 25, 1956. The meeting was attended by about 325 persons including 286 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings Professor Everett Pitcher of Lehigh University delivered an address entitled *Inequalities of critical point theory* at a general session presided over by Professor R. E. Langer. Sessions for contributed papers were held in the morning and afternoon, presided over by Professors J. H. Barrett, A. A. Bennett, R. H. Bing, G. K. Kalisch, D. E. Kibbey, and C. H. W. Sedgewick.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, the paper was presented by that author whose name is followed by "(p)". Dr. Banaschewski was introduced by Professor D. B Sumner, Drs. Bremermann and Huckemann and Miss Weiss by Professor R. D. Schafer, Mr. Friedberg by Dr. Hartley Rogers, Jr., Professor Kalish and Dr. Montague by Professor Alfred Horn, Professor Kreyszig and Dr. Nitsche by Professor Stefan Bergman, Dr. Laugwitz by Professor E. R. Lorch, and Mr. Robinson by Professor R. F. Rinehart.

ALGEBRA AND THEORY OF NUMBERS

283*t*. R. M. Baer: *Certain homomorphisms of partially ordered sets.*

It is well known that for an arbitrary partially ordered set X there exists a chain C and a homomorphism σ which carries X onto C. [Tarski, Kuratowski, Szpilrajn.] The authors calls a homomorphism of X onto C *thorough* with respect to a collection of chains (D_α) in X if the homomorphism maps every chain D_α *onto* C. In terms of a notion of *gap* in a partially ordered set X, those $(X, (D_\alpha))$ are characterized for which, given C, there exists a homomorphism which is thorough with respect to a nonempty collection of maximal chains in X and which maps X onto C. (Received January 13, 1956.)

284. D. W. Blackett: *Simple near-rings of differentiable transformations.*

This paper examines simple near-rings of transformations of a finite dimensional real vector space V into itself. The particular transformations considered have the origin θ as a fixed point and are continuously differentiable at θ. A simple near-ring N of such transformations is faithfully represented as a ring of linear transformations by restriction to the invariant subspace of V generated by the images of a fixed vector not annihilated by N. This statement must be modified by the restriction that at least one transformation in N has some nonvanishing first partial derivative at θ. One consequence is that the ring of linear transformations of V is a maximal simple near-ring in the near-ring of all transformations of V with θ as a fixed point and a point of continuous differentiability. Another corollary is that the simple near-rings of entire functions over the complex plane which vanish at 0 are the rings of transformations $z \to bz$ where b ranges over a subfield of the complex numbers. (Received January 3, 1956.)

285. J. R. Büchi: *On the representation of Boolean rings by sets.* Preliminary report.

$\langle B, +, 0\rangle$ is a *Boolean group* (B.G.) if it is a group satisfying the law $x+x=0$, it is a *Boolean group of sets* if B is a class of sets closed under the operation $\oplus$ of symmetric sum and '+' denotes $\oplus$. Let $x \leftrightarrow x^*$ be a one-to-one mapping of a $+-$basis S into sets x^* such that, (1) $S^* = [x^* \mid x \in S]$ is $+-$independent. (2) $h(x_1 + \cdots + x_n) = x_1^* \oplus \cdots \oplus x_n^*$ then defines an isomorphism h of the B.G. onto the B.G. of sets, $\oplus$-generated by S^*. By Zorn's lemma there exist $+-$ bases for every Boolean group, furthermore $x^* = [x]$ satisfies (1). Therefore, (I) *every B.G. is isomorphic to a B.G. of sets (of all finite subsets of a set)*. To represent Boolean rings $\langle B, +, \cdot, 0\rangle$ an additional condition is required, (3) $x, y, u_1, \cdots, u_n \in S$, $x \cdot y = u_1 + \cdots + u_n \to x^* \odot y^* = u_1^* \oplus \cdots \oplus u_n^*$. (II) *If S is a $+-$basis of a B.R. and $x \leftrightarrow x^*$ satisfies* (1) *and* (3), *then* (2) *defines a representation of the B.R. by a B.R. of sets.* Combined with the following fact this yields rather economical representations for a wide class of B.R.'s, including such which have a chain-basis. (III) *If S is a $+-$ basis of a B.R., if $S \cup [0]$ is closed under $\cdot$, and if $D \subset S$ is "dense" in S, then $x^* = [u \mid u \in D, u \subseteq x]$ satisfies* (1) *and* (3). (Received January 13, 1956.)

286t. Harvey Cohn: *Equivalence of theta reciprocity and Gaussian sum reciprocity.* I.

The method of Cauchy and Hecke establishes Gaussian sum reciprocity as a consequence of theta reciprocity through the asymptotic value of both members of the latter relation. To go in the opposite direction we note a complete expansion is equivalent to the following identity: $\exp\,[\pi i S(\operatorname{sgn}\omega)/4] g^{-n} a^{-1} \sum \exp\,[-\pi i S(\omega\lambda^{*2})] \cos 2\pi S \cdot [\mu\lambda^*/g] = N(\omega)^{1/2} G^{-n} \sum \exp\,[\pi i S(\mu^{*2}/\omega) + \pi i S(\omega\lambda^2)/G^2] \cos 2\pi S[\lambda\mu^*/G]$. Here $\mathfrak{a}, \mathfrak{A}$ are reciprocal algebraic modules, ($\mathfrak{a}$ of determinant a) and ω is an algebraic number of the corresponding field with S the trace operator; g, G are integers depending only on $\mathfrak{a}$, $\mathfrak{A}$, ω; $\lambda^* \in \mathfrak{a}$, $\mu^* \in \mathfrak{A}$ are summed over g^n, G^n residue classes mod $g\mathfrak{a}$ and mod $g\mathfrak{A}$; and $\lambda \in \mathfrak{a}$, $\mu \in \mathfrak{A}$ are connected by $\mu/g = \pm\omega\lambda/G$ a vacuous sum being represented by zero. The identities hold for such λ, μ provided they hold for only a finite number of residue classes. This is another way of saying $\mathfrak{a}$, $\mathfrak{A}$ are reciprocal modules if the basis relations satisfy the reciprocal relation modulo M for a finite integer M. (Received January 11, 1956.)

287. M. P. Epstein: *Derivations of differential fields.* Preliminary report.

Let F be an ordinary differential field of characteristic zero with field of constants C, Ω a universal extension of F with field of constants C^*. Denote by δ the differentiation operator in Ω. Let G be a differentially algebraic extension of F of transcendence degree n. Define a (differential) derivation of G/F to be a map $D: G \to \Omega$ satisfying: (1) for $\alpha, \beta \in G$, $D(\alpha+\beta) = D\alpha + D\beta$, $D(\alpha\beta) = \alpha D\beta + \beta D\alpha$; (2) for $a \in F$, $Da = 0$; (3) $D\delta = \delta D$. Call D restricted if $D(G) \leq G$. Then the set of derivations of G/F forms a

Lie algebra over C^* of dimension exactly n. However there may be no nontrivial restricted derivations of G/F. But when G is a Picard-Vessiot extension of F with the same field of constants C then there are n linearly independent restricted derivations of G/F so the set of all restricted derivations of G/F is a Lie algebra $\mathfrak{g}$ over C. Kolchin showed that the group of automorphisms of G/F may be identified with an algebraic matric group $\mathfrak{G}$ over C. When this is done, $\mathfrak{g}$ may be naturally identified with the Lie algebra of $\mathfrak{G}$. When G is strongly normal over F again there are n linearly independent derivations of G/F. (Received January 11, 1956.)

288t. G. D. Findlay and Joachim Lambek: *On rings with projective ideals.*

Let R be a ring having a unit element and assume that every left ideal of R is projective. Then every submodule of any projective unitary left R-module is a direct sum of modules isomorphic to left ideals of R, and is therefore also projective. This generalizes a theorem by Everett (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 312–316). If M and N are any two left R-modules, then the canonical homomorphism from the tensor product $M^* \cdot N$ into Hom (M, N) is faithful. This generalizes a theorem by Whitney (Duke Math. J. vol. 4 (1938) pp. 495–528). (Received January 16, 1956.)

289. H. K. Flesch: *Finite elementary nilpotent groups of class* 2.

Let EN2-group denote a group in which every element except identity is of the same odd prime order p and the commutator group and center coincide. If G is the EN2-group with minimal basis of length k whose commutator group C has the full order $p^{\binom{k}{2}}$, all EN2-groups are contained in the classes $G(p, k, d)$ of factor groups G/S with S of order p^d in C. The effect on $c \in G$ of an automorphism α of G is described by a congruence $\tilde{c} \to A\tilde{c}A'$, where A is the matrix of the transformation induced by α on the vectorspace G/C, and $\tilde{c}$ a skew-symmetric matrix isomorphically corresponding to c. Since every automorphism of G/C can be lifted to G, it follows that $G/S \cong G/T$ if and only if there exists A such that $A\tilde{S}A' = \tilde{T}$. Hence classification of $G(p, k, d)$ reduces to that of the homogeneous d-parameter families of skew k by k matrices over $GF(p)$ under congruence and linear parameter transformation. This is trivial for $d = 1$ and manageable for $d = 2$ by reference to invariant factor theory. Enumerations verify in particular all previous results for $d \leqq 2$. An example shows that EN2-groups exist for d up to $\binom{k}{2} - 1$ when k is even. (Received January 10, 1956.)

290. I. N. Herstein: *Lie and Jordan systems in simple rings with involution.*

Let A be a simple ring with an involution $*$, and let S be the set of self-adjoint elements of A and K the set of skew elements of A. S is a Jordan ring under the product $a \circ b = ab + ba$ and K is a Lie ring under $[a, b] = ab - ba$. In this paper it is shown that if the characteristic of A is not 2 then S is a simple Jordan ring; also, if in addition A is more than 16 dimensional over its center, the only Lie ideals of K are either in the center of A or contain $[K, K]$. It is also shown that if A is more than 4 dimensional over its center S generates A; also, in this situation the subring generated by K is A. (Received January 11, 1956.)

291. D. R. Hughes: *A class of non-Desarguesian projective planes.*

In 1907 Veblen and Wedderburn gave an example of a projective plane of order nine, none of whose planar ternary rings are Veblen-Wedderburn systems (with either distributive law); this plane is sometimes known as the "Carmichael plane." This plane is shown to be self-dual (thus establishing that there are exactly four planes of order nine known at the present time) and to possess at least 78 distinct Fano subplanes. Furthermore, the Veblen and Wedderburn technique is generalized to give a non-Desarguesian plane π of order p^{2n}, for any odd prime p and any positive integer n, with the following properties: none of the planar ternary rings for π are Veblen-Wedderburn systems, but (at least) one of the rings has the elementary abelian group of order p^{2n} as its additive loop. The question of self-duality is still open for every case excepting $p^{2n}=9$. (Received December 14, 1955.)

292t. S. A. Jennings and Rimhak Ree: *On a family of Lie algebras of characteristic p.*

Let A be the group algebra over an algebraically closed field Φ of characteristic p of an abelian p-group of order p^n and of the type $(p, p, \cdots, p)$, $D_0, \cdots, D_m$ derivations of A satisfying the conditions (i) $D_i \circ D_j = 0$ for all i, j; (ii) $\sum f_i D_i = 0$, where $f_i \in A$, implies $f_i = 0$ for all i; (iii) If $f \in A$ is such that $D_i f = \lambda_i f$, where $\lambda_i \in \Phi$, for all i, then $f = 0$ or f is a unit in A; (iv) $D_i f = 0$ for all i implies $f \in \Phi$. Let $a_0, \cdots, a_m \in A$ be such that $D_i a_j = D_j a_i$ for all i and j. Then the set L of all derivations of A of the form $f_0 D_0 + \cdots + f_m D_m$, where $f_i \in A$ satisfy $\sum (D_i f_i - a_i f_i) = 0$, forms a subalgebra of the derivation algebra of A. It is shown that dim L is either $mp^n + 1$ or mp^n. For the former case L is said to belong to the family F_{I}, and for the latter case, the family F_{II}. It is shown that algebras in a certain subfamily of F_{II} are simple, that algebras in a subfamily of F_{I} have simple first derived algebras of dimension $m(p^n - 1)$, where m, n may be arbitrary integers such $1 \leq m < n$, and that if $m = 1$ then certain algebras in F_{I} have simple second derived algebras of dimension $p^n - 2$, where $n > 1$, $p > 2$. The dimension $m(p^n - 1)$ is in general new. The simple algebras constructed by M. S. Frank [Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954)] are contained in F_{I}. However, the connection between the algebras in F_{I} and those constructed by A. A. Albert and Mrs. M. S. Frank [Bull. Amer. Math. Soc. Abstract 61-4-530] is undecided. (Received January 13, 1956.)

293t. Naoki Kimura: *Decomposition of homogeneous semigroups.*

A semigroup S is called homogeneous if for any two elements $x, y \in S$ there exists an automorphism of S sending x to y. A semigroup S is called idempotency-simple or commutativity-simple, if any homomorphic image of S which is idempotent or commutative is reduced into one element semigroup respectively. Under these definitions any homogeneous semigroup S is decomposed in the following manner: $S \cong A \times B \times C \times D \times E$, where A is a left singular semigroup, i.e., for any elements $x, y \in A$, $xy = x$, B is a right singular semigroup, C is a homogeneous semilattice, D is an idempotency-simple homogeneous commutative semigroup and E is both an idempotency-simple and a commutativity-simple homogeneous semigroup. (Received January 13, 1956.)

294t. Joseph Landin and Irving Reiner: *Automorphisms of the Gaussian unimodular group.*

Let G_n denote the group of $n \times n$ unimodular matrices over the ring of Gaussian integers. It is proved that the automorphism group $\mathfrak{A}_n$ of G_n is generated by the following automorphisms: (i) inner, (ii) $X \to X'^{-1}$, (iii) $X \to \overline{X}$ (conjugate), (iv) $X \to (\det X)^k X$ where $k = 1$ for even n, and $k = 2$ for odd n, and (v) $(P, S, T) \to (P, -S, -T)$ for $n = 2$ only, where $(x, y)P = (ix, y)$, $(x, y)S = (-y, x)$, $(x, y)T = (x, x+y)$. For $n = 2$, the automorphism (ii) is superfluous. The proofs given depend upon canonical forms for involutions in G_n under conjugacy in G_n, a method due to C. E. Rickart (Amer. J. Math. vol. 72 (1950) pp. 451–464) for characterizing the $(1, n-1)$ involutions, and the use of maximal sets of involutions (I. Reiner, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 459–476) to distinguish between different kinds of $(1, n-1)$ involutions. (Received January 11, 1956.)

295t. M. D. Marcus: *An extension of the Minkowski determinant inequality.*

Theorem 1: Assume (i) A and B are n-square non-negative Hermitian matrices with complex entries; (ii) the eigenvalues of A, B, and $A+B$ are $0 \leq \alpha_1 \leq \cdots \leq \alpha_n$, $0 \leq \mu_1 \leq \cdots \leq \mu_n$, $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ respectively; (iii) $E_r(a_1, \cdots, a_k)$ denotes the rth elementary symmetric function of the numbers $a_1, \cdots, a_k$. Then for $1 \leq r \leq k \leq n$, $E_r^{1/r}(\lambda_1, \cdots, \lambda_k) \geq E_r^{1/r}(\alpha_1, \cdots, \alpha_k) + E_r^{1/r}(\mu_1, \cdots, \mu_k)$. Theorem 2: Assume (i) and (ii). Let $p(A) = x^n + \sum_{j=1}^{n} p_j(A)x^{n-j}$ denote the characteristic polynomial of A. Then $|p_j(A+B)|^{1/j} \geq |p_j(A)|^{1/j} + |p_j(B)|^{1/j}$. The case $j = n$ is the classical Minkowski determinant inequality. (Received December 19, 1955.)

296t. M. D. Marcus: *Eigenvalue inequalities for finite matrices.* I.

Let A and B be n-square Hermitian matrices with complex entries. Let λ_j, α_j, μ_j, ω_j, $j = 1, \cdots, n$, be the eigenvalues of $C = A + iB$, A, B and $(C^*C)^{1/2}$ respectively, so arranged that $|\lambda_j| \leq |\lambda_{j+1}|$, $\alpha_j \leq \alpha_{j+1}$, $\mu_j \leq \mu_{j+1}$, $\omega_j \leq \omega_{j+1}$. Let $E_r(a_1, \cdots, a_k)$ be the rth elementary symmetric function of the a_j and let $p(A) = x^n + \sum_{j=1}^{n} p_j(A)x^{n-j}$ be the characteristic polynomial of A. Results: $0 \leq \alpha_1$ and $1 \leq r \leq k \leq n$ imply (i) $\prod_{j=1}^{k} |\lambda_j| \geq \prod_{j=1}^{k} \alpha_j$; (ii) $|E_r(\lambda_1, \cdots, \lambda_n)| \geq C_{n,r} \prod_{j=1}^{r} \alpha_j$; (iii) $E_r(\omega_1, \cdots, \omega_k) \geq E_r(\alpha_1, \cdots, \alpha_k)$; (iv) $|p_r((C^*C)^{1/2})| \geq |p_r(A)|$; (v) if A is real symmetric and B is real antisymmetric then $|p_r(A+B)| \geq |p_r(A)|$. In case $r = k = n$, (i)–(v) specialize to a result of Ostrowski and Taussky (Neder. Akad. Wetensch. A. vol. 54 (1951) p. 383). If $0 \leq \rho \leq 1$ and $\sigma_j \leq \sigma_{j+1}$ are the eigenvalues of $\rho A + (1-\rho)B$ and $0 \leq \alpha_1$, $0 \leq \mu_1$ then (vi) $E_r^{1/r}(\sigma_1, \cdots, \sigma_k) \geq \rho E_r^{1/r}(\alpha_1, \cdots, \alpha_k) + (1-\rho)E_r^{1/r}(\mu_1, \cdots, \mu_k) \geq E_r^{\rho/r}(\alpha_1, \cdots, \alpha_k)E_r^{(1-\rho)/r}(\mu_1, \cdots, \mu_k)$; (vii) $|p_r(\rho A + (1-\rho)B)| \geq |p_r(A)|^{\rho}|p_r(B)|^{1-\rho}$. (vi) and (vii) generalize a result of Fan (Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) p. 31). (Received December 19, 1955.)

297t. M. D. Marcus and B. N. Moyls: *Eigenvalue inequalities for finite matrices.* II.

Let $0 \leq \lambda_i \leq \lambda_{i+1}$; $0 \leq a_i \leq a_{i+1}$ for $i = 1, \cdots, k-1$, and assume $\sum_{i=1}^{j} (a_i - \lambda_i) \geq 0$ for $j = 1, \cdots, k$. (i) If f is a concave monotone nondecreasing function defined on $[\lambda_1, \max(\lambda_k, a_k)]$ then $\sum_{i=1}^{k} (f(a_i) - f(\lambda_i)) \geq 0$. Let A be a positive definite n-square Hermitian matrix with eigenvalues $0 < \alpha_1 \leq \cdots \leq \alpha_n$. Let $E_r(a_1, \cdots, a_k)$ denote the rth elementary symmetric function of the a's for $1 \leq r \leq k \leq n$. Set $\phi = E_r(f((Ax_1, x_1)), \cdots, f((Ax_k, x_k)))$ where $x_1, \cdots, x_k$ are an orthonormal (o.n.) set. Then if the function f in (i) satisfies $f(xy) = f(x)f(y)$ and is monotone increasing it follows that (ii) $\min \phi = E_r(\alpha_1, \cdots, \alpha_k)$, $\max \phi = C_{k,r}(f(\sum_{j=1}^{k} \alpha_{n-j+1}/k))^r$ where $x_1, \cdots, x_k$ run over all sets of k o.n. vectors. (ii) is applied to obtain inequalities connecting the eigenvalues of $(A+B)^{1/2}$, $A^{1/2}$ and $B^{1/2}$. (Received December 28, 1956.)

298t. M. D. Marcus and B. N. Moyls: *Eigenvalue inequalities for finite matrices.* III.

For notation see above abstract. It is known that minimizing and maximizing sets of k o.n. vectors for ϕ span an invariant subspace under A when A is positive definite (M. Marcus and J. L. McGregor *Extremal properties of Hermitian matrices*—to appear). It is shown here that for $f(x)=x$, $r=k$ (i.e. the product of k forms) the assumption on A need only be Hermitian nonsingular, to conclude the invariance. From this it follows that for $r=k=2$, max ϕ and min ϕ are max $((\alpha_i+\alpha_j)/2)^2$, min $\alpha_i\alpha_j$ respectively taken over all $i<j$ where $\alpha_i \leqq \alpha_{i+1}$ are the eigenvalues of A. For $r=2$, $k=n$ and A simply Hermitian it is also proved that min $\phi=p_2(A)$, the 2nd coefficient in the characteristic polynomial of A. (Received January 3, 1956.)

299. R. F. Rinehart: *Generalization of the concept of derivative of a matric function.*

Let $f(z)$ be a function of a complex variable and A a square matrix over the complex field. Let $f(A)$ be defined in the sense of Giorgi or any of the equivalent senses. In an earlier paper a definition of the derivative of the matrix function $f(Z)$ at $Z=A$ was given, in which the "incremental matrix" H was required to be commutative with A. In the present paper the following extension of that definition is given which removes that restriction: If $f(A+H)-f(A)$ is expressible, for $|h_{rs}|$ sufficiently small, in the form $\sum_{i=1}^{m} P_iHQ_i$, and if $\lim_{H\to 0} \sum_{i=1}^{m} P_iQ_i$ exists uniquely for any manner of approach of H to zero (i.e. $h_{rs}\to 0$), then this limit is called the derivative of $f(Z)$ at $Z=A$. It is shown that a necessary and sufficient condition that this derivative exist, is that the scalar function $f(Z)$ be analytic at each of the characteristic roots of A. The result $f'(A)=g(A)$, where $g(Z)=f'(Z)$ remains valid for this broader definition. (Received January 3, 1956.)

300. D. W. Robinson: *Functions of commutable linear transformations.*

A study is made of a defined composition, in a finite linear space over the field of complex numbers, between functions of several complex variables and sets of pairwise commutable linear transformations. The functions considered are not restricted to be simply power series, nor are the transformations required to be normal, as in previous studies. Equivalent forms of the composition are obtained, and some algebraic and topological properties are investigated. The results reveal the composition to be in both form and properties an extension of the previously studied case of a single variable. This is accomplished primarily by the formulation of the concept of projections of a set of pairwise commutative linear transformations, corresponding to prescribed eigenvalue arrangements. The eigenvalues of the defined linear transformations are also exhibited, generalizing and illuminating a result of Frobenius for rational functions. Finally, an extension is made of the class of admissible functions by means of the notion weak convergence. (Received January 6, 1956.)

301*t*. Robert Steinberg: *Prime power representations of some classical finite groups.*

There are five well-known two-parameter families of simple finite groups: the unimodular projective, the symplectic, the unitary, and the first and second orthogonal groups, acting on a vector space of a finite number of elements. The following is proved: Let G be one of the groups above. Let p be the characteristic of the base field, let d be the order of a p-Sylow subgroup P of G, and let m be the index of the normalizer of P in G. Let Σ be any vector space of dimension d over a field of characteristic 0 or prime to m. Then G has an irreducible representation of degree d with Σ as the representation space. The method is constructive and yields the representation space and the representing matrices explicitly. (Received January 10, 1956.)

302. Harold Widom: *Approximately finite algebras.*

Let $\mathfrak{M}$ be a factor of type II_1 on a (not necessarily separable) Hilbert space $\mathfrak{H}$. We introduce three notions of approximate finiteness for $\mathfrak{M}$. (A1) Given $A_1, \cdots, A_n \in \mathfrak{M}$ and $\epsilon > 0$ there exists a subfactor $\mathfrak{N}$ of $\mathfrak{M}$ of type I containing elements $B_1, \cdots, B_n$ such that $[[A_i - B_i]] < \epsilon$ $(i = 1, \cdots, n)$. (A2) If $\mathfrak{U}$ is a collection of subfactors of $\mathfrak{M}$ with the properties (a) if $\mathfrak{N}_\lambda$ is an increasing chain of members of $\mathfrak{U}$ then $R(\mathfrak{N}_\lambda) \in \mathfrak{U}$, and (b) $\mathfrak{U}$ contains the subfactors of type I, then $\mathfrak{M} \in \mathfrak{U}$. (B) There exist mutually commuting subfactors $\mathfrak{N}_\lambda$ of $\mathfrak{M}$ of type I such that $R(\mathfrak{N}_\lambda) = \mathfrak{M}$. In case $\mathfrak{H}$ is separable, (A1)–(B) are equivalent to the usual notion of approximate finiteness. In general (A1) and (A2) are equivalent, both being weaker than (B). Let $\chi(\mathfrak{M})$ be the density character of $\mathfrak{M}$ relative to the metric $[[\ \ ]]$. If $\mathfrak{M}$ and $\mathfrak{M}_1$ are approximately finite (B) and $\chi(\mathfrak{M}) = \chi(\mathfrak{M}_1)$ then $\mathfrak{M}$ and $\mathfrak{M}_1$ are algebraically isomorphic. If $\mathfrak{M}$ is approximately finite (A1) or (A2) and $\chi(\mathfrak{M}) = \aleph_0$ then $\mathfrak{M}$ is approximately finite (B). Extensions of these results hold for AW*-algebras of type II_1 possessing a central trace. (Received January 6, 1956.)

303t. Harold Widom: *Embedding in algebras of type* I.

Starting with order convergence in a complete lattice, a notion of convergence is introduced in a commutative AW*-algebra. Using this as a replacement for the ordinary topology on the complex numbers, various topologies are obtained for AW*-modules and AW*-algebras of type I analogous to the usual topologies on Hilbert space and the algebra of bounded operators on Hilbert space. Extending the methods of Dixmier and Feldman to the more general situation, the following results are obtained: Let A be an AW*-algebra with center Z, and assume there exists a complete set of positive linear mappings $f: A \rightarrow Z$ satisfying (a) $x \in A$, $a \in Z$ imply $f(ax) = af(x)$, (b) if e_λ are orthogonal projections in A with sup e and $x \in A$ then $f(x^* ex) = \sum f(x^* e_\lambda x)$. Then A is AW*-embeddable in an algebra of type I with center Z. Let B be an algebra of type I with center Z, A an AW*-subalgebra of B containing Z. Then $A = A''$ in B under any of the following conditions: (a) A is of type I, (b) I is the sup of countably many finite projections of A, (c) Z is locally countably decomposable. (Received January 6, 1956.)

304t. J. L. Zemmer: *A new class of nonassociative algebras.* Preliminary report.

The class $\mathfrak{G}(F)$ of linear algebras over a field F is defined as follows: $K \in \mathfrak{G}(F)$ if and only if K possesses an invariant bilinear form $k(x, y)$, and $(xy)z - x(yz) = k(y, z)x - k(x, y)z$, for all x, y, z in K. A mapping of $\mathfrak{G}(F)$ onto the set of all associative algebras with identity is defined. This mapping and the equivalence relation determined by it are the main tools used in studying $\mathfrak{G}(F)$. Some of the results are: (1) every right and left ideal in $K \in \mathfrak{G}(F)$ is associative; (2) let $K \in \mathfrak{G}(F)$ have finite dimension over F, if $a \in K$, r, s positive integers, then there exist α_i in F such that $a^r a^s = \alpha_1 a + \alpha_2 a^2 + \cdots + a^{r+s}$, where a^n is defined by $a^1 = a$, $a^n = a^{n-1}a$, for $n > 1$; (3) if $K \in \mathfrak{G}(F)$ is not associative and has finite dimension over F then its radical (as defined by A. A. Albert, *The radical of a non-associative algebra*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 891–897) is its maximal ideal. This last result facilitates the construction of an algebra in $\mathfrak{G}(F)$ whose radical is any preassigned associative algebra with identity. (Received January 11, 1956.)

ANALYSIS

305. Joseph Andrushkiw: *Polynomials with prescribed values at critical points.*

Let D_n, $D_n(x)$, D'_{n-1} be the discriminants of the polynomials $f(s) = t_0 + t_1 s + \cdots + t_{n-1}s^{n-1} + s^n$, $f(s; x) = f(s) - x$ and their derivative $f'(s)$, respectively. The discriminants are expressed in the form of determinants of the order $2n$, $2n$, $2(n-1)$, respectively, with the coefficients t_k as their elements. D_k, $D_k(x)$, D'_k are their principal diagonal minors of order $2k$. Further, let $a_1, a_2, \cdots, a_{n-1}$ denote the zeros of $f'(s)$. It follows that: 1. $f(a_k) = m_k$, $m_k \neq m_j$, $k \neq j$, $k, j = 1, 2, \cdots, n-1$, if and only if $D_n(m_k) = 0$; 2. p among m_k equal m if and only if $D_r(m) = 0$, $r = n, n-1, \cdots, n-p+1$; 3. m_k represent maxima and minima if 1. (and eventually 2.) and $D'_r > 0$, $r = 1, 2, \cdots, n-1$; 4. if all maxima equal M and all minima equal m, M and m are the roots of quadratic equation $D_{m+1}(x) = 0$ in case $n = 2m-1$, and m is the root of linear equation $D_{m+1}(x) = 0$ and M a root (namely the immediately larger than m) of quadratic equation $D_{m+2}(x)/(x-m) = 0$ in case $n = 2m$. (Received December 30, 1955.)

306. R. B. Barrar: *On the inverse Sturm-Liouville problem.*

Consider the differential equation (1) $y'' + [\lambda - P(x)]y = 0$ for complex $P(x) \in L(0, \pi)$ with the boundary condition (2) $y(0) \cos \alpha + y'(0) \sin \alpha = y(\pi) \cos \beta + y'(\pi) \sin \beta = 0$ and (3) $y(0) \cos \alpha + y'(0) \sin \alpha = y(\pi) \cos \gamma + y'(\pi) \sin \gamma = 0$; $\sin(\beta - \gamma) \neq 0$. It is shown that if the spectrum of (1) is known for each of the given boundary conditions (2) and (3) then $P(x)$ is uniquely determined. This result is an extension of that given by Levinson for real $P(x)$ in Mat. Tidsskr. B, 1949, pp. 25–30 (1949). The proof utilizes the integral introduced by Gelfand and Levitan for this type of problem [see Levinson, Physical Review vol. 89 (1953) pp. 755–757]. (Received January 12, 1956.)

307. Anatole Beck (p) and J. T. Schwartz: *A vector-valued random ergodic theorem.*

Let $\mathfrak{X}$ be a reflexive B-space and (S, Σ, m) a σ-finite measure space. Let $L_p(S, \Sigma, m, \mathfrak{X})$ denote the space of all strongly measurable functions f defined on S with values in $\mathfrak{X}$ such that $\int_S \|f(s)\|^p m ds < \infty$. Let there be defined on S a strongly measurable function T_s with values in the B-space $B(\mathfrak{X})$ of operators on $\mathfrak{X}$. Suppose that $\|T_s\| \leq 1$ for all $s \in S$. Let $s \to h(s)$ be a measure-preserving transformation in (S, Σ, m). Then, for each $X(s) \in L_1(S, \Sigma, m, \mathfrak{X})$, there is a strongly measurable function $\overline{X}(s) \in L_1(S, \Sigma, m, \mathfrak{X})$ such that $\lim_{n\to\infty} (1/n) \sum_{i=1}^{n} T_s T_{h(s)} \cdots T_{h^{i-1}(s)}(X(h^i(s))) = \overline{X}(s)$ a.e. in S, and the limit on the left also exists in the mean of order 1. Moreover, if $m(S) < \infty$, $\overline{X}(s) = T_s \overline{X}(h(s))$ a.e. in S. Under special conditions, it is possible to evaluate the limit $\overline{X}$ and thus derive theorems which are extensions of the Law of Large Numbers and the Mean Ergodic Theorems. (Received January 5, 1956.)

308. H. J. Bremermann: *Holomorphic functionals and complex convexity in Banach spaces.*

A complex-valued functional is "Gateaux holomorphic" in a domain D of a complex Banach space B_n if it is single-valued and its restriction to an arbitrary analytic plane $\{z \mid z = z_0 + \lambda a$, $(z_0 \in D$, $a \in B_n$, λ a complex parameter) is a holomorphic function of λ in the intersection of the plane with D. A notion of holomorphic continuation is defined and a lemma on the simultaneous continuation of holomorphic functionals is proved. A consequence of the lemma is that not every domain is a domain of existence of a holomorphic function (domain of holomorphy). Let $d_D^{(N)}(z)$ denote the distance of the point z from the boundary of D measured in the norm N. Then, if D is a domain of holomorphy, the functional $-\log d_D^{(N)}(z)$ is plurisubharmonic in D. The property of $-\log d_D^{(N)}(z)$ to be plurisubharmonic is invariant with respect to all norms N that generate equivalent topologies. The domains for which $-\log d_D^{(N)}(z)$ is plurisubharmonic are called "pseudo-convex." Some properties of the pseudo-convex domains are studied and compared with those of the convex domains. In particular, tube domains are pseudo-convex if and only if they are convex. (Received January 13, 1956.)

309*t*. R. C. Buck: *A theorem of Bochner and Martin.*

In their book on functions of several complex variables, Bochner and Martin obtain a number of results dealing with formal power series and "inner transformations." This note approaches these from a different point of view. Let P be the linear space of complex polynomials in n variables. The completion of P in a suitable topology τ is $\mathfrak{F}$, the space of formal power series. Let $\mathfrak{L}(\mathfrak{F})$ be the algebra of continuous linear transformations of $\mathfrak{F}$ into itself. Theorem 1. The "inner transformations" are precisely the $T\in\mathfrak{L}$ which are multiplicative. Theorem 2. An inner transformation T has an inverse if and only if the linear part of T is nonsingular. The proof of the latter uses the following simple (and probably well known) criterion: let E be a linear space and $T=I-U$ a transformation in $\mathfrak{L}(E)$; if lim $U^nx=0$ for all $x\in E$, then T is one-to-one, and its range is dense; if $x+Ux+U^2x+\cdots$ converges for each x, then T is an automorphism of E. (Received January 13, 1956.)

310. R. C. Buck: *Topologies defined by transformations.* Preliminary report.

Let E be a real linear space with a locally convex topology τ. Denote by $\mathfrak{L}(E)$ the algebra of continuous linear transformations T of $\langle E, \tau\rangle$ into itself, and let $\Gamma\subset\mathfrak{L}$. Let $\beta(\Gamma)$ be the smallest topology on E such that each $T\in\Gamma$ is continuous from $\langle E, \beta\rangle$ to $\langle E, \tau\rangle$. Example: Take $E=C(S)$, the space of bounded continuous functions on a locally compact space S, with τ the (normed) topology σ of, uniform convergence on S. Let $A\subset C(S)$ and take Γ as the transformations $x\rightarrow ax$ for $a\in A$. If $A=C_\infty$, the subspace of functions with compact support, then $\beta(\Gamma)=k=$compact-open; if $A=C_0$, the subspace of functions which are zero at infinity, then $\beta(\Gamma)$ becomes the "strict" topology β. Theorem: β and σ have the same bounded sets; $\beta=k$ on (σ) bounded sets; the dual space of $\langle C(S), \beta\rangle$ is $\langle C_\infty, \sigma\rangle^*$. (Received January 13, 1956.)

311. Lamberto Cesari: *Retraction of surfaces.*

A Fréchet continuous (path) surface S: (T, J) of graph $[S]=T(J)$ in E_n and boundary (path) curve θS: (T, J^*) (J a closed Jordan region of boundary J^*) is said to be nondegenerate if no maximal continuum of constancy g for T in J separates J. It is proved that: (I) Given a simple closed curve C in E_n and $\epsilon>0$ there is a $\delta=\delta(C, \epsilon)>0$ such that every surface S with $\|\theta S, C\|<\delta$ has a nondegenerate retraction S_0 with $\|\theta S_0, C\|<\epsilon$ ($\|\quad\|$ Fréchet distance). Given S let g_i denote any continuum g as above (if any) such that gJ^* is not connected. A surface S with $[\theta S]\subset C$ is said to have property P if for every g_i there is at least one arc $\lambda=w_1w_2=\lambda(g)\subset J^*$ with $\lambda\supset gJ^*$, w_1, $w_2\in g$, such that the closed curve c: (T, λ) is relatively homotopic to zero in C, i.e. $c\simeq 0(C)$. (II) Every S with $[\theta S]\subset C$ having property P has a retraction S_0

whose boundary θS_0 is freely homotopic to θS in C, i.e., $\theta S_0 \simeq \theta S(C)$, and $[\theta S_0] \subset [\theta S] \subset C$. Also, S_0 is nondegenerate if θS is not nulhomotopic in C. These and other theorems are proved in the author's book *Surface area* (Annals of Mathematics Studies) and are used there in the discussion of the representation problem of continuous mappings (surfaces) from any finitely connected Jordan region, or any plane open set (admissible sets). (Received January 11, 1956.)

312*t*. Paul Civin and Bertram Yood: *Maximal subalgebras in commutative Banach algebras.*

Let B be a complex commutative B^*-algebra with identity e and space of maximal ideals $\mathfrak{M}$. It is shown that if A is a maximal proper closed subalgebra containing e, then either (1) A distinguished between pairs of points of $\mathfrak{M}$, or (2) $A = \{x \mid x(M_1) = x(M_2)$, where $M_1 \neq M_2$ and $M_i \in \mathfrak{M}\}$. All sets of the type (2) are maximal proper closed subalgebras. If $\mathfrak{M}$ is countable, then any A must be of type (2). Some generalizations are considered. (Received December 2, 1955.)

313. Chandler Davis: *A device for studying Hausdorff moments.*

Numbers μ_k, with $k = 1, 2, \cdots$, form a (normalized) Hausdorff sequence provided they can be obtained as $\mu_k = \int x^k dm(x)$, where integration is over $[0, 1]$ and $m([0, 1]) = 1$. Given n numbers $\mu_1, \cdots, \mu_n$, here is a necessary and sufficient condition for them to be the first n members of some Hausdorff sequence. There must exist an $m \times m$ matrix A (where $n = 2m-2$ or $n = 2m-1$), subject to certain requirements, such that $\mu_k = (A^k)_{11}$ for each given μ_k. The requirement on A can be, for instance, that (for $i = 1, \cdots, m$) $A_{ii} = (1-\theta_{i-1})t_i + (1-t_i)\theta_i$, $A_{i,i+1} = A_{i+1,i} = ((1-t_i)(\theta_i - \theta_i^2)t_{i+1})^{1/2}$, and all other $A_{ij} = 0$; with $0 \leq \theta_i \leq 1$, $0 \leq t_i \leq 1$, and $t_1 = 0$. If only some of $\mu_1, \cdots, \mu_n$ are given, the condition for extendability still holds. If an infinite sequence is given, only the evident alteration is needed: the indices of A must run $1, 2, \cdots$. These easily proved facts yield some of the basic properties of Hausdorff sequences, as well as inequalities between moments which are convenient in statistics. (Received January 12, 1956.)

314*t*. Albert Edrei: *Logarithmic densities and gap theorems of Pólya.*

Consider the entire function $F(z) = \sum a_n z^n$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of all the subscripts $\lambda (\geq 1)$ such that $a_\lambda \neq 0$. If $L(t) = \sum_{\lambda_n \leq t} \lambda_n^{-1}$ and $0 \leq \xi < 1$, the logarithmic density $\Delta(\xi) = \limsup_{x \to +\infty} \{L(x) - L(x^\xi)\}/(1-\xi) \log x$ is analogous to Pólya's density $D(\xi)$ [Math. Zeit. vol. 29 (1929) p. 557]. In particular $\Delta(1) = \lim_{\xi \to 1-0} \Delta(\xi)$ exists and $\Delta(1) \leq D(0)$. Many of Pólya's results remain true with $D(1)$ replaced by $\Delta(1)$. For instance, let ρ ($\leq +\infty$) be the order of $F(z)$ and $\Delta(1)$ the logarithmic density associated with $\{\lambda_n\}$. Then $F(z)$ cannot be of order less than ρ in an angle of magnitude greater than $2\pi\Delta(1)$. (Received January 30, 1956.)

315*t*. Jacob Feldman: *The uniformly closed ideals in an AW* algebra.*

Let $\mathfrak{A}$ be an AW* algebra, $\mathfrak{L}$ its lattice of projections. All ideals will here be two-sided and uniformly closed. F. Wright has shown a 1-1 correspondence between the ideals of $\mathfrak{A}$ and the equivalence-closed lattice ideals of $\mathfrak{L}$ ("p-ideals"), the ideal $\mathfrak{J}$ corresponding to the p-ideal $\mathfrak{J} \cap \mathfrak{L}$. The inverse correspondence can be given by $\mathfrak{J} = \{A \mid E^{\perp}_{A^*A}(\lambda) \in \mathfrak{J} \cap \mathfrak{L}$ for all $\lambda > 0\}$, E_{A^*A} being the spectral resolution of A^*A. There is a canonical splitup of $\mathfrak{A}$ into a c^* sum of AW* algebras $\{\mathfrak{A}_i\}$. One can completely classify the ideals of $\mathfrak{A}$ by a more or less natural 1-1 correspondence with certain objects associated with the collection $\{Z_i\}$ of centers of the summands. The situation is clearest if $\mathfrak{A}$ is a II_1. Let $\mathcal{Y}=\{Z|Z\in$ center of $\mathfrak{A}, 0\leqq Z\leqq 1\}$. For Z_1, Z_2 in $\mathcal{Y}$, say $Z_1 \ll Z_2$ if $Z_1 \leqq \alpha Z_2$, α real. This induces an equivalence map π on $\mathcal{Y}$, and a natural order among the equivalence classes, making them into a lattice $\mathfrak{L}$. Then if d is the central dimension function on $\mathcal{L}$, $\pi \circ d$ induces a 1-1 order-preserving correspondence between the p-ideals of $\mathcal{L}$ and the lattice-ideals of $\mathfrak{L}$. There are unexpectedly many ideals; for example, for each maximal ideal $\mathfrak{M}$ of $\mathfrak{A}$, the set of ideals under $\mathfrak{M}$ but under no other maximal ideal is nondenumerable. (Received November 3, 1955.)

316. Leonard Gillman: *A continuous exact set.*

A linearly ordered set is *exact* if it is not similar to any of its proper subsets. All examples hitherto have been sets with gaps (Cuesta, Revista Mat. Hisp.-Amer. vol. 14 (1954) pp. 237–268; Ginsburg, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 341–361; Bagemihl-Gillman, Fund. Math. vol. 42.1 (1955) pp. 141–165; and references cited therein). Cuesta (loc. cit.) has asked whether there can exist a *continuous* exact set. In the present paper, the author constructs a continuous exact set A of power $\mathfrak{c}$; the proof of exactness, however, requires the hypothesis that $\mathfrak{c}$ be a *regular* cardinal. The set A is a certain subset of the lexicographically ordered set of all ω-sequences (x_n) of reals; the coordinates x_n are restricted inductively, according to a scheme too complicated to be described here. (Received February 10, 1956.)

317. A. O. Huber: *A characteristic property of polynomials.*

An entire analytic function $w=f(z)$ is a polynomial if and only if there exists a number $\lambda>0$ such that $\int_\sigma |f(z)|^{-\lambda}|dz|=+\infty$ for every locally rectifiable path σ tending to infinity. (Received January 11, 1956.)

318. G. K Kalisch: *On reducing subspaces and unitary invariants of certain Volterra operators.*

If $f\in L=L_p(0, 1)$, $T_K f=\int_0^x K(x, y)f(y)dy\in L$ for all $f\in L$, then the subspaces $L_\lambda=L_p(0, \lambda)(0\leqq\lambda\leqq 1)$ of L all are reduced by T_K. $\{L_\lambda\}$ is the totality of reduced subspaces if $K\in C^1$, $K(x, x)\neq 0$, and if $K(x, y)=K(y-x)$ where $K(t)=t^n K_1(t)$, $K_1(0)\neq 0$, $K_1\in C^{n+2}$ in a neighborhood of 0. This theorem is used to find conditions for unitary equivalence of operators T_K of the above kind defined in a Hilbert space. (Received January 13, 1956.)

319. N. D. Kazarinoff: *Asymptotic forms for Whittaker functions with both parameters large.*

The behavior of the Whittaker functions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ for both $|k|$ and $|m|$ large, $|x|$ unrestricted, is studied under the hypothesis that $(m^2-k^2)/k$ be bounded. The variable and parameters may be either real or complex. The analysis consists of three parts. For $|x/2k|>1$, the asymptotic forms are obtained by means of the classical theory of asymptotic solutions of differential equations near an ordinary point, the solutions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ being identified by their behavior as $|x|\to\infty$. Near the turning point $x=2k$ the theory of R. W. McKelvey [Trans. Amer. Math. Soc. vol. 79 (1955) pp. 103–123] is used. For $|x/2k|<1$, use is made of the paper of McKelvey and the author on the asymptotic solutions of differential equations near a regular singular point [Canadian Journal of Mathematics vol. 8 (1956) no. 1]. This research was supported by the USAF through the OSR-ARDC. (Received January 9, 1956.)

320. George Klein: *A theorem on mixed derivatives.*

Let $\Delta_{hk}F(x, y) = F(x+h, y+k) - F(x, y+k) - F(x+h, y) + F(x, y)$. Suppose that everywhere in a closed rectangle R, $\Delta_{hk}F(x, y)/hk$ tends to zero with h^2+k^2. It is shown that this implies that $F(x, y) = \phi(x) + \psi(y)$ in R, and that the argument can be extended to analogous results for functions of several variables. An application of these results establishes the fundamental theorem of calculus for multiple integrals without the use of the concept of uniform continuity. (Received January 13, 1956.)

321*t*. Erwin Kreyszig: *Decomposition of solutions of partial differential equations.*

Let (1) $w(z, z^*) = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} w_{\mu,\nu} z^{\mu} z^{*\nu}$ be a solution of (2) $w_{zz^*} + a(z, z^*) w_z + b(z, z^*) w_{z^*} + c(z, z^*) w = 0$, represented by a Bergman operator of the first kind in the form (3) $w(z, z^*) = \exp(-\int_0^{z^*} a(z,\xi)d\xi)\left[g(z) + \sum_{n=1}^{\infty} q_n(z, z^*)\int_0^z\int_0^{z_1} \cdots \int_0^{z_{n-1}} g(\xi)d\xi dz_{n-1} \cdots dz_1\right]$ (see Bergman, Trans. Amer. Math. Soc. vol. 57 (1945) p. 299 ff.). In (2), a, b, and c are entire functions of two independent complex variables z, z^*. In (3), $q_n(z, 0) = 0$, $n = 1, 2, \cdots$, and the "associated function" $g(z)$ is analytic and is regular at $z = 0$. A particular solution of (2) is called an elementary solution if its associated function is of the form $g(z) = k(z - z_0)^s$, $s = 0, \pm 1, \pm 2, \cdots$. Theorem I: $w(z, z^*)$ can be represented by a finite sum of elementary solutions and has at most finitely many singularity planes if and only if the determinants $w_\sigma^{(q)} = \det(w_{ik}^{(q,\sigma)})$, $w_{ik}^{(q,\sigma)} = w_{i+k+\sigma-2,0}$, $i, k = 1, 2, \cdots, q+1$, $\sigma = 0, 1, \cdots$ vanish, with the exception of at most finitely many of them. Theorem II: $w(z, z^*)$ possesses exactly one singularity plane (which is a pole surface of first order and, for all values of $z^* \neq 0$, a logarithmic branch surface) if and only if $\limsup_{\mu\to\infty} \left|w_{\mu,0}w_{\mu+2,0} - w_{\mu+1,0}^2\right|^{1/\mu} < r^{-2}$ where r denotes the radius of convergence of the power series development of $w(z, 0)$ at $z = 0$. (Received February 13, 1956.)

322*t*. Erwin Kreyszig: *Solutions of partial differential equations satisfying ordinary differential equations.*

As has been recently shown [Journal of Rational Mechanics and Analysis vol. 4 (1955) pp. 907–923] there exist interesting classes of partial differential equations of the form (1) $w_{zz^*} + a(z, z^*)w_z + b(z, z^*)w_{z^*} + c(z, z^*)w = 0$ possessing particular solutions which satisfy also an ordinary differential equation in z. From this fact various conclusions on properties of those solutions can be drawn, e.g. concerning their behavior in the neighborhood of singularities etc. It is of interest that the problem of obtaining solutions of (1) satisfying ordinary differential equations can be formulated as a coefficient problem. Theorem: A particular solution $w(z, z^*) = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} w_{\mu,\nu} z^{\mu} z^{*\nu}$ of (1) satisfies, for every fixed value $z^* = z_0^*$, in addition to (1) also an ordinary linear differential equation in z if the determinants $W^{(\sigma)} = \det(W_{ik}^{(\sigma)})$, $W_{ik}^{(\sigma)} = (i+k-2)! \cdot \sum_{\nu=0}^{\infty} w_{i+k-2,\nu} z_0^{*\nu}$, $i, k = 1, 2, \cdots, \sigma$; $\sigma = 1, 2, \cdots$, are zero, with the exception of at most finitely many of them; the coefficients of that ordinary differential equation are independent of z, but depend on z_0^*, in general. This theorem can be proved by using Bergman operators of the first kind [see Duke Math. J. vol. 6 (1940) p. 537], while in the above-mentioned paper Bergman operators of the third kind [Rec. Mat. vol. 2 (1937) p. 1193] have been used when representing the particular solutions under consideration. (Received February 13, 1956.)

323. R. E. Langer: *On the asymptotic solutions of a class of ordinary differential equations of the fourth order, with special reference to an equation of hydrodynamics.*

The paper is concerned with the differential equation $w^{iv}+\lambda^2[P(z,\lambda)w''+Q(z,\lambda)w'+R(z,\lambda)w]=0$, in which P, Q, and R are power series in $1/\lambda$. The matter at issue is the forms of the solutions when $|\lambda|$ is large and the z-region in question contains a simple zero of $P(z,\infty)$ (a turning point). The limiting equation as $\lambda\rightarrow\infty$ in general has a regular singularity at the turning point. When the exponents do not differ by an integer the differential equation is referred to as a regular type. A general theory of the solution forms to arbitary powers of $1/\lambda$ is given. When the exponents differ by an integer the equation is called irregular. There are several categories of irregular equations, and for these, with one exception, the theory is given. The Orr-Sommerfeld equation of hydrodynamics is of this type and is irregular. The application of the general theory to this equation is discussed. (Received January 10, 1956.)

324. Solomon Leader: *Functions integrable with respect to a finitely additive measure.*

Let μ be a finitely additive measure on a Boolean algebra $\mathfrak{A}$ of sets with X as unit. A real-valued function $f(x)$ is integrable if the approximating sums over finite partitions of X converge uniformly with respect to refinement of partition. An integrable function must therefore be bounded except on a set of measure zero. Through consideration of inner and outer measure $\mathfrak{A}$ can be completed in X with respect to μ to obtain the algebra of measurable sets. A function with values 0 and 1 is integrable if, and only if, it is the indicator of a measurable set. From a characterization of the completed algebra it follows that a bounded function is integrable if, and only if, it is a uniform limit of linear combinations of indicators of measurable sets. This result is used to prove Lebesgue's theorem on Riemann integrable functions. (Received December 29, 1955.)

325. J. J. Levin: *On singular perturbations of nonlinear systems of differential equations related to conditional stability.*

Previously announced results [Bull. Amer. Math. Soc. Abstract 61-6-723] are extended to the systems: (1) $dx/dt=f(t,x,y,\epsilon)$, $\epsilon dy/dt=g(t,x,y,\epsilon)$; (2) $dx/dt=f(t,x,y,0)$, $0=g(t,x,y,0)$, where x and y are real vectors of m and n components respectively. Where $x=p(t)$, $y=q(t)$ is a solution of (2) for $0\leqq t\leqq T$, it is assumed that k of the characteristic roots of the matrix $g_y(t)=(\partial g_i(t,p(t),q(t),0)/\partial y_j)$ have negative real parts and that the remaining $n-k$ have positive real parts for $0\leqq t\leqq T$. The existence of a k-dimensional initial manifold in y space, which depends on ϵ and the initial x vector, such that if the initial y vector lies on the manifold then the solution of (1) exists over $0\leqq t\leqq T$ for all sufficiently small $\epsilon>0$ is established. Several properties of this manifold are obtained. The proofs are similar to those of [Abstract 723] and employ some recent results of L. Flatto and N. Levinson. (Received January 10, 1956.)

326t. W. S. Loud: *On a nonlinear differential equation of second order.* Preliminary report.

The differential equation (1) $x''+cx'+g(x)=e(t)$ is considered, where $c>0$, $e(t)$ is continuous with $|e(t)|\leqq E$, $g(x)$ is differentiable with $g'(x)\geqq b>0$ and $g'(x)$ bounded on any finite interval, and $g(0)=0$. It is shown for the more general equation (2) $x''+f(x)x'+g(x)=e(t)$ with $f(x)$ continuous, and $f(x)\geqq c$ that ultimately any solution will satisfy $|x|\leqq Eb^{-1}+4Ec^{-2}$, $|x|\leqq Eb^{-1}+4Ec^{-1}b^{-1/2}$, $|x'|\leqq 4Ec^{-1}$. A growth theorem for solutions of the linear equation (3) $y''+a(t)y=0$ is obtained to the effect that if $|a(t)|\leqq A^2$, then no solution of (3) and no first derivative of a solution can grow more rapidly than $\exp(At)$. If $a(t)$ is periodic, this shows that in the Floquet representation of the solution of (3), $y=\exp(\rho t)p_1(t)+\exp(-\rho t)p_2(t)$, $\rho^2\leqq A^2$. The growth theorem is applied to obtain a convergence theorem for solutions of (1). Two solutions are said to converge in case both of $|x_1-x_2|$ and $|x_1'-x_2'|$ approach zero as t becomes infinite. If all solutions of (1) satisfy ultimately $|x|\leqq A$, then all solutions converge in case $c^2>2H(A)$, where $H(A)$ is the maximum of $g'(x)$ for $|x|\leqq A$. If $e(t)$ is periodic, then in this case there is a unique periodic solution of (1) to which all solutions converge. The work for this paper was supported in part by the Office of Ordnance Research. (Received January 11, 1956.)

327. W. S. Loud: *On periodic solutions of second-order linear differential equations with periodic coefficients.*

The differential equation (1) $y''+a(t)y=0$, with $a(t)$ periodic is considered in the case that there exists a periodic solution $p(t)$. It is known in the case of Mathieu's equation that when such a $p(t)$ exists, no other linearly independent solution can be periodic unless $a(t)$ is a constant (Ince, Proc. Cambridge Philos. Soc. vol. 21 (1922) p. 117). Here (1) is studied as a variation equation for (2) $x''+g(x)=e(t)$ with $e(t)$ small at a periodic solution $x_0(t)$ of (3) $x''+g(x)=0$. The variation equation is (4) $y''+g'(x_0(t))y=0$, and $p(t)=x_0'(t)$. If $g(x)$ is nondecreasing in a neighborhood of zero, is differentiable, and satisfies $g(0)=0$, $xg(x)>0$ otherwise, then all solutions of (4) are periodic if and only if $\int_{-B}^{A}[E-G(x)]^{-1/2}[2^{-1}-G(x)g'(x)(g(x))^{-2}]dx=0$, where $G(x)=\int_0^x g(u)du$, E is the constant value of $2^{-1}x_0'^2+G(x_0)$, and A and $-B$ are the maximum and minimum values of x_0. The period of a solution of (3) depends on amplitude in general, and the above condition is equivalent to the period being stationary as a function of amplitude. The case in which $g(x)$ is allowed to vanish in an interval about zero is considered. It is shown also that if $p(t)$ has no zeros, (1) can not have all solutions periodic, and that if $p(t)$ is a trigonometric polynomial, no other solution of (1) is, unless $a(t)$ is a constant. The work for this paper was supported in part by the Office of Ordnance Research. (Received January 11, 1956.)

328. M. A. Martino: *Applications of a certain measure of non-analyticity.*

Let D be a simply-connected domain in the complex plane, and L the set of Lebesgue-summable complex-valued functions on D. If $f\in L$, and $I=[a_1, b_1; a_2, b_2]$ is a rectangle in D, then $J_f(I)=\oint_{(I)}f(z)dz$ is well-defined provided none of the coordinates, a_1, b_1, a_2, b_2, belong to a certain set N of measure zero. J_f can be linearly extended to a functional of finite unions of nonoverlapping rectangles, all of whose side-coordinates do not belong to N. If the exceptional set N can be chosen so the resulting functional is bounded, then J_f extends to a complex-valued measure μ_f. Let B be the subspace of those functions in L for which μ_f is so obtainable. Letting A be the subspace of those functions equal to an analytic function on D, almost everywhere, the factor space B/A is isomorphic with the linear space of complex-valued bounded measures on D. For each $f\in B$, $f(z)=a(z)-(1/2\pi i)\iint_D d\mu_f/(\zeta-z)$, for some $a\in A$. This representation allows function-theoretic inferences for f from assumptions on μ_f. Examples: An extension of Vitali's compactness theorem follows from Egoroff's theorem. For continuous $f(z)$, the existence of $\iint_D d\mu_f/|\zeta-z_0|^2$ implies $f(z)$ is approximately derivable at z_0. (Received January 11, 1956.)

329. C. J. Neugebauer: *A characterization of the Lebesgue area.*

Let A be an admissible set of the Euclidean plane E_2, i.e., A is either an open subset of E_2 or a finite union R of disjoint finitely connected Jordan regions or an open subset of R. Denote by $\mathfrak{T}$ the class of all continuous mappings (T, A) from an admissible set $A\subset E_2$ into the Euclidean 3-space E_3. A mapping $(T, R)\in\mathfrak{T}$ will be termed elementary provided it is Fréchet equivalent to a quasi-linear mapping. Let now $\mathfrak{F}$ be the class of all functionals $\Phi(T, A)$ defined for each $(T, A)\in\mathfrak{T}$ such that (a) $\Phi(T, A)$ is non-negative; (b) $\Phi(T, A)$ is lower-semicontinuous; (c) $\Phi(T, A)$ satisfies the Kolmogoroff principle; (d) $\Phi(T, A')\leq\Phi(T, A)$, if A' is an admissible subset of A; (e) $\Phi(T, A)$ agrees with the Lebesgue area $L(T, A)$ for elementary mappings. For each $\Phi\in\mathfrak{F}$, denote by $\mathfrak{T}(\Phi)$ the class of all mappings $(T, A)\in\mathfrak{T}$ for which $\Phi(T, A)=L(T, A)$, and let $\mathfrak{T}^*=\bigcap\mathfrak{T}(\Phi)$, $\Phi\in\mathfrak{F}$. Let now $\mathfrak{T}'$ be the class of mappings $(T, A)\in\mathfrak{T}$ satisfying: for every $\epsilon>0$ there exists an elementary mapping (T', R), $R\subset A$, such that (1) $|T'(w)-T'(w')|\leq|T(w)-T(w')|$, $w, w'\in R$, (2) $|L(T, A)-L(T', R)|<\epsilon$, if $L(T, A)<\infty$; $L(T', R)>\epsilon^{-1}$, if $L(T, A)=+\infty$. It is proved that $\mathfrak{T}^*=\mathfrak{T}'$. If the condition (e) is replaced by (e′) $\Phi(T, A)$ agrees with the Lebesgue area for all quasi-linear mappings, then an example of a functional Φ is given which satisfies (a), (b), (c), (d), and (e′) but which does not coincide with the Lebesgue area. (Received January 12, 1956.)

330*t*. Johannes Nitsche: *On solutions of differential equations of mixed type.*

The analytic solutions of $U_{xx}+x^pK(x)U_{yy}=0$ are considered in the region where x^p is real ($p>0$) and $K(x)=1+\sum_1^\infty k_nx^n>0$. Putting $\xi=\int_0^x[x^pK(x)]^{1/2}dx$, $\eta=y$, $N(\xi)=(1+2\gamma_1)[1+\sum_1^\infty N_n\xi^{2n\gamma_2}]/4\xi$ and $\zeta=\xi+i\eta$ one obtains $U_{\zeta\bar\zeta}+N(\xi)(U_\zeta+U_{\bar\zeta})=0$ $(\gamma_j=(-1)^j/(p+2))$. The solutions U can be represented as real parts of functions $u(\zeta, \bar\zeta)=\int_{-1}^{+1}E(\zeta, \bar\zeta, \tau)\cdot f(\zeta(1-\tau^2)/2)(1-\tau^2)^{-1/2}d\tau$ (cf. S. Bergman, Amer. J. Math. vol. 70, p. 856; vol. 74, p. 444), $f(\sigma)$ an arbitrary analytic function of one complex variable, integration along $|\tau|=1$. The integraloperator E has the form $E(\zeta, \bar\zeta, \tau)=\exp\,[-2\int^\xi N(\xi)d\xi]\sum_{j=1}^2\,[\zeta(1-\tau^2)/2]^{\gamma_j+1/2}G^{(j)}(s, t)$. Here $s=2\xi/\zeta\tau^2$, $t=\zeta\tau^2$ and $G^{(j)}$ are solutions of a system of integral equations with $G^{(j)}(s, 0)=s^{\gamma_j+1/2}F(\gamma_j+1/2, \gamma_j+1, 2\gamma_j+1; s)$ (F hypergeometric series). They converge for all t and $|s|<1$, that is, in the sector S: $\{3\xi^2<\eta^2\}$ containing the parabolic line, and can be analytically continued along each path avoiding $s>0$. The properties of $f(\sigma)$ permit statements about the properties of the corresponding solution, independent of the special form of $x^pK(x)$. If e.g. $f(\sigma)$ has a pole of mth order at σ_0, then $u(\zeta, \bar\zeta)$ is regular in the intersection of S and $|\zeta|<|2\sigma_0|$, and near $\zeta=2\sigma_0$ has a development $\sum_{\nu=1}^{2m-1}P_\nu(\zeta, \bar\zeta)\cdot(\zeta-2\sigma_0)^{-\nu/2}+$reg. function. So (analogous to S. Bergman, ZaMM vol. 32, pp. 33) many theorems of the theory of analytic functions can be transferred to solutions $u(\zeta, \bar\zeta)$ resp. $U(x, y)$. (Received January 16, 1956.)

331*t*. Johannes Nitsche: *On solutions of Cauchy's problem for equations of mixed type with data on the parabolic line.*

The author considers the analytic solutions of $U_{xx}+x^pK(x)U_{yy}=0$ satisfying initial conditions $U(0, y)=\sum_0^\infty\alpha_ny^n$, $U_x(0, y)=\sum_0^\infty\beta_ny^n$ (in the region where x^p is real ($p>0$) and $K(x)=1+\sum_1^\infty k_nx^n>0$). Using the notation and the results of the preceding abstract these solutions can be represented as the real part of $\int_{-1}^{+1} E(\zeta, \bar{\zeta}, \tau) \cdot f(\bar{\zeta}(1-\tau^2)/2)(1-\tau^2)^{-1/2}d\tau$. For the coefficients a_n of $f(\sigma) = \sum_0^\infty a_n\sigma^n$ one obtains an expression, which asymptotically has the form $(i/2)^n a_n = A\alpha_n n^{\gamma_2}[1+O(n^{2\gamma_1})] + B\beta_n n^{\gamma_1}[1+O(n^{-1})]$ where A and B are complex constants depending only on p. So from the behavior of the power series $\sum \alpha_n y^n$ and $\sum \beta_n y^n$ one can infer the properties of the solution, especially the region of its regularity and the type of its singularities. The analogy of the Hadamard-Mandelbrojt classification of the functions of one complex variable by means of their singularities can also be developed. (Concerning the case of elliptic equations compare S. Bergman, Trans. Amer. Math. Soc. vol. 57, p. 299.) If, for instance, the α_n, β_n have the property that $|(a_{n+1}/a_n)-(1/a)| < C\cdot\rho^n$, $\rho<1$ and the point a is situated in the sector S then the solution is regular in the intersection of S and $|\zeta| < |2a|$ and on $|\zeta| = |2a|$ has only a single singularity which is of the type Re $[P(\zeta, \bar{\zeta})(\zeta-2a)^{-1/2}]$+reg. function. (Received January 16, 1956.)

332. L. A. Rubel: *An optimal gap theorem.*

Theorem. Given a sequence $\{p_n\}$ of positive integers, in order that there exist a non-null function $f(z)$ which has a power series expansion of the form $f(z) = \sum a_n z^{p_n}$ and whose region of regularity includes the full negative real axis $-\infty \leqq z \leqq 0$, it is necessary and sufficient that lim inf $n/p_n > 0$. The theorem is proved by translating the problem into one on entire functions by using an interpolation theorem of Buck [Duke Math. J. vol. 13 (1946) pp. 541–549]. The corresponding problem on entire functions has recently been solved by the author. [Bull. Amer. Math. Soc. Abstract 61-3-417]. (Received January 10, 1956.)

333t. Morris Schreiber: *Representation of non-normal operators.*

Pursuing the investigation reported in [Bull. Amer. Math. Soc. Abstract 62-2-290] and following the terminology of that abstract, it is shown that for operators A with $\|A\| \leqq 1$ whose operator measure F is equivalent to Lebesgue measure on the unit circle C and whose spectrum contains C, the map $f \rightarrow f(A) = \int_C f(t)dF(t)$ is isometric and isomorphic between the algebra of continuous boundary values of functions analytic for $|z| < 1$, with supremum norm, and the uniform closure of the algebra generated by A and I. An example of such an operator is the unilateral shift operator on sequence space. (Received January 16, 1956.)

334t. Morris Schreiber: *Unitary dilations.*

If A is a contraction ($\|A\| \leqq 1$) on a Hilbert space H and U is a unitary operator on a larger space $K \supset H$ such that (i) $PU^n x = A^n x$ for all n and $x \in H$, where P is the projection onto H, and (ii) no subspace of $K \ominus H$ reduces U, then it follows that U is uniquely determined within unitary equivalence by A, and it is known that every contraction does determine such a unitary operator [Nagy, Acta Szeged vol. 15 (1953) pp. 87–92. See also the author's abstract, Bull. Amer. Math. Soc. Abstract 62-2-290]. It shall be called "The" unitary dilation $U(A)$ of A. The structure of $U(A)$ is determined in the following cases. (1) If $\|A\| < 1$ and dim $(H) = n \leqq \aleph_0$, then $U(A)$ is the n-fold copy of the bilateral shift operator on sequence space. (2) If A is a projection with dim $(A) = m$, codim $(A) = n$, then $U(A)$ is the direct sum of the m-dimensional identity operator and the n-fold copy of the bilateral shift operator on sequence space. From these results it follows that (3) all A with $\|A\| < 1$ on separable space have unitarily equivalent unitary dilations, and (4) two projections P and Q are unitarily equivalent if and only if $U(P)$ and $U(Q)$ are. (Received January 16, 1956.)

335*t*. Maurice Sion: *Variational measure.*

Let F be a family of sets, σF their union, and $\mathcal{P}(F)$ the set of all finite, disjointed subfamilies P with $\sigma P=\sigma F$. For f a function on σF and ν a measure on rng f, the variational measure $\mu=V(F, f, \nu)$ is defined by: $\mu(A)=\sup_{P\in\mathcal{P}(F)}\sum_{\alpha\in P}\nu(f(A\alpha))$, for $A\subset\sigma F$. If σF is a topological space and F satisfies certain conditions, $f(\alpha)$ is ν-measurable for $\alpha\in F$ and $f^{-1}\{y\}$ is closed for ν-almost all y, A is μ-measurable, and for every $A'\subset A$ with $\mu(A')>0$ there is $B\subset A'$ with $0<\mu(B)<\infty$, then $f(A)$ is ν-measurable. As an application, if $\mathfrak{M}_0$ denotes the set of outer measures on the line under which intervals are measurable, f is real-valued, continuous on the irrationals and $f^{-1}\{y\}$ is countable for all y, then f maps a set, ν-measurable for all $\nu\in\mathfrak{M}_0$, into one of the same kind. Let $\mathfrak{M}_1$ be the set of all $\nu\in\mathfrak{M}_0$ for which there is no sequence S such that: S_n is a finite, disjointed family; $\sigma S_n=\sigma S_0$; S_{n+1} is a refinement of S_n; $\nu(\sigma S_0)>0$; if $B\subset\sigma S_0$ and $\nu(B)>0$ then $\sum_{A\in S_n}\nu(AB)\xrightarrow{n}\infty$. Then a real-valued function continuous on the irrationals maps a set, ν-measurable, for all $\nu\in\mathfrak{M}_1$, into one of the same kind. (Received January 9, 1956.)

336*t*. Mary C. Weiss: *On the Littlewood series* $\sum n^{-1/2}\exp(i\beta n\log n+in\theta)$.

The series (denote it by S and its partial sums by S_N) exhibits a number of phenomena resembling those of series of independent random variables. The following two properties are typical: (1) for almost every θ we have $\limsup|S_N|/\{\log n\cdot\log\log\log n\}^{1/2}=\alpha$ where $\alpha=\alpha(\beta)$ is a finite positive constant, (2) On every set $E\subset(0, 2\pi)$ and of positive measure, the real and imaginary parts of S_N obey asymptotically (for $N\to\infty$) independent Gaussian Laws with mean value 0 and dispersion $C(\log n)^{1/2}$. Similar results apply to Abel means and to a family of series related to S. (January 26, 1956.)

337*t*. Mary C. Weiss: *The law of the iterated logarithm for lacunary trigonometric series.*

Let $S_k(x)$ denote the partial sums of a lacunary series (*) $\sum_1^\infty\rho_k\cos(n_kx+\alpha_k)$, $n_{k+1}/n_k>q>1$, and let $B_k^2=1/2\sum_1^N\rho_j^2$. Salem and Zygmund have shown that (**) $\limsup|S_N(x)|/\{2B_N^2\log\log B_N\}^{1/2}\leq 1$ with the provision that (***) $\operatorname{Max}_{1\leq k\leq N}\rho_k=o\{B_N/(\log\log B_N)^{1/2}\}$. Recently Erdős and Gal proved the inequality opposite to (*) for the series $\sum_1^\infty\exp i(n_kx+\alpha_k)$. It can be shown that the inequality opposite to (**) for the series (*) satisfies the general condition (***). (Received January 26, 1956.)

338. Albert Wilansky: *The inverse matrix in summability.*

Let $'A$, A', A^{-1} denote any matrices such that $'AA=AA'=AA^{-1}=(A^{-1})A=I$; A being a fixed conservative $(=K)$ matrix. THEOREM. *If* $\|A^{-1}\|<\infty$ (i.e. A is K_r) *then* A^{-1} *is conservative.* Here $\|A\|=\sup_n\sum_k|a_{nk}|$. Known Tauberian theorems follow easily, e.g. $a_{nk}\leq 0$ for $k<n$, $\neq 0$ for $k=n$, $=0$ for $k>n$, imply A equivalent to convergence: for these hypotheses imply A^{-1} has non-negative entries. Therefore $\|A^{-1}\|<\infty$. Examples show the theorem false for $'A$ and A'; e.g. A may be multiplicative 0 yet $\|A'\|<\infty$ (but A co-null implies A' not conservative and $\|'A\|=\infty$.) Other results: If A is row-finite, sums no divergent sequences, then A^{-1} is conservative if it exists; there exists regular triangle A summing divergent sequences with regular $'A$; and a regular triangle A summing no divergent sequence with $\|'A\|=\infty$. If A is reversible and $\|A'\|<\infty$ then A sums no divergent sequence, A^{-1} exists and is conservative. If A is reversible, not necessarily conservative, then A^{-1} exists iff $c_n=0$ (see Banach, p. 50). (Received January 6, 1956.)

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339. R. J. Arms, L. D. Gates, Jr. (p), and Ber
sive block overrelaxation.

This generalizes D. Young's result on the "successive over
solving linear systems (Trans. Amer. Math. Soc. vol. 76 (19
system with matrix A be given by (1) $\sum_{j=1}^{N} A_{ij}x_j=y_i$ ($i=$
blocks of A, with $A_{ii}\neq 0$, x_i and y_i vectors whose orders equal
Matrix A has *block property A* with respect to the partition
disjoint sets S and T whose sum is the set of first N integers,
$i\neq j$, then $i\in S$, $j\in T$ or $j\in S$, $i\in T$. An iterative algorithm
designated as *successive block over-relaxation* (SBOR) follows f
of Young's paper hold in generalized form for the new method
A An application of the SBOR method is to elliptic differe
components of x_i are unknown values on the ith grid line, al
inverted. It has also been shown that in the usual situation, t
verges faster than the SOR method. (Received January 11, 1

340t. A. A. Blank: *Axiomatics of the binocular v*

Since Luneburg's work, *The mathematical analysis of bino*
1947), it has generally been conceded on insufficient physical e
sensory space may be taken as metric and even homogeneou
there is exhibited a set of axioms sufficient to sustain these c
are qualitative or nonmetric in nature and have already been
ments of Hardy, Rand, and Rittler in collaboration with
December 29, 1955.)

341t. Margaret F. Conroy: *Stresses at the boundar shaped hole in an infinite plate.*

In this paper the elastic stresses at the boundary of a sym
in an infinite plate, due to bands of force or concentrated force
boundary and in the plane of the plate, are determined. The c
sidered are those contours in the z-plane into which the unit
mapped by the function $z=\zeta[A_0+\sum_{n=1}^{n=m} A_n\zeta^{-kn}]$ where $k\geq$
coefficients restricted so that the roots of $dz/d\zeta$ lie within the u
The force distribution has an angular periodicity of $2\pi/k$. T
consists of a combination of the conformal mapping techniqu
method for solving plane elasticity problems. The results obt
finite plates of such an extent that the stresses at the outer
loading at the inner boundary, or hole contour, vanish. The r
paper were obtained in the course of research conducted at t
Laboratory, Watertown, Mass. (Received January 13, 1956.)

342. J. B. Diaz and G. S. S. Ludford (p): *On the of Bergman and Le Roux.*

In a previous note (Quarterly of Applied Mathematics vol.
a correspondence was found between the two representations
the linear hyperbolic equation $u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x,$
integral operator methods of S. Bergman, Math. Sbornik N.S.
1198, and J. Le Roux, Ann. École Norm. (3) vol. 12 (1915)

respondence has the disadvantage that to some regular Bergman kernels $E(x, y, t)$ there may correspond singular Le Roux kernels $U(x, y, \alpha)$. The purpose of the present note is to give an alternative correspondence which avoids this difficulty. An extension and an example are given. (Received January 11, 1956.)

343. Peter Henrici: *Remark on the quotient-difference algorithm of H. Rutishauser.*

Let $F(\lambda)=\sum_{n=0}^{\infty} a_n\lambda^{-n-1}$ be meromorphic in $|\lambda|>\Lambda$ with simple poles at the points λ_i $(i=1, 2, \cdots)$, $|\lambda_1|>|\lambda_2|>\cdots>\Lambda$. Let $H_k^{(n)}=\det\ (a_{ij})$, where $a_{ij}=a_{n+i+j-2}$ $(i, j=1, 2, \cdots, k)$, and $H_k^{(n)}(\lambda)=\det\ (b_{ij})$, where $b_{ij}=a_{n+i+j-2}$ $(i=1, 2, \cdots, k+1;$ $j=1, 2, \cdots, k)$ and $b_{i,k+1}=\lambda^{i-1}$ $(i=1, 2, \cdots, k+1)$. Put $p_k^{(n)}(\lambda)=H_k^{(n)}(\lambda)/H_k^{(n)}$. It is known that for $n\rightarrow\infty$ and fixed k, $p_k^{(n)}(\lambda)=\prod_{i=1}^{k}(\lambda-\lambda_i)+O((\lambda'_{k+1}/\lambda_k)^n)$, where $\lambda'_{k+1}=|\lambda_{k+1}|+\epsilon$, $\epsilon>0$ arbitrary. Lemma: For $\lambda=\lambda_i$ $(i<k)$ this relation can be sharpened as follows: $p_k^{(n)\prime}(\lambda_i)=O((\lambda_{k+1}/\lambda_i)^n)$, where λ_{k+1} is the same as before. This yields a simple proof of the following theorem of H. Rutishauser (Z. Angew. Math. Physik vol. 5 (1955) p. 389): Let A be a matrix of order N with simple eigenvalues λ_i, $|\lambda_1|>|\lambda_2|>\cdots>|\lambda_N|>0$. For $n=0, 1, \cdots$ let $x_k^{(n)}$, $y_k^{(n)}$ $(k=0, 1, \cdots, N-1)$ be the vectors obtained by biorthogonalising the two systems $A^{n+k}x$, $(A^T)^{n+k}y$ $(k=0, 1, \cdots, N-1)$, where x and y are two almost arbitrary vectors. Then, as $n\rightarrow\infty$, $x_k^{(n)}/|x_k^{(n)}|\rightarrow v_{k+1}$, $y_k^{(n)}/|y_k^{(n)}|\rightarrow w_{k+1}$, where v_i and w_i are the eigenvectors of A and A^T belonging to λ_i. (Received January 11, 1956.)

344. J. R. Isbell (p) and W. H. Marlow: *Hotspot problems.*

Several authors have considered the problem: determine a number v and optimal strategies such that the game with payoff matrix $(a_{ij}+b_{ij}v)$, each $b_{ij}<0$, has value zero. (See Shapley, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 1095–1100). The present starting point is such a game, due to C. B. Tompkins (George Washington University Logistics Papers, Issue 2, 1950) concerning assignment of military forces to a disputed area, "Hotspot." It is noted that there is a formal connection (via the description of combat) between the matrix game, $(-a_{ij}/b_{ij})$ and Tompkins' game; furthermore, whenever one has a solution in pure strategies then so does the other, with the same good strategies and value. For the second simplification of Tompkins' game, rather than having new assignments at stated intervals, changes are permitted whenever a unit on either side is destroyed. This new game has a solution in pure strategies; hence neither player would wish to change his assignment at any other time than those prescribed. The best strategy for either player in any case is to assign all, none, or exactly one of his units to Hotspot. Research supported (in part) by the Office of Naval Research. (Received January 12, 1956.)

345t. Erwin Kreyszig: *On the Fresnel integrals for complex argument.*

The Fresnel integrals $C(z)=2^{-1}\int_0^z t^{-1/2}\cos t\,dt$, $S(z)=2^{-1}\int_0^z t^{-1/2}\sin t\,dt$ are of importance in connection with many problems in physics and technics. The zeroes $z_n=x_n+iy_n$ of $C(z)$ and $z_n^*=x_n^*+iy_n^*$ of $S(z)$ are asymptotically located on one and the same logarithmic curve, in alternating order; this curve can be represented in the form $y=\pm 2^{-1}\log 2\pi x$. For numerical calculation of these zeroes the following approximation formulas are useful: $x_n\sim(4n-1)\pi/2-\log\ \pi(4n-1)^{1/2}/(4n-1)\pi$, $y_n\sim\log\ \pi(4n-1)^{1/2}$, $x_n^*\sim 2n\pi-\log\ 2\pi n^{1/2}/4n\pi$, $y_n^*\sim\log\ 2\pi n^{1/2}$, $n=1, 2, \cdots$. The functions $z^{1/2}C(z)$ and $z^{1/2}S(z)$ are entire functions of order 1 of divergence class as follows from the distribution of the zeroes. The Weierstrassian products are

$C(z) = z^{1/2}\prod_{n=1}^{\infty}(1-z^2/z_n^2)(1-z^2/\bar{z}_n^2)$, $S(z) = 3^{-1}z^{3/2}\prod_{n=1}^{\infty}(1-z^2/z_n^{*2})(1-z^2/\bar{z}_n^{*2})$. (Received February 13, 1956.)

346. Walter Littman: *On the existence of gravity waves near critical speed.*

A new proof is given for the existence of two-dimensional gravity waves near critical speed, using a method originally devised by Friedrichs and Hyers for the purpose of proving the existence of solitary waves. The problem is first reduced to solving a nonlinear boundary value problem for the potential equation in an infinite horizontal strip, with boundary conditions which are nonlinear on top and linear on the bottom. This problem, in turn, is reduced to that of solving a nonlinear integral equation. The essential step in the procedure is a stretching of the independent variable $\hat{\phi} = a\phi$. There results a one parameter family of integral equations all depending on a. For $a = 0$ the equation becomes very simple, and is, in fact, solved explicitly, the solution (which contains two free parameters) giving rise to the first approximation of the sought-after flow, and agreeing essentially with the approximate description of J. B. Keller. By perturbing about this solution, the original integral equation is solved for small positive a. This amounts to proving the existence of periodic flows for sufficiently large wavelengths and for near-critical speeds. This method yields both sub- and supercritical flows. (Received December 30, 1955.)

347. Imanuel Marx: *On the characteristic values of a complex quadratic form.*

For a (*non-Hermitean*) diagonalizable complex quadratic form $Q = (Az, z)$ (*complex symmetric inner product*), each characteristic value is a stationary value of $(Az, z)/(z, z)$, or equivalently of (Az, z) for vectors z such that $(z, z) = 1$, and stationary vectors are characteristic vectors (Comm. Pure Appl. Math. vol. 7 (1954) p. 625). For an arbitrary complex quadratic form, each characteristic value whose characteristic vector lies in an m-dimensional maximal cyclic invariant subspace relative to A is a stationary value of $\left|(Az_1, z_1) + \cdots + (Az_m, z_m)\right| / \left|(z_1, z_1) + \cdots + (z_m, z_m)\right|$ subject to the conditions $(z_1, z_1) = (z_2, z_2) = \cdots = (z_m, z_m)$ and $(z_j, z_k) = 0$ $(j \neq k)$, or equivalently of $(Az_1, z_1) + \cdots + (Az_m, z_m)$ for an orthonormal set. Every orthonormal basis of the invariant subspace furnishes a solution of the variational problem. Each isotropic characteristic vector z makes (Az, z) stationary subject to the condition $(z, z) = 0$. All characteristic values and corresponding invariant subspaces and characteristic vectors can be determined by the solution of a succession of variational problems in suitably chosen subspaces of admissible vectors. (Received December 27, 1955.)

348. L. E. Payne: *Inequalities for eigenvalues of a free plate.*

Upper bounds for the eigenvalues Λ_i of a free plate may be obtained by application of the Rayleigh-Ritz technique in the minimum principle. However, there appears to be no practical method for estimating Λ_i from below. In this paper the following lower bound for Λ_i is obtained: $\Lambda_{i+2} \geq (1-\sigma)\mu_2\mu_i$, ($\sigma$ is Poisson's ratio), where μ_i are the eigenvalues of the free membrane of the same shape. (The plate equation has been taken in the form $\Delta^2 u - \Lambda u = 0$, where Δ denotes the Laplace operator.) If the region possesses certain symmetry properties it is known that μ_2 may be bounded from below, the bound depending only on the geometry of the region (see Payne and Weinberger, Tech. Note BN-56, University of Maryland). One obtains in this case a lower bound for the first nonzero eigenvalue of the free plate which depends only on the geometry of the plate. (Received January 12, 1956.)

349. I. F. Ritter: *A compact method for solving a system of linear equations of any condition.*

A compact method is described for solving the equation $AX=R$ for the vector or matrix X by triangularization of the numerical matrix A. In this method, an upper triangle C is determined so that $AC=B$ is a lower triangle, the *post*-multiplication of A by the explicitly computed C being in contrast to the *pre*-multiplication of A in the Gauss elimination. Decisive advantages over other compact methods based on factoring A are secured by the following features of the method employing C explicitly: (1) The right side R enters into the computations only after B and C are established and are available for a convenient error correction process. (2) With back-substitution avoided by the use of the Aitken "below-the-line" bookkeeping device, the numerically largest component of each nonzero column of B can be chosen as pivot. (3) Columns of C corresponding to zero-columns of B represent the complete solution of the homogeneous system $A\bar{x}=0$. (4) The error in any pair of corresponding columns of B and C can be corrected immediately upon the construction of these columns. (5) With any loss in significant digits avoided by the error correction, the solution of a nearly singular, or ill-conditioned, system can be computed as accurately as desired. (6) If A is modified slightly, the pivot columns need not be adjusted for solving the new system. (7) Features (3), (4), and (6) form the basis of an effective approximation method for solving the characteristic value problem $A(\lambda)\mathfrak{x}(\lambda)=0$ with a matrix $A(\lambda)$ not necessarily linear in the parameter λ. (Received January 9, 1956.)

350t. H. E. Salzer: *Numerical integration of $y''=\phi(x, y, y')$ using osculatory interpolation.*

This present paper presents several new procedures, with formulas, for the stepwise integration of $y''=\phi(x, y, y')$, starting from some initial values of y, y' and y''. These methods are based upon different types of osculatory interpolation formulas. One method uses an approximation to the next y'' by a formula for osculatory extrapolation for the derivative, and then finds the next y' and y using formulas for $n\frac{1}{2}$-point osculatory interpolation and $(n+1)$-point osculatory quadrature respectively. Another method is applied to the special case of $y''=\phi(x, y)$, employing some new interpolation formulas based upon the function and its second derivative. Finally, for $y''=\phi(x, y, y')$ the most accurate set of formulas utilizes "hyperosculatory" interpolation, which is based upon the function with its first and second derivatives. Predicting formulas, which are purely extrapolatory, give the next y and y' (then y''). Refining formulas employ the next y'' in "$n\frac{1}{2}$-point" hyperosculatory formulas to find a new y', and then they employ y'' and this new y' in "$n\frac{1}{2}$-point" hyperosculatory formulas to find a new y (then a new y''). This refining process may be repeated if necessary. (Received November 23, 1955.)

GEOMETRY

351. Louis Auslander (p) and Masatake Kuranishi: *On the holonomy group of locally Euclidean spaces.*

Let $M^n(\pi)$ be a compact n dimensional locally euclidean space. (A space is called locally euclidean if it has a Riemann metric with curvature and torsion zero.) Then Bieberbach (Math. Ann. vol. 70 (1911) pp. 297–336) proved that the holonomy group $h(\pi)$ of $M(\pi)$ is a finite group. The authors have recently proven, using the methods of cohomology theory of groups, that any finite group G can be faithfully represented as the holonomy group of a compact n dimensional locally euclidean space for sufficiently large n. (Received December 19, 1955.)

352t. James Eells, Jr.: *The differential geometry of mapping spaces.* I.

For the purposes described in II, a detailed study of differentiable manifolds $\mathfrak{M}$ of (possibly) infinite dimension is made; i.e., each point of $\mathfrak{M}$ has a neighborhood U (called a *coordinate patch*) homeomorphic with an open set in a complete locally convex linear space (real or complex), and coordinate transformations are required to be of class C^k in the sense of Hildebrandt-Graves (Trans. Amer. Math. Soc. vol. 29, pp. 127–153) (with certain extensions). Tangent vectors are defined, and the formalities of tensor bundles and Grassmann bundles are carried over to the infinite dimensional case by means of the topologized tensor product; completeness of the tangent spaces and of their tensor products is used essentially. Using the method of sheaves, we prove a de Rham theorem relating the real (singular) cohomology of $\mathfrak{M}$ to that derived from the exterior algebra of differential forms on $\mathfrak{M}$. If $\mathfrak{M}$ is locally a Hilbert space, the sheaf of germs C^k forms on $\mathfrak{M}$ is fine as usual; otherwise, the proof of de Rham's theorem requires more detailed study of the integration of differential forms. Also, a study is made of the relation of the cohomology of $\mathfrak{M}$ to that of the nerves of certain open covers of $\mathfrak{M}$. (Received January 11, 1956.)

353. James Eells, Jr.: *The differential geometry of mapping spaces.* II.

Let M be a C^k Riemannian manifold with metric ρ, and let S be a compact Hausdorff space. The totality of continuous maps $x:S\rightarrow M$ form a function space $\mathfrak{M}$ metrized by $R(x, y)=\sup\{\rho(x(s), y(s)):s\in S\}$. Using the Riemannian structure of M (both the metric and directional properties of geodesics are used) it is shown that $\mathfrak{M}$ has a differentiable structure (in the sense of I) of class C^{k-1}. The coordinate relations are made by considering the geodesic equations on M as functions of the initial conditions. If M is a Lie group, then the function space $\mathfrak{M}$ is an analytic group (of infinite dimension, in general); relations to the associated Lie algebra are considered. In the case that $\mathfrak{M}$ is the space of paths on M with fixed origin m_0, then the space of loops at m_0 is a closed submanifold of $\mathfrak{M}$; Serre's fibre space over M is then studied with cohomology elements represented by differential forms. Several variants of the manifold $\mathfrak{M}$ are developed; e.g., spaces of differentiable maps and of absolutely continuous maps, spaces of mappings of a compact pair (S, S_0) into (M, M_0), where M_0 is a closed submanifold of M. (Received January 11, 1956.)

354. S. I. Goldberg: *Projectively Euclidean Hermitian manifolds.* I.

In an Hermitian manifold M^n a curve $C:\xi^i=\xi^i(t)$ is called a geodesic if the tangent vector $\dot{\xi}^i$ at a point remains tangent under a parallel displacement along C. It can be shown that the n.a.s. conditions are $\ddot{\xi}^\alpha+\Gamma^{*\alpha}_{\beta\gamma}\dot{\xi}^\beta\dot{\xi}^\gamma+2g_{\gamma\rho^*}g^{\alpha\sigma^*}S_{\tau^*\sigma^*}{}^{\rho^*}\dot{\xi}^{\tau^*}\dot{\xi}^\gamma=f(t)\dot{\xi}^\alpha$. (For definitions and notation cf. the author's paper *Tensorfields and curvature in Hermitian manifolds with torsion*, Ann. of Math., to appear). In analogy to the Riemannian case, it is asked if it is possible to transform the connection Γ^i_{jk} of M^n in such a way that the geodesics remain geodesics. A group G_{SP} of "special" projective transformations $\Gamma^i_{jk}\rightarrow\Gamma'^i_{jk}$ is defined as follows: (i) the Γ'^i_{jk} are self-adjoint and vanish for indices of opposite parity, (ii) $S'^i_{jk}=S^i_{jk}$. If under a "special" projective transformation M^n is mapped into a locally Euclidean space, M^n is said to be G_{SP}-flat. It is shown that an Hermitian manifold is G_{SP}-flat if and only if it has "constant" holomorphic curvature. This generalizes a result due to S. Bochner (*Curvature in Hermitian metric*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 179–195. Bochner's theorem, as well as its generalization, is a complex analogue of a well-known theorem due to H. Weyl (*Zur Infinitesimalgeometrie: Einordnung der projectiven und der konformen Auffassung*, Gottinger Nachrichten (1921) pp. 99–112). (Received January 9, 1956.)

355*t*. S. I. Goldberg: *Projectively Euclidean Hermitian manifolds*. II.

This is a continuation of the author's previous abstract in this bulletin. The following additional results are obtained: 1. The only manifolds whose geodesics correspond to the geodesics of a manifold of "constant" holomorphic curvature are manifolds of "constant" holomorphic curvature. This is the complex analogue of a well-known theorem due to Beltrami. 2. A G_{SP}-flat manifold is conformal with some Kaehler (Einstein) space. For projectively flat manifolds this result is trivial. 3. A G_{SP}-flat compact manifold is projectively flat. This last statement is proved with the help of a well-known lemma due to S. Bochner. (Received January 9, 1956.)

356. C. C. Hsiung: *On the differential geometry of hypersurfaces in the large.*

Let V^n (V^{*n}) be an orientable hypersurface of class C^3 imbedded in a Euclidean space E^{n+1} of dimension $n+1 \geqq 3$ with a closed boundary V^{n-1} (V^{*n-1}) of dimension $n-1$. Suppose that there is a one-to-one correspondence between the points of V^n, V^{*n} such that at corresponding points V^n, V^{*n} have the same normal vectors. Let M_α ($\alpha = 1, \cdots, n$) be the αth mean curvature of V^n at a point P defined by $\binom{n}{\alpha} M_\alpha = \sum \kappa_1 \cdots \kappa_\alpha$, where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of V^n at P, and the expression on the right side is the αth elementary symmetric function of $\kappa_1, \cdots, \kappa_n$. Let dA be the area element of V^n at P, and p^* the oriented distance from a fixed point O in E^{n+1} to the tangent hyperplane of V^{*n} at the point P^*, which corresponds to P under the given correspondence. The purpose of this paper is first to derive some expressions for the integrals $\int_{V^n} M_\alpha p^* dA$ ($\alpha = 1, \cdots, n$), and then using one of these expressions to prove that under some further conditions V^n and V^{*n} are congruent or symmetric. (Received January 11, 1956.)

357*t*. Detlef Laugwitz: *Riemannian metrics associated with convex bodies and normed spaces.*

It is well known that with a bounded convex body C containing the origin O of a real vector space L as an inner point there is associated a convex positive definite functional $F(x)$ (the norm or Minkowski metric). Recently, E. R. Lorch has begun the study of another geometry closely related to C, namely the Riemannian geometry defined by $g_{ik}(x) = 2^{-1}\partial^2 F^2(x)/\partial x^i \partial x^k$ (See Bull. Amer. Math. Soc. Abstract 622–247). The results are valid for spaces of finite as well as of infinite dimension, for the latter cases by the use of the authors notational convention (See Math. Zeit. vol. 61 (1954) pp. 100–118). The curvature tensor R_{ijkl} of this geometry as computed by Lorch and the tensor S_{ijkl} of the tangent Minkowski space in Cartan's theory of Finsler spaces are related by $R = F^2 S$. *For two-dimensional spaces, R vanishes identically; this implies $S = 0$ for any two-dimensional Finsler space.* A necessary and sufficient condition that all the straight lines of L be geodesics for g_{ik} is that C be an ellipsoid with 0 as its center, in other words, that the Minkowski metric F be euclidean. (Received January 11, 1956.)

358t. T. S. Motzkin: *L'Hospital's rule for nonzero characteristic.* Preliminary report.

An algebroid branch at the origin of affine n-space over a ground field of characteristic p, given by a reduced representation $x=\sum_{k=1}^{\infty} a_k t^k$, $a_k=(a_{1k}, \cdots, a_{nk})$ (i.e. not similarly representable by any parameter $\tau=\sum_{l=2}^{\infty} c_l t^l$), has a straight line of highest intersection multiplicity or "0-tangent" $x=a_{k_0}t$, where k_0 is the smallest k with $a_k\neq 0$; a "1-tangent" or specialized generic tangent $x=a_{k_1}t$, $k_1=\min k$, $ka_k\neq 0$; a "λ-tangent" $x=a_{k_\lambda}t$, $k_\lambda=\min k$, $C_{k_{\lambda-1}}a_k\neq 0$. These definitions are independent of the reduced representation selected. There may be up to k_0-q+2 different tangents, where $q=\prod(r_i+1)$ and the r_i are the digits of k_0 to base p. (Received January 13. 1956.)

359. D. K. Pease: *Four-space representation of the complex plane.* I. *The line at infinity.*

If (X, Y) is a point of the complex plane, C_2, with $X=x+iu$, $Y=y+iv$, let (x, y, u, v) be the corresponding point of R_4. A complex line of C_2 is represented by a *regular plane* of R_4. The regular planes of R_4 intersect R_3^∞, the hyperplane at infinity of R_4, in *regular lines*, which comprise the real lines of a linear congruence determined by the *regular directrices* D_1: $x+iu=y+iv=0$ and D_2: $x-iu=y-iv=0$. To each point of C_1^∞, the complex line at infinity of C_2, there corresponds a regular line of R_3^∞. Any selected real plane, R_2^∞, of R_3^∞ contains a unique point of each regular line except for the unique regular line lying in R_2^∞. Since this regular line is equivalent to one point of C_1^∞, R_2^∞ is topologically equivalent to a complete Argand plane. This confirms a conjecture of J. S. Taylor. (Received January 6, 1956.)

360t. D. K. Pease: *Four-space representation of the complex plane.* II. *Affinities.*

An affinity, $X=AX'+BY'+E$, $Y=CX'+DY'+F$, on C_2 induces a *type* A affinity on R_4. (See part I for notation and terms.) The type A affinities are just those real affinities on R_4 for which the regular directrices, D_1 and D_2, are invariant. If the characteristic roots of the collineation induced on C_1^∞ are P and Q, those on R_3^∞ are P, Q, $\overline{P}$, $\overline{Q}$. The following is a list of canonical forms together with the Segre characteristics. $\{\ \ \}$ indicates complex conjugates. $\widehat{\ }$ indexes the root $\rho=1$ corresponding to the fixed R_3^∞. P and Q are complex; p and q are real. Non-parabolic: $X=PX'$, $Y=QY'$, $[\{11\}\{11\}\hat{1}]$; $X+PX'$, $Y=qY'$, $[(11)\{11\}\hat{1}]$; $X=PX'$, $Y=PY'$, $[\{(11)(11)\}\hat{1}]$; $X=PX'$, $Y=\overline{P}Y'$, $[\{(11)(11)\}\hat{1}]$; $X=PX'$, $Y=Y'$, $[(11\hat{1})\{11\}]$; $X=PX'$, $Y=Y'+F$, $[(\hat{2}1)\{11\}]$; $X=pX'$, $Y=qY'$, $[(11)(11)\hat{1}]$; $X=pX'$, $Y=pY'$, $[(1111)\hat{1}]$; $X=pX'$, $Y=Y'$, $[(11\hat{1})(11)]$; $X=pX'$, $Y=Y'+F$, $[(\hat{2}1)(11)]$; $X=X'$, $Y=Y'$, $[(1111\hat{1})]$; $X=X'$, $Y=Y'+F$, $[(\hat{2}111)]$. Parabolic: $X=PX'+Y'$, $Y=PY'$, $[\{22\}\hat{1}]$; $X=pX'+Y'$, $Y=pY'$, $[(22)\hat{1}]$; $X=X'+Y'$, $Y=Y'$, $[(22\hat{1})]$; $X=X'+Y'$, $Y=Y'+F$, $[(\widehat{32})]$. Of the 69 types of real affinities on R_4, 15 occur as type A. (Received January 6, 1956.)

361t. D. K. Pease: *Four-space representation of the complex plane.* III. *Anti-affinities.*

An anti-affinity, $X=A\overline{X}'+B\overline{Y}'+E$, $Y=C\overline{X}'+D\overline{Y}'+F$, on C_2 induces a *type* B affinity on R_4. (See parts I and II for notation and terms.) The type B affinities are just those real affinities on R_4 for which the regular directrices, D_1 and D_2, are interchanged. The type A and type B affinities together comprise the *regular affinities* on R_4. The following is a list of the canonical forms together with the Segre characteristics. Hyperbolic: $X=p\overline{X}'$, $Y=q\overline{Y}'$, $[1111\hat{1}]$; $X=p\overline{X}'$, $Y=\overline{Y}'$, $[(1\hat{1})111]$; $X=p\overline{X}'$, $Y=p\overline{Y}'$, $[(11)(11)\hat{1}]$; $X=\overline{X}'$, $Y=\overline{Y}'$, $[(11\hat{1})(11)]$; $X=p\overline{X}'$, $Y=\overline{Y}'+F$, $[\hat{2}111]$; $X=\overline{X}'$, $Y=\overline{Y}'+F$, $[(\hat{2}1)(11)]$. Parabolic: $X=p\overline{X}'+\overline{Y}'$, $Y=p\overline{Y}'$, $[22\hat{1}]$; $X=\overline{X}'+\overline{Y}'$, $Y=\overline{Y}'$, $[(2\hat{1})2]$; $X=\overline{X}'+\overline{Y}'$, $Y=\overline{Y}'+F$, $[\widehat{32}]$. Elliptic: $X=Q\overline{Y}'$, $Y=P\overline{X}'$, $[\{11\}\{11\}\hat{1}]$; $P\overline{Q}$ real, $[\{(11)(11)\}\hat{1}]$; $P\overline{Q}=1$, $[(11\hat{1})(11)]$; $X=Q\overline{Y}'+E$, $Y=P\overline{X}'$, $P\overline{Q}=1$, $[(\hat{2}1)(11)]$. Of the 69 types of real affinities on R_4, eight occur as type B and 17 as regular affinities. (Received January 6, 1956.)

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LOGIC AND FOUNDATIONS

362*t*. R. M. Friedberg: *The solution of Post's problem by the construction of two recursively enumerable sets of incomparable degrees of unsolvability.*

Kleene and Post have constructed (*The upper semi-lattice of degrees of recursive unsolvability*, Ann. of Math. vol. 59 (1954) pp. 379–407, esp. pp. 385–390) two functions β_1 and β_2 of incomparable degrees. In the present paper this construction is made recursive so that β_1 and β_2 are characteristic functions of recursively enumerable sets. Functions α_1 and α_2 and conditions $\{2\}$ and $\{3\}$ are dropped. (Notation here and below is from Kleene-Post, *op. cit.*) Each condition $\{0, e\}$ or $\{1, e\}$ cannot be treated in one step since the choice between subcases cannot be made effectively. Therefore treatment of condition $\{0, e_1\}$ may require indefinite indecision as to the value of $\beta_1(a)$, while treatment of $\{1, e_2\}$ requires immediate decision. In the present construction, if $e_1 \leq e_2$, condition $\{1, e_2\}$ is treated on the assumption that $\beta_1(a)$ will never be set equal to 0; this treatment is afterwards corrected if condition $\{0, e_1\}$ requires $\beta_1(a)=0$. If $e_2 < e_1$, condition $\{1, e_2\}$ is treated fully, and treatment of $\{0, e_1\}$ is shifted from a to another argument. Similarly for β_2. It is shown that the treatment of each condition is shifted or corrected only a finite number of times. (Received January 10, 1956.)

363*t*. Richard Montague: *Zermelo-Fraenkel set theory is not a finite extension of Zermelo set theory.*

By Zermelo-Fraenkel set theory (ZF) is understood the theory, formulated in the lower predicate calculus with identity, whose only non-logical symbol is the two-place predicate '$\in$' and whose non-logical axioms are (1) the axiom of extensionality, (2) the power-set axiom, (3) the sum-set axiom, (4) the axiom of infinity, (5) the axiom of replacement (which comprises every instance of the schema '$(x, y, z)((x\in a\cdot\phi(x, y)\cdot\phi(x, z))\supset y=z)\supset(\exists b)(y)(y\in b\equiv(\exists x)(x\in a\cdot\phi(x, y)))$'), and (6) the Fundierungsaxiom. For precise formulations of (1)–(4), (6), and the pair-axiom see Mostowski, Fund. Math. vol. 37 (1950), p. 111. Zermelo set theory (Z) differs from ZF by containing, in place of (5), the pair-axiom and the Aussonderungsaxiom (the schema '$(\exists x)(y)(y\in x\equiv(\phi\cdot y\in a))$'). McNaughton (Proc. Amer. Math. Soc. vol. 5 (1954) pp. 505–509) has shown that neither theory is finitely axiomatizable. Further, it is well known that ZF is an extension of Z. It can be shown, however, that ZF, if consistent, is not equivalent to any theory obtained by adding finitely many axioms to Z; that is, ZF is not a *finite extension* of Z. In fact, a stronger result holds: if T is any consistent extension without new constants of ZF, then T is not a finite extension of Z. (Received November 28, 1955.)

364*t*. Richard Montague and Donald Kalish: *A simplification of Tarski's formulation of the predicate calculus.*

Tarski (Bull. Amer. Math. Soc. Abstract 57-1-74) has given the following formulation, which does not employ the notion of free variable or that of proper substitution on free variables, of the lower predicate calculus with identity. Axiom-schemata: (1) If ϕ is tautologous, then ϕ is an axiom. (2) $\ulcorner(\alpha)(\beta)\phi\supset(\beta)(\alpha)\phi\urcorner$ is an axiom. (3) $\ulcorner(\alpha)(\phi\supset\psi)\supset((\alpha)\phi\supset(\alpha)\psi)\urcorner$ is an axiom. (4) $\ulcorner\phi\supset(\alpha)\phi\urcorner$ is an axiom if α does not occur in ϕ. (5) $\ulcorner(\alpha)\phi\supset\phi\urcorner$ is an axiom. (6) $\ulcorner\sim(\alpha)\sim\alpha=\beta\urcorner$ is an axiom. (7) $\ulcorner\alpha=\beta\supset(\phi\supset\psi)\urcorner$ is an axiom if ϕ is atomic and ψ is obtained from ϕ by replacing an occurrence of α by β. Rules: Modus Ponens, Generalization. Schemata (2) and (5) can be eliminated; the resulting system is complete and independent. The Rule of Generalization can be eliminated by taking as axioms all universal generalizations of the axioms comprised under (1), (3), (4), (6), (7); the resulting system is also complete and independent. (Received November 28, 1955.)

365*t*. Richard Montague and Donald Kalish: *An extension of Tarski's notion of satisfaction.*

Let L be a language, i.e., a set of predicates and operation symbols. Let $\langle DxRFA\rangle$ be a model of L; i.e., let $x\in D$, R and F be functions such that if π is an n-place predicate in L then $R(\pi)$ is a set of n-tuples of members of D, and if δ is an n-place operation symbol in L then $F(\delta)$ is a function from the class of n-tuples of members of D into D, and A be a function from the class of variables into D. Then, by simultaneous recursion: (i) if α is a variable, α is a *term* (of L) and *val* (α) (for $\langle DxRFA\rangle$) is $A(\alpha)$; (ii) if δ is an n-place operation symbol in L and $\zeta_1, \cdots, \zeta_n$ are *terms*, then $\ulcorner\delta\zeta_1, \cdots, \zeta_n\urcorner$ is a *term* and *val* $(\ulcorner\delta\zeta_1\cdots\zeta_n\urcorner)$ is $[F(\delta)]$ (*val* $(\zeta_1)\cdots$ *val* (ζ_n)); (iii) if ϕ is a *formula* (of L), then $\ulcorner\iota\alpha\phi\urcorner$ is a *term* and *val* $(\ulcorner\iota\alpha\phi\urcorner)$ is the unique s such that $\langle DxRFA^{\alpha}_{s}\rangle$ *satisfies* ϕ if there is such a s, and x otherwise (where A^{α}_{s} is that function whose value for α is s and which otherwise coincides with A); (iv) if π is an n-place predicate in L, ζ, η, $\zeta_1, \cdots, \zeta_n$ are *terms*, and ϕ, ψ are *formulas*, then $\ulcorner\zeta=\eta\urcorner$, $\ulcorner\pi\zeta_1\cdots\zeta_n\urcorner$, $\ulcorner(\alpha)\phi\urcorner$, $\ulcorner\sim\phi\urcorner$, $\ulcorner(\phi\supset\psi)\urcorner$, etc. are *formulas*, and $\langle DxRFA\rangle$ *satisfies* each of them under the usual conditions. (Received December 13, 1955.)

366*t*. Richard Montague and Donald Kalish: *Formulations of the predicate calculus with operation symbols and descriptive phrases.*

For terminology see abstract 365. In abstract 364 is given a simplified formulation of the lower predicate calculus with identity for languages which do not contain operation symbols. A completely analogous formulation can be given for an arbitrary language L. Axiom schemata (here ϕ, ψ, and ζ, η are respectively formulas and terms of L without descriptive phrases): (1), (3), (4) of abstract 1; (6′) $\ulcorner\sim(\alpha)\sim\alpha=\zeta\urcorner$, where α does not occur in ζ; (7′) $\ulcorner\zeta=\eta\supset(\phi\supset\psi)\urcorner$, where ϕ is a quasi-atomic formula of L (i.e., $\ulcorner\zeta_1=\zeta_2\urcorner$ or $\ulcorner\pi\zeta_1\cdots\zeta_n\urcorner$), and ψ is obtained from ϕ by replacing an occurrence of ζ by η. Rules: Modus Ponens, Generalization. This system is complete: if ϕ is a formula of L without descriptive phrases, then ϕ is a theorem if and only if ϕ is satisfied by every model of L. To extend this formulation to include descriptive phrases, omit the restriction which excludes them from formulas and terms, and add (for β not occurring in $\ulcorner\iota\alpha\phi\urcorner$): (8) $\ulcorner(\alpha)(\phi\equiv\alpha=\beta)\supset\beta=\iota\alpha\phi\urcorner$; (9) $\ulcorner(\beta)\sim(\alpha)\cdot(\phi\equiv\alpha=\beta)\supset\iota\alpha\phi=\iota\gamma\gamma=\gamma\urcorner$. This system also is complete and is formulated without the notions of free variables and proper substitution. (Received December 13, 1955.)

367*t*. J. P. Roth: *An algorithm for the problem of Quine.*

It has been seen that a truth-functional formula ϕ defines a cubical complex K (*A combinatorial topological method for the synthesis of switching systems in n variables*, Bull. Amer. Math. Soc. Abstract 62-2-281), any equivalent formula yielding an isomorphic complex; the problem of Quine (Amer. Math. Monthly vol. 62 (1955) pp. 627–631), finding a shortest normal equivalent, is interpreted as that of finding a *cover* consisting of q_k k-cubes of K of various dimensions k such that the sum $\sum_1^n q_k(n-k)$ is a minimum. An algorithm designed for automatic machine computation yields covers which e.g. cannot be improved by Quine's subsuming and consensus operations. Elaborations over previous results for going below the Quine minimum are also given. (Received January 10, 1956.)

368*t*. A. R. Schweitzer: *Mathematics and literary composition.* I.

This paper is concerned with a composition "C" consisting of a set of statements "A," the foundation of C, and a set of statements "B" elaborating A in accordance with a plan of thought "D" leading to a concept or system or concepts. Two types of the composition C are considered, namely, mathematical composition C_1 (Mathematics) and literary composition C_2 (literature) understood as a play or a novel or a poem. Corresponding to C_1, A is a set of postulates, B is the set of theorems derived from A in accordance with D, a system of logic. (Received January 9, 1956.)

369*t*. A. R. Schweitzer: *Mathematics and literary composition.* II.

In continuation of the preceding abstract; corresponding to C_2 A is a plot, B elaborates A in accordance with D, a system of literary technique. Examples are given of each type of composition; in particular for C_2 the author selects Shakespearean plays including King Lear, Hamlet, Othello, Macbeth, Merchant of Venice leading respectively to the concepts ingratitude, indecision, jealousy, ambition, inhumanity. Reference is made to Martin Johnson, *Art and scientific thought*, New York, 1949; G. D. Birkhoff, *Aesthetic measure*, Collected Mathematical Papers, vol. III, p. 382; Paul Goodman, *The structure of literature*, Chicago, 1954. The author interprets Goodman's treatise as a study of the whole-part relation with application to the literature in the sense indicated in the preceding abstract. (Received January 9, 1956.)

370*t*. A. R. Schweitzer: *Mathematics and literary composition.* III.

An example of the analogy of mathematics with literary composition is found in the occurrence of the betweenness relation in the foundations of geometry and in the use, in effect, of this relation by Aristotle in his treatise on poetry (Poetics: beginning, middle, ending). (Received January 9, 1956.)

371*t*. R. L. Vaught: *On the axiom of choice and some metamathematical theorems.*

Consider the following three known theorems concerning the existence of models of sentences (of the first-order predicate logic): (1) a sentence having a model of power $\mathfrak{b}$ has also a model of every power $\mathfrak{a}$ such that $\aleph_0 \leqq \mathfrak{a} \leqq \mathfrak{b}$; (2) a sentence having a model of power $\aleph_0$ has also a model of every power $\mathfrak{c} \geqq \aleph_0$; (3) if Q is a set of sentences, in which the set of individual constants involved may have an arbitrary power $\mathfrak{b}$, and every finite subset of Q has a model, then Q has a model, whose power is not greater than $\mathfrak{b}+\aleph_0$. Theorem: *Each of* (1), (2), *and* (3) *implies the Axiom of Choice in its general* *form.* As regards (3), this answers a question, raised by Henkin (Trans. Amer. Math. Soc. vol. 74 (1953) p. 420), as explicitly formulated. (On the other hand, the statement, obtained from (3) by dropping the final phrase ("whose . . . ") has been shown by Henkin and Łos to be equivalent to the existence of prime ideals in Boolean algebras, and its exact relationship to the Axiom of Choice is still not determined.) The Theorem is derived from the known result (of Tarski) that the Axiom of Choice is implied by the statement: if $\aleph_0 \leqq \mathfrak{b}$, then $\mathfrak{b}^2 = \mathfrak{b}$. (Received January 13, 1956.)

STATISTICS AND PROBABILITY

372. D. G. Austin: *On second derivatives of Markoff functions—denumerable case.* Preliminary report.

The author has shown that if i and j are stable states in a Markoff process with conditional transition functions $p_{ij}(t)$, $i, j = 1, 2, \cdots$, then each of the functions $p_{kj}(t)$ and $p_{ik}(t)$ $(k = 1, 2, \cdots)$ possess continuous first derivatives (Tech. Report AF 18(600)-760). The inequality $D_*^{(2)} p_{ij}(t+s) \geqq \sum_k Dp_{ik}(t)Dp_{kj}(s)$ is verified. It is shown that $Dp_{ij}(t)$ is absolutely continuous and that, for almost all points (t, s), $(^*)$ $D^2p_{ij}(t+s) = \sum_k Dp_{ik}(t)Dp_{kj}(s)$. If for some s the $Dp_{kj}(s)$ are bounded then $Dp_{ij}(t)$ possesses a continuous derivative on the interval $t \geqq s$. In the case where all states are stable the following result is obtained: (the notation of Doob is used): If $\sum_{j \neq i} q_{ij} = q_i$ and $\sum_j q_{ij}q_j$ converges, then the $p_{ij}(t)$ $(j = 1, 2, \cdots)$ possess finite continuous second order derivatives on the interval $0 \leqq t < \infty$ which satisfy the differential equation $(^*)$; if in addition $\sum_j q_{ij}q_j^2$ converge the $p_{ij}(s)$ possess continuous third order derivatives on the infinite interval. (Received January 13, 1956.)

373*t*. A. A. Blank: *The existence and uniqueness of a uniformly most powerful unbiased randomized test for the binomial.*

The Neyman-Pearson theory of tests was utilized by K. D. Tocher in Biometrika vol. 37 (1950) for discrete distributions to obtain necessary conditions for the uniformly most powerful tests among unbiased randomized tests. In particular, he exhibited the necessary form of such tests for the binomial. A proof of existence and uniqueness for a test of the requisite form is exhibited herein. It is a corollary that the test satisfies the stated criteria. (Received December 29, 1955.)

374. R. V. S. Chacon: *Markov transition probability functions which are not necessarily measurable.*

Let $p_{ij}(t)$ denote the transition probability functions of a denumerable Markov process, with no further assumptions. It is proved that $\liminf_{t\to\infty} p_{ij}(t) \leqq \liminf_{t\to\infty} p_{jj}(t)$ and $\liminf_{t\to\infty} p_{ij}(t) = \liminf_{t\to\infty} p_{jj}(t)$ or $p_{ji}(t) \equiv 0$ and also the same equations for the limit superior, with no reversal of the inequalities. Using this theorem it is proved that a necessary and sufficient condition that $\lim_{t\to\infty} p_{ij}(t)$ exist for all i, j is that $\lim_{t\to\infty} p_{kk}(t)$ exist for all k. It is also proved that if the $p_{ij}(t)$ are so that each is either positive or identically zero, then $\lim_{t\to\infty} p_{ij}(t)$ exists for all i, j. (Received February 1, 1956.)

375*t*. D. G. Kendall and G. E. H. Reuter: *The determination of the ergodic projection for Markov chains and processes.* I.

Let $(p_{ij}: i, j = 0, 1, 2, \cdots)$ be the one-step transition probabilities for a Markov chain with a countable infinity of states; let N (N_+) be the set of vectors (positive vectors) x such that $\sum_\alpha |x_\alpha| < \infty$ and $x_j = \sum_\alpha x_\alpha p_{\alpha j}$ (all j); and let $N^*(N_+^*)$ be the set of vectors (positive vectors) y such that $\sup_\alpha |y_\alpha| < \infty$ and $y_i = \sum_\alpha p_{i\alpha} y_\alpha$ (all i). A state j is "positive" if and only if it lies in the support of an element of N; two positive states are in different "classes" if and only if there exists an element of N whose support contains one and not the other. Let the positive state j be in the class C^ρ; then among the elements of N_+ such that $x_j = 1$ there is a least, x^j, and $\pi^\rho = x^j/\|x^j\|$ depends only on C^ρ and has C^ρ as its exact support. Also among the elements of N_+^* such that $(y, \pi^\rho) = 1$ there is a least, $\breve{\omega}^\rho$, say. Theorem: $\pi_{ij} = (C, 1) \lim_{n\to\infty} p_{ij}^{(n)} = (\breve{\omega}^\rho)_i(\pi^\rho)_j$ for all i. (Of course $\pi_{ij} = 0$ for all i when j is not a positive state.) (Received December 19, 1955.)

376t. D. G. Kendall and G. E. H. Reuter: *The determination of the ergodic projection for Markov chains and processes*. II.

(1) If $(P_t : t \geq 0)$ is the transition semigroup of operators on the Banach space l associated with a Markov process having transition probabilities $p_{ij}(t)$ then the above construction gives $\pi_{ij} = \lim_{t\to\infty} p_{ij}(t)$ when N (N_+) is the set of (positive) vectors annihilated by the infinitesimal generator Ω and N^* (N_+^*) is the set of (positive) vectors annihilated by the adjoint operator Ω^*. (2) If $Q = (q_{ij} : i, j = 0, 1, 2, \cdots)$ is a matrix of finite real numbers nonnegative save on the principal diagonal and having zero row-sums and if the Markov process defined by $p_{ij}'(0+) = q_{ij}$ is unique, then the problem of determining the π's when the q's are known is solved as follows. Write N_+ for the set of positive vectors x such that $\sum_\alpha x_\alpha < \infty$ and $\sum_\alpha x_\alpha q_{\alpha j} = 0$ (all j), and write N_+^* for the set of positive vectors y such that $\sup_\alpha y_\alpha < \infty$ and $\sum_\alpha q_{i\alpha} y_\alpha = 0$ (all i). The positive states and classes can be found as before but now the supports of the elements of N_+ are to be used. The minimality properties determining π^ρ and $\breve{\omega}^\rho$ as elements of the new N_+ and N_+^* still hold and we have as a solution to the problem the Theorem: $\pi_{ij} = \lim_{t\to\infty} p_{ij}(t) = (\breve{\omega}^\rho)_i(\pi^\rho)_j$ for all i when $j \in C^\rho$, and $\pi_{ij} = 0$ for all i when j is not a positive state. (Received December 19, 1955.)

377. Emanuel Parzen: *A law of large numbers for identically distributed random variables with identically distributed increments*.

Let the random variables $X(t)$, defined for t in $T = \{0, \pm 1, \pm 2, \cdots\}$, be identically distributed, with common distribution function $F(x)$ and characteristic function $\phi(u)$. For every τ, let the increments $X(t+\tau) - X(t)$ be identically distributed, with characteristic function $\phi(u; \tau)$. Then for every Borel function g, defined on the real line, such that $E|g| = \int_R |g(x)| dF(x) < \infty$, the sample means $M_n(g)$, defined by $(n+1)M_n(g) = \sum_{t=0}^{n} g(X(t))$, converges as a limit in first mean, and if $E|g|^2 < \infty$, then the $M_n(g)$ converge as a limit in quadratic mean. If, further, for every real u, $(n+1)^{-1}\sum_{t=0}^{n} \phi(u; t) \to |\phi(u)|^2$ as $n \sim \infty$, then $M_n(g) \to Eg = \int_R g(x) dF(x)$ as a limit in first mean if $E|g| < \infty$, and as a limit in quadratic mean if $E|g|^2 < \infty$. (Received January 13, 1956.)

TOPOLOGY

378. R. D. Anderson: *A continuous curve admitting monotone open maps onto all locally connected metric continua*.

The universal curve M is a one-dimensional continuous curve which is an analog of the Cantor middle third set and is obtainable by punching rectangular "holes" out of a cube in a particular regular fashion. In a forthcoming paper the author characterizes the universal curve. The principal theorem of this paper asserts that for any locally connected continuum S there exists a monotone open map f of M onto S with the additional property that for each $s \in S$, $f^{-1}(s)$ is homeomorphic to M. It has not previously been known whether any continuum admits either monotone or open maps onto all locally connected continua. (Received January 11, 1956.)

379*t*. Steve Armentrout: *On spirals in the plane.*

It is shown that there exist in the plane an arc M, a point O not belonging to M, and a collection G of arcs such that (1) every arc of G has O as one end point and some point of M as its other end point and no two arcs of G have any point, other than O, in common, (2) if X is a point of M, there is an arc of G which contains X and spirals down on X, (3) if g is an arc of G and P is a point such that g spirals down on P, then P belongs to M, (4) G^* is bounded. (Received December 19, 1955.)

380*t*. Hubert Arnold: *Closed level curves.*

Denote by $\bar{D}$ the closed disk $x^2+y^2 \leqq a^2$ and by L and D its circumference and interior. Denote by $f(p)$ the transformation (single- or multiple-valued) $u=u(x, y)$, $v=v(x, y)$ from $\bar{D}$ of the $x-y$ plane into the $u-v$ plane. Assume $f(p)$ continuous for p in $\bar{D}$ and locally a homeomorphism for p in D, and assume that $f(p)=(0, 0)$ if and only if $p=(0, 0)$. Then the set of points (x, y) for which $|f(p)|^2=u^2+v^2=k^2$ is a single simple closed continuous curve, for each constant k for which $0<k<\min_{p \in L}|f(p)|$. (Received January 11, 1956.)

381*t*. Bernhard Banaschewski: *On the Weierstrass-Stone approximation theorem.*

By the Weierstrass-Stone approximation theorem, any ring R of bounded continuous real valued functions on a compact space K containing (i) the constants and (ii) sufficiently many functions (that is: to any x, $y \neq x$ in E an f such that $fx \neq fy$) is dense in the ring of all bounded continuous real valued functions on K relative to uniform convergence. An example is given, obtained by refining the topology of the Čech compactification βE of an arbitrary completely regular E at the points of $\beta E-E$ without change of the topology of the subspace E, which shows that the validity of this theorem extends beyond the category of compact spaces. In fact, the following is proved: Let E have sufficiently many bounded continuous real valued functions. Then, the Weierstrass-Stone approximation theorem holds for E if and only if any completely regular filter $\mathfrak{A}$ on E has nonvoid intersection $\bigcap A$, $A \in \mathfrak{A}$. (Complete regularity of $\mathfrak{A}$: $\mathfrak{A}$ has a basis $\mathfrak{B}$ of open sets such that for any $B \in \mathfrak{B}$ there exists a $B' \subseteq B$ in $\mathfrak{B}$ and a continuous function f mapping E onto the real unit interval, vanishing on B' and equal to 1 outside B.) In particular: If the approximation theorem holds for a completely regular E, then E is compact. (Received January 12, 1956.)

382. R. H. Crowell: *Invariants of alternating link types.*

If $\Delta(t_1, \cdots, t_\mu)$ is the Alexander polynomial of a tame link type L of multiplicity μ, the *reduced Alexander polynomial* $\tilde{\Delta}(t)$ of L is equal to: $\Delta(t)$ if $\mu=1$; $(t-1)\Delta(t, \cdots, t)$ if $\mu \geqq 2$. It is known that degree $\tilde{\Delta}(t) \leqq 2h+\mu-1$, where h is the genus of L (G. Torres, *On the Alexander Polynomial*, Ann. of Math. vol. 57 (1953)). A link type L is *admissible* iff there exists a connected, alternating, directed link projection P of type L. Three theorems are proved: (I) For any admissible link type L of multiplicity μ and reduced Alexander polynomial $\tilde{\Delta}(t)$, degree $\tilde{\Delta}(t)=2h+\mu-1$. For admissible link types then, the genus is an effectively calculable invariant. Included in the proof of (I) is a proof of: (ii) The coefficients of the reduced Alexander polynomial of an alternating link type alternate in sign. This result provides the easiest proof of the existence of nonalternating link types. Let L be an admissible link type with a connected, alternating, directed link projection P. Then there exists an effective algorithm, which when applied to P yields a link projection P' also of type L. The algorithm is intuitively a process of untwisting. Then: (iii) If the number of crossings of P' is d and the determinant of L is D, then $D \geqq d$. A corollary is: The group of an alternating knot type K is infinite cyclic if and only if K is trivial. Result (iii) was stated for $\mu = 1$ by Bankwitz (C. Bankwitz, *Ueber die Torsionszahlen der Alternierenden Knoten*, Math. Ann. vol. 103 (1930) pp. 145–161). (Received October 24, 1955.)

383*t*. Eldon Dyer: *One-dimensional open maps onto Cantorian manifolds.*

In this paper it is shown that if f is a monotone open (continuous) mapping of a compact metric space X onto an n-dimensional Cantorian manifold Y and for each point y of Y, $f^{-1}(y)$ is one-dimensional and lc^1 (with coefficient group the reals mod 1), then X is an $(n+1)$-dimensional Cantorian manifold. The proof relies on category arguments and homological dimension theory. Also use is made of an earlier result of the author announced in an abstract entitled *Open mappings and regular convergence* (Bull. Amer. Math. Soc. Abstract 62-2-275). This theorem generalizes certain results of Hamstrom and Anderson (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 766–769) and strengthens a result of the author (to appear soon in Ann. of Math.). (Received January 12, 1956.)

384. Eldon Dyer: *Open mappings and dimension*. Preliminary Report.

The following result is established in this paper. Suppose f is an open (continuous) map of a compact metric space X onto an n-dimensional space Y and that for each point y of Y, $f^{-1}(y)$ is homologically locally connected (with coefficient group the reals mod 1) in all dimensions. Suppose further that there are integers p and q, $p<q$, such that for each point y of Y, $p \leqq \dim (f^{-1}(y)) \leqq q$. For each integer i, let Y_i denote the set of all points y of Y such that $\dim (f^{-1}(y)) = i$. Then there is a greatest integer r such that Y_r intersects every dense G_δ subset of Y, and $r+n \leqq \dim (X) \leqq q+n$. In case $\dim (f^{-1}(y)) = q$ for each point y of Y, $\dim (X) = q+n$. The arguments depend on results announced by the author in an abstract entitled *Open mappings and regular convergence* (Bull. Amer. Math. Soc. Abstract 62-2-275) and use homological dimension theory. (Received January 12, 1956.)

385*t*. Herbert Federer: *A theorem on principal fibre maps.*

Suppose $f: E \rightarrow B$ is a fibre map in the sense of Serre, F is an (arcwise connected) fibre of f, and $g: E \times F \rightarrow E$ is a continuous, fibre preserving, associative multiplication with a unit $1 \in F$. (If $x \in E$, $y \in F$, $z \in F$, then $f(g(x, y)) = f(x)$, $g(g(x, y), z) = g(x, g(y, z))$, $g(x, 1) = x$, $g(1, y) = y$.) The corresponding Pontrjagin product makes the cubical singular complex M of E (using cubes with vertices at 1) a differential graded right module over the Pontrjagin cubical chain ring A of F, in the sense of H. Cartan (École Normale Supérieure, Séminaire 1954/1955). Let I be the kernel of the usual augmentation of A and consider Cartan's associated quotient module $\overline{M} = M/MI$. It is proved that *the natural homomorphism induced by f maps the homology groups of $\overline{M}$ isomorphically onto the cubical singular homology groups of B*. (The proof uses a general theorem of J. C. Moore to compare two filtrations of M, namely the fibre space filtration of Serre and the differential graded module filtration of Cartan.) One consequence of this result is a simplification of the homology theory of spaces of type $K(\pi, n)$, because the geometric significance of the "bar construction" can now be established directly without use of the "construction W." (Received December 27, 1955.)

386. Jesús Gil de Lamadrid: *The bounded topologies in locally convex topological vector spaces.* Preliminary report.

Let E be a locally convex topological vector space and $\mathfrak{T}$ its topology. We denote by $\mathfrak{T}'$ the finest (not necessarily locally convex) topology which coincides with $\mathfrak{T}$ on bounded sets. We call $\mathfrak{T}'$ the bounded topology. The family of convex sets, open in $\mathfrak{T}'$, is a base of a locally convex topology $\mathfrak{T}''$, called the locally convex bounded topology. A mapping (linear mapping) of E into a topological space (into another locally convex space) is continuous in $\mathfrak{T}'$ ($\mathfrak{T}''$) if and only if it is continuous relative to—that is when restricted to—bounded sets. $\mathfrak{T}=\mathfrak{T}'$ ($\mathfrak{T}=\mathfrak{T}''$) if and only if mappings (linear map pings), continuous relative to bounded sets, are continuous. Examples of $\mathfrak{T}=\mathfrak{T}''$ are bornological spaces [Dieudonné, Bull. Amer. Math. Soc. vol. 59 (1953) pp. 495–512]. Among them are normed spaces, for which further $\mathfrak{T}=\mathfrak{T}'$. If $\mathfrak{T}=\mathfrak{T}''$ the space of continuous linear transformations of E into any locally convex space F is a closed (in the bounded-open topology) subspace of the space of all mappings of E into F. (Received November 28, 1955.)

387*t*. Jesús Gil de Lamadrid: *Spaces of mappings.* Preliminary report.

The terminology and notation of Abstract No. 387 is used. The purpose of this note is to establish certain topological results regarding subspaces of the space of all mappings of E into F. If this space of mappings is given the topology of uniform convergence over the bounded sets of E, the main results state that the following vector subspaces are closed: the space of bounded mappings (mapping bounded sets of E into bounded sets of F), the space of mappings that are continuous with respect to the bounded topology of E, and the space of compact mappings (the latter in case F is complete). For a closely related result see J. W. Brace, Bull. Amer. Math. Soc. vol. 60 (1954) p. 152. The proofs of these results are all similar. The case of compact mappings depends on the lemma: if A is a bounded subset of E and Φ is a maximal filter of A, then for every compact mapping f of E into F, $f(\Phi)$ is a convergent filter of F. From the above and other results it follows that a mapping is compact if and only if it is the uniform limit over bounded sets of bounded finite dimensional mappings (mapping bounded subsets of E into finite dimensional vector subspaces of F). (Received December 30, 1955.)

388. Friedrich Huckemann: *On the line-complex representation of Riemann surfaces.*

The representation of simply connected Riemann surfaces, which carry singularities only over a finite number of base points, by a line-complex, is well known [Speiser, R. Nevanlinna, Elfving] and widely used with success in problems within the theory of distributions of values. It seems desirable, therefore, to extend the notion of the line-complex as to cover a larger class of Riemann surfaces, in particular in order to deal with critical points of Riemann surfaces. Steps in this direction have been taken already, an axiomatic definition of a generalized notion of a line-complex, however, was still missing. A topological mapping now leads to the definition of a line-complex which permits to represent all those (simply connected) Riemann surfaces which have all their singularities over a closed Jordan curve without multiple points. A necessary and sufficient condition determines whether a line-complex does represent such a Riemann surface. A certain structure, called "boundary element," in a line-complex determines uniquely a critical point of the surface represented, while between different boundary elements a certain relation holds if and only if these elements determine the same critical point. (Received January 11, 1956.)

389. B. H. McCandless: *Test spaces for dimension n.*

If n is a non-negative integer, a separable metric space Y^n is called a *test space for dimension n* if it satisfies the statement: A separable metric space X has dimension $\leq n$ if, and only if, given a closed subset C of X and mapping $f: C \rightarrow Y^n$, there is an extension of f over X. The purpose of this paper is to characterize test spaces for dimension n. The following result is obtained: Let n be a non-negative integer. A compact, n-dimensional n-LC space Y^n is a test space for dimension n if and only if (i) $\pi_i(Y^n) = 0$, $i = 0, 1, \cdots, n-1$. (ii) Y^n is LC^{n-1}, (iii) $\pi_1(Y^1)$ is infinite cyclic if $n = 1$; if $n \geq 2$, there exists an integer $k_n > 0$ such that $\pi_n(Y^n)$ is the sum of k_n infinite cyclic groups. (Here $\pi_0(Y) = 0$ means that Y is arc-wise connected.) Results due to Kuratowski and Hurewicz are utilized in the proof. (Received January 9, 1956.)

390. Paul Olum: *The second level in obstruction theory.*

Given $f, g: X \rightarrow Y$ with $f|A = g|A$, this paper defines effectively computable invariants at the "second level" of depth bearing on the homotopy of f and g rel. A. Let n and $q > n$ be integers, Π and G be abelian groups; take elements $y \in H^n(Y; \Pi)$, $w \in H^{q+1}(\Pi, n; G)$ such that $w \vdash y = 0$, and $x \in H^{n-1}(X, A; \Pi)$; (if $n = 1$, with appropriate modifications, Π may be non-abelian and operate on G). Assume the vanishing of the "first level" invariants, namely (†): $(f-g)^* y = 0$ in $H^n(X, A; \Pi)$ and $(f-g)^*: H^q(Y; G) \rightarrow H^q(X, A; G)$ has image 0. Following the ideas of Steenrod's functional operations, a function $\lambda_{f,g}(y, w, x)$ is defined purely homologically in terms of certain mapping cylinders. Denote by $\Lambda_{f,g}(y, w)$ the image set of $\lambda_{f,g}$ for fixed y, w. Then $f \simeq g$ rel. A implies that, for all (y, w) as above, (†) is satisfied and $0 \in \Lambda_{f,g}(y, w)$. For the second obstruction (i.e. $\pi_r(Y) = 0$ for $r < n$ and $n < r < q$, $\pi_n(Y) = \Pi$, $\pi_q(Y) = G$), and (X, A) polyhedral of dimension q, the converse is true. Corresponding definitions and results hold for the extension problem. For $Y = n$-sphere, $X = (n+1)$-complex, the results reduce to those found earlier by Massey (unpublished). (Received January 12, 1956.)

391*t*. Paul Olum: *Classification of mappings of compact 2-manifolds.*

Let $f: M \rightarrow N$ map one compact connected 2-manifold into another, inducing $\theta_f: \pi_1(M) \rightarrow \pi_1(N)$ and having twisted degree $\deg f$ (cf. *Mappings of manifolds and the notion of degree*, Ann. of Math. vol. 58 (1953) pp. 458–480). Let P be the projective plane. Then θ_f and $\deg f$ are known to be a sufficient set of invariants for determining the homotopy class of f provided either that $N \neq P$ or that $N = P$ and θ_f is orientation-true (loc. cit. p. 464). The present paper solves this problem in the remaining case. Let $f, g: M \rightarrow P$ be order-preserving simplicial maps with θ_f, θ_g not orientation-true. Suppose now (1) $\theta_f = \theta_g$ and (2) $\deg f = \deg g$ (mod 2). Using integers mod 2 as coefficients throughout, let z^1 be a nontrivial 1-cocycle in P; there are then two 0-cochains c^0 and $c^{0\prime}$ such that $f^* z^1 - g^* z^1 = \delta c^0 = \delta c^{0\prime}$. Define a 2-cocycle z^2 in M by $z^2 = f^* z^1 \cdot f^* z^1 \cdot c^0 + f^* z^1 \cdot c^0 \cdot g^* z^1 + c^0 \cdot g^* z^1 \cdot g^* z^1$; similarly for $z^{2\prime}$ using $c^{0\prime}$. Define $\mu = \sum z^2(\sigma_2)$, summed (mod 2) over all 2-simplexes $\sigma_2 \in M$; similarly for μ'. Theorem: $f \simeq g$ if and only if (1) and (2) hold and $0\in\{\mu, \mu'\}$. The proof is an application of the results announced in the preceding abstract. (Received January 12, 1956.)

392. J. P. Roth: *A function-space formulation of the switching-system problem.*

The actual physical problem facing the designer of switching systems is considerably deeper than its logical formulation (cf. abstract 367); it is given here the following topological formulation. A truth function ϕ defines a cubical complex K contained in an n-cube Q, also determined by ϕ. A *pair* (X, A) consists of a linear graph X and an *assignment* A which attaches to each branch (1-cell) of X an $(n-1)$-cube of Q. Let p, q be distinct 0-cells of X and α an (acyclic) geometrical 1-chain linking p and q; A defines a corresponding algebraic 1-chain α^*, the coefficient of a cell σ of 0^* being the $(n-1)$-cube assigned to σ by A. The *Kronecker index* of α^* is the set-theoretic product of its coefficients. Let $\Omega(X, p, q; A)$ be the space of all such α^*; the pair (X, A) shall be *admissible* if the union of the Kronecker indices of the elements of Ω is precisely the complex K. The *switching-system problem* is now to find an admissible pair such that the number of branches of X is a minimum. The multiple-terminal problem —a logical formulation of which is unknown to the writer—admits of a similar though more complex interpretation. The topological problem has in general a lower minimum than the logical problem. Routine algebraic methods are given for recognizing whether the bridge circuit is admissible. (Received January 10, 1956.)

393. Sol Schwartzman: *Asymptotic cycles.* I.

Let X be a locally connected compact metric space whose first Betti number is finite. Suppose the real line acts as a topological transformation group on X. Recall that $x_0\in X$ is called quasi-regular if $\lim_{t\to\infty}(1/T)\int_0^T f(x_0t)\,dt$ exists for every continuous $f(x)$. It is well known that the set of all points that are not quasi-regular has measure zero with respect to every finite invariant measure. Take any fixed quasi-regular point x_0. For each $t>0$, let K_t be an arc going from x_0t to x_0. We suppose the K_t chosen so they can be parametrized equi-uniformly continuously. The orbit from x_0 to x_0t followed by the arc K_t is a closed curve. Let C_t be the corresponding element of the first Betti group. Theorem: $\lim_{t\to\infty} C_t/t$ exists and is independent of the choice of the arcs K_T. Thus we have assigned to each quasi-regular x_0 an element of $B^1(X)$. If μ is any finite invariant measure we call the (μ)-average of this vector valued function the (μ)-asymptotic cycle C_μ. General formulas are obtained for computing C_μ. Applications to theorems about continuous eigenfunctions, cross sections in the large and related problems will be given at a later date. (Received January 12, 1956.)

394t. Sol Schwartzman: *Asymptotic cycles.* II.

Let X be a compact differentiable manifold of class C^3 and dimension $2M$. Assume given on X a closed nonsingular differentiable 2-form ω. An admissible coordinate system will mean a local C^3 coordinatization from a $2n$-dimensional sphere with coordinates p_i, q_i such that $\omega=\sum dp_i dq_i$ in this coordinate system. Finally let α be a differentiable closed one-form. In any admissible coordinate system, choose a function H such that $dH=\alpha$. Set up the Hamiltonian equations $dq_i/dt=\partial H/\partial p_i$, $dp_i/dt=-\partial H/\partial q_i$. Obviously these equations depend only on α and not on the choice of H. The restriction to admissible coordinates guarantees that a change of coordinate systems gives a compatible set of equations; so we obtain a one-parameter flow on X. By Liouville's theorem the exterior product of ω with itself n times gives an invariant measure μ. Theorem: The μ-asymptotic cycle for the flow defined by the one-form α

depends only on the cohomology class to which α belongs. The map which sends each cohomology class into the corresponding μ-asymptotic cycle is a homomorphism. In particular, if we insist on a single-valued Hamiltonian function, so that $\alpha = dH$, the μ asymptotic cycle is zero. (Received January 12, 1956.)

395. Stephen Smale: *A Vietoris mapping theorem for homotopy.*

Let X and Y be compact metric spaces and $f:X\to Y$ be onto. Vietoris proved (Math. Ann. (1927)) that if for all $r\leq n-1$ and all $y\in Y$, $H_r(f^{-1}(y))=0$ (Vietoris homology mod 2) then $f_*:H_r(X)\to H_r(Y)$ is an isomorphism onto for $r\leq n-1$ and onto for $r=n$. Begle (Ann. of Math. (1950)) gave generalizations to nonmetric spaces and more general coefficient groups. Examples show that this theorem does not hold directly for homotopy. However, suppose X and Y are LC^n and that for each $y\in Y$, (1) $f^{-1}(y)$ is LC^{n-1} and (2) $\pi_r(f^{-1}(y))=0$ for $0\leq r\leq n-1$; then it is proved that $f_\#:\pi_k(X)\to\pi_k(Y)$ is an isomorphism onto for $k\leq n-1$ and onto for $k=n$. Actually the hypotheses can be weakened in some respects. For example, if instead of assuming $\pi_{n-1}(f^{-1}(y))=0$, one only has that $i_\#:\pi_{n-1}(f^{-1}(y))\to\pi_{n-1}(X)$, the homomorphism induced by inclusion, is zero, then still $f_\#:\pi_{n-1}(X)\to\pi_{n-1}(Y)$ is an isomorphism onto. (Received December 14, 1955.)

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RESEARCH PROBLEM

9. Richard Bellman. *Minimization problem.*

We are given a region R and a random point P within the region. Determine the paths which

(a) Minimize the expected time to reach the boundary, or

(b) Minimize the maximum time required to reach the boundary.

Consider, in particular, the cases

(a) R is the region between two parallel lines at a known distance d apart.

(b) R is the semi-infinite plane and we are given the distance d from the point P to the bounding line. (Received November 18, 1955.)

BOOK REVIEWS

Begründung der Funktionentheorie auf alten und neuen Wegen. By L. Heffter. Berlin, Springer, 1955. 8+63 pp. 12.60 DM.

The "foundations of function-theory" are understood here to consist of the study of the minimal possible set of assumptions under which a complex function $f(z)$ can be expanded into a power series. Generally such assumptions fall into three broad categories: the existence and continuity of the derivative (Cauchy), the unique existence of the derivative (Goursat), and the unique integrability of $f(z)$ (Morera-Osgood). In the present monograph, the author pictures the foundations of the theory as consisting of six distinct categories with the equivalence of the Cauchy and Weierstrass theories forming one such category, the Cauchy-Goursat theory a second, and the Morera-Osgood theory a third, which is asserted to be "*. . . nur aus historischem Interesse.*" Into a fourth category falls the Looman-Menchoff Theorem and the theory associated with it. The fifth and sixth categories were devised by the author himself in 1936 and 1951, respectively, and each, in its way, can be considered as a variation or extension of the idea of unique integrability of $f(z)$. More specifically, the fifth category consists of showing that if, in a domain G, the real and imaginary parts of $f(z)$ satisfy Cauchy-Riemann difference equations [in terms of mean values of integrals] over every rectangle in G with sides parallel to the axes, then $f(z)$ is regular in G, while the sixth consists of showing that every integral of $f(z)/(z-\zeta)$ vanishes over a certain class of closed paths in G [the point ζ must lie outside the closed path].

Except for the Looman-Menchoff Theorem, the book is completely self-contained, beginning with the notion of the convergence of a sequence and of a Dedekind cut, and proceeding, with all necessary proofs, to Gauss' theorem and the usual properties of functions of a complex variable needed at this stage. All line integrals which are considered are taken either over polygonal paths with segments parallel to one of the axes, or over arcs of circles, a fact which does not compromise generality, but which, in this instance, permits a clearer presentation of the underlying principles. The only theorem which is not proved is the full Looman-Menchoff Theorem which is stated on page 38 together with something of its background; however, a diluted form of the theorem is stated and proved.

The introductory material and the development of the six categories comprise two chapters; in a third (and last) chapter there is a selected list of original papers in chronological order together with the author's comments on these papers. For example, it is of interest to note that the first entry in the list is a letter from Gauss to Bessel, dated 18 December 1811, in which there is communicated the substance of Cauchy's theorem and certain consequences of it. [According to the second entry in the list, this was some three years before Cauchy announced his result and fourteen years before he published it.] When one realizes that the author's first papers in this field appeared almost simultaneously with various papers of Goursat and Morera on the subject, one appreciates the author's connection with the development of the field at the turn of the century.

A. J. LOHWATER

Altes und Neues über konvexer Körper. By H. Hadwiger. (Elemente der Mathematik von höheren Standpunkte aus. Band III.). Basel, Birkhäuser, 1955. 115 pp. 13.50 Swiss fr.

This small volume, containing less than one hundred pages of actual text, gives an elegant and concise account of convex bodies from the standpoint of the geometry of sets. The general approach is to consider the collection of all convex bodies in ordinary space as themselves forming a metric space $\mathfrak{C}$ with convex polyhedra as a dense subset. $\mathfrak{C}$ also has algebraic structure, namely, addition (the Minkowski sum of convex sets) and multiplication by positive scalars (dilation). On this space the volume V, surface area A, and integral mean curvature M are functionals defined in the first instance for convex polyhedra and then extended to $\mathfrak{C}$. Thus it is unnecessary to make any assumption beyond convexity itself on the class of bodies considered. In this context many questions concerning, for example, the differential geometry of convex surfaces become unnatural; but, on the other hand, the study of the functionals V, F, M and their properties, Steiner's symmetrization, and so on, are briefly and elegantly treated. Thus the author easily proves a theorem of Gross and Lusternik to the effect that by repeated symmetrization it is possible to gradually transform any convex body into a sphere; he proves Steiner's formula $V(A_\rho) = V(A)+\rho F(A)+\rho^2 M(A)+4\pi\rho^3/3$ for the volume of a convex body A_ρ parallel to A at distance ρ; and he demonstrates the classical Brunn-Minkowski inequalities: $F^2-3MV \geqq 0$ and $M^2-4\pi F \geqq 0$. In this latter connection an interesting discussion is given of the unsolved problem of determining what further inequalities three real numbers V, F, M must satisfy in order to be the values of $V(A)$, $F(A)$, $M(A)$ for some convex body A. The final twenty pages deal in a general fashion with the integral geometry of convex bodies. The book ends with a detailed thirteen page bibliography. Some original results of the author are included, in particular various contributions to the unsolved problem above and a neat characterization of the class of functionals generated by V, F, and M in terms of functional properties alone. In general, of course, the book is expository, developing in particular Blaschke's idea of introducing a metric into $\mathfrak{C}$. However, the thoroughness with which this idea is exploited is due to the author, who throughout has kept a nice balance between his abstract approach and the concrete results achieved.

WILLIAM M. BOOTHBY

Méthodes d'algèbre abstraite en géométrie algébrique. By P. Samuel. Berlin, Springer, 1955. 9+133 pp. 23.60 DM.

The goal of the present work is, according to the author, "to give as complete an exposition of the foundations of abstract algebraic geometry as is possible," and to be useful to the practitioner ("l'usager" as Samuel calls him). Actually the main use of this book will be found as a handbook for one who wishes a less abrupt and difficult introduction to the abstract methods of algebraic geometry than is afforded by Weil's *Foundations* (which, it is too often forgotten, was not meant as an introduction). This latter book begins with three arduous chapters on pure algebra, whose use does not become apparent until much further in the book. Such a barrier does not exist in Samuel's exposition, because he assumes known all the needed basic algebra, or rather refers as he goes along to an appendix containing purely algebraic basic results, or references to those in the literature. (This of course could not have been done by Weil, for the good reason that most of these results were not in the literature at the time.) Samuel proceeds immediately with the geometric language and hence the reader's first contact with abstract methods is reasonably soft.

The book is divided into two parts. The first one gives the general theory of algebraic varieties, defined as either affine or projective varieties. It begins with the notion of algebraic set (set of zeros of polynomial ideals), union and intersection of these, and continues with the notion of dimension, generic points, products, projections, correspondences, rational and birational maps. The deepest theorem in this part asserts that every component in the intersection of two varieties V and W of dimension r and s in projective n space has dimension $\geqq r+s-n$. Elimination Theory is discussed as a special case of the theory of projections (as it should be) and is derived elegantly from the basic and elementary theorem on the extension of specializations. So is the Hilbert Nullstellensatz.

There is a section on properties which hold almost everywhere (i.e. at least on the complement of some proper algebraic subset of a given variety), followed by a section on Chow coordinates. The notion of normality is also introduced and two useful results are proved: a point on a variety V/k which is k-normal remains normal over a separably generated extension of k, and the product of two normal points is normal on the product variety.

The second part deals mostly with local problems. It begins with a further discussion of normal points (see further comments below concerning ZMT) and with the theory of simple points. It is mostly devoted to an account of the intersection theory. The proofs are sketched (but can be read without difficulty) and follow more or less closely those given by the author in his thesis. They belong to what Severi calls the static method (local rings and multiplicity of a primary ideal being the fundamental notions around which all others revolve).

The enumeration which precedes will suffice to show that the book contains a wealth of basic material, and as we have said earlier, will be of great use to one desiring to learn this material. (It will also suffice to show that the expert will find little in the book which he does not know already.)

Granting the author's point of view, which was to write a complete foundational book on the abstract methods of algebraic geometry, the reviewer believes nevertheless that the book has the serious defect of not being complete. We shall illustrate this by some specific examples.

a. *Abstract varieties.* No mention is made of abstract varieties. Of course no one wishes to ban projective varieties, and we are not promoting a competition between abstract and projective varieties. One should note however that notions like products, intersections, projections, are all intrinsic and do not require an embedding in projective space. (No differential geometer would define the product of two differentiable manifolds by first requiring that they be imbedded in Euclidean space.) More seriously, certain problems are best treated, and others can only be treated by dealing directly with abstract varieties. (For instance, algebraic groups, homogeneous spaces, fibre spaces. It is of course useful, and highly desirable, to have an embedding theorem, because we gain thereby an additional tool for studying them. In other topics (we are thinking particularly of Serre's theory of coherent sheaves) the main theorems are all concerned with projective varieties, but the theory of abstract varieties is a technical necessity, partly because the embedding changes so often, that one needs some underlying structure which does not change.

b. *Properties holding almost everywhere.* The Zariski topology should have been defined in the section dealing with properties holding almost everywhere. The closed sets on a variety are its algebraic subsets, and a set is k-closed if it can be defined by equations with coefficients in k. Furthermore, one should note that in practice it is usually not sufficient to know that a property holds almost everywhere (on a set containing a nonempty open set) but one must know that the property holds exactly on an open set. Many of Samuel's properties which are stated to hold almost everywhere in fact hold on an open set. For precise results in this direction, see for instance Weil's *Critères d'equivalence*, Math. Ann. (1954) referred to as *Crequ.* and also the Appendix to his *Groups of transformations*, Amer. J. Math. (1955) referred to as GT.

c. *Chow coordinates.* The treatment here given follows closely the original one given by Chow and van der Waerden. One notable addition is Chow's theorem that for a divisor on a variety, the smallest field of rationality coincides with the field generated by the Chow coordinates. Although this conclusion does not remain valid for an arbitrary cycle (in characteristic p), it should have been remarked that it does if the coefficients of the components of the cycle are not divisible by p. See for instance GT, Appendix. Other useful properties not included here are given in Crequ. It would be highly desirable to have once for all a *complete* list of the properties of these coordinates, and of the theorems which relate them with the theory of intersection and specialization of cycles. The results given here are only fragmentary.

d. *Normalization and ZMT.* We come here to one of the main omissions, that of Zariski's Main Theorem on birational correspondences (referred to as ZMT). Let T: $V \rightarrow W$ be a rational map having the property that the total image of a point P normal on V has one component consisting of a single point Q. Then this total image is exactly Q, and T is well defined at P. Only a trivial special case of ZMT is stated in II, §§3, 4. This is all the worse because a complete proof of it could have been given easily with the techniques used in this second part of the book. The theorem concerning the analytical irreducibility of a variety at a normal point is proved in the adjoining section, and one of its most interesting applications is precisely ZMT itself.

Apropos of normalization, we note in passing that the statement on page 125 that the Italian geometers had the notion is quite misleading. They took as definition the property that the system of hyperplane sections is complete. This is a global property, depending on the embedding, whereas Zariski's notion of a normal point is local, depending on the local ring of the point.

e. *Intersection theory.* A strong warning should have been given here (but was not) that the definition of cycles does not coincide with that of Weil. Weil defines an s-cycle on a given variety V to be an element of the free abelian group generated by the simple subvarieties of V, of dimension s. For Samuel, a projective space P_n is fixed, and his s-cycles are cycles on P_n in the Weil sense. A cycle on a variety V in Samuel's terminology simply means that the point set of the cycle is carried by V. This difference in the basic definitions entails a technical difference in the results of the intersection theory. Samuel's is admittedly more general, and at least once has been used in a more advanced investigation (Néron's thesis), but its use is seriously impaired by the incompleteness of his list of theorems. This incompleteness is most striking in those parts dealing with divisors of functions. Very few of the results stated by Weil in F-VIII are included in Samuel's book. Some of them which look formally alike actually differ, as for instance the theorem relating the divisor of a function f on a variety V and the divisor of its induced function on a subvariety W. This is due to the fact that Samuel's definitions allow components of the divisor of f to be singular on V. One consequence of this is that Theorem 10 of F-VIII$_3$ does not remain valid with Samuel's definition (to cite only one example). (It is the theorem stating essentially that when a divisor X is rational over k, then the space of functions whose divisors are $>X$ has a basis defined over k.) Other examples of useful theorems not included by Samuel are those dealing with the relation between the divisors of a function and its norm over a subfield (F-VIII$_3$ Th. 7) and functions on product varieties (F-VIII$_2$ Th. 1 and its corollaries).

The principal avowed purpose of *Foundations* was to furnish the working algebraic geometer with a *complete* and secure encyclopedia of basic "obvious" results, and to spare him the task of having to reconsider constantly foundational lemmas in the course of a more advanced investigation. The above comments show clearly that Samuel's book cannot be used in a similar capacity. In the preceding list of topics which might have been more completely treated, we have even limited ourselves to those which fit in directly with Samuel's own choice of general material for his book, and whose inclusion would not have materially increased the length of the book, but would have greatly increased its value.

On the other hand, we can think of a number of other topics which might also be characterized as belonging to the foundations of abstract algebraic geometry, and might have been included. For instance, linear systems and Bertini's theorems, and differential forms. These would have found a natural place in the book, at the cost of a quite small increase in size, and their inclusion would have been doubly useful since at present the theory is somewhat scattered in the literature.

The need for a book (essentially a sequel to *Foundations*) which will include at least all the above mentioned topics is now seriously felt. The material for such a book is accumulating in the literature, and we would like to conclude by expressing the hope that someone will have the devotion to collect it together.

SERGE LANG

BRIEF MENTION

Theory of groups. Vol. I. By A. G. Kurosh. Trans. from the 2d Russian edition (1952) of Kuros' *Teoriya Grupp* and edited by K. A. Hirsch. New York, Chelsea, 1955. 272 pp. $4.95.

This is the first volume of a projected two volume work. It includes Part I: The elements of group theory, and Part II: Abelian groups. For a review of the German translation of the first edition of this work, cf. this Bulletin, vol. 61, p. 362.

Théorie des opérations linéaires. By S. Banach. New York, Chelsea, 1955. 7+254 pp. $3.95.

Reprinted from Monografje Matematyczne, Tom I, 1932.

Famous problems. By F. Klein et al. New York, Chelsea, 1955. 11+321 pp. $3.25.

This consists of reprints of the following monographs: Famous problems of elementary geometry, by F. Klein. From determinant to tensor, by W. F. Sheppard. Introductory combinatory analysis, by P. A. Macmahon. Three lectures on Fermat's last theorem, by L. J. Mordell.

Opere. By U. Dini. Vol. 3. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Rome, Cremonese, 1955. 6+660 pp. 6000 lire.

For vols. 1 and 2, cf. this Bulletin vol. 60, p. 288 and vol. 61, p. 260. This third, and final, volume contains papers on ordinary and partial differential equations.

NOTES

The following individuals were among those who received doctorates during 1955 in the mathematical sciences and related subjects from universities in the United States and Canada. The university, month in which the degree was conferred, minor subjects (other than mathematics), and title of the dissertation are given in each case if available.

S. S. Abhyankar, Harvard, March, *Local uniformization on algebraic surfaces over modular ground fields.*

J. W. Addison, Jr., Wisconsin, January, *On some points of the theory of recursive functions.*

M. W. Al-Dhahir, Michigan, June, *Configurational characterizations of commutativity in projective spaces.*

A. R. Amir-Moez, California, Los Angeles, June, *Extreme properties of eigenvalues of a Hermitian transformation and singular values of the sum and product of linear transformations.*

R. J. Aumann, Massachusetts Institute of Technology, June, minor in humanities, *Asphericity of alternating linkages.*

H. R. Bailey, Purdue, August, minor in electrical engineering, *Problems of existence and stability of solutions of differential systems.*

W. L. Baily, Jr., Princeton, June, *On the quotient of a complex analytic manifold by a discontinuous group of complex analytic self-homeomorphisms.*

J. Y. Barry, Yale, June, *Generation of abstract Markov processes.*

R. W. Bass, Johns Hopkins, June, minor in physics, *On the singularities of certain non-linear systems of differential equations.*

E. H. Batho, Wisconsin, January, *On a class of rings with ideal nuclei.*

W. A. Beck, Purdue, June, *On commutators and generalizations of commutators in a Hilbert space.*

Barbara J. Beechler, Iowa, June, *Complete ideals in polynomial rings.*

Stoughton Bell, II, California, Berkeley, June, *Aspects of the boundary-layer problem near the leading edge of a flat plate.*

R. H. Bender, Boston, June, *An investigation of the general second order linear partial differential equation of elliptic type.*

J. P. Benzecri, Princeton, June, *Variété localement plate.*

Sterling Berberian, Chicago, December, *The ring of closed operators affiliated with a finite AW^*-algebra.*

D. D. Betts, McGill, October, *Theoretical investigation of resonance electron-capture cross sections.*

P. P. Billingsley, Princeton, June, *The invariance principle for dependent random variables.*

E. A. Bishop, Chicago, June, *Spectral theory for operators on a Banach space.*

C. H. Boll, Stanford, April, *Comparison of experiments in the infinite case and the use of invariance in establishing sufficiency.*

W. E. Boyce, Carnegie Institute of Technology, June, *Vibrations of rotating beams.*

D. L. Boyer, Kansas, October, minor in physics, *Enumeration theorems in infinite abelian groups.*

Helen Bozivich, Iowa State, June, minor in animal breeding, *Power of test procedures for certain incompletely specified random and mixed models.*

L. R. Bragg, Wisconsin, August, *The solutions of a linear ordinary differential equation of the third order about a simple turning point of the second order.*

T. R. Brahana, Michigan, June, *The local Betti numbers in topological product spaces.*

C. C. Braunschweiger, Wisconsin, August, *The C-cone and geometric constructions of (L) and (M) spaces.*

R. L. Brock, Oregon State, minor in physics, *On the mathematical theory of detection by electronic instruments.*

Julian Brody, Princeton, January, *The Fox invariant.*

S. H. Brooks, Johns Hopkins, *Comparison of methods for estimating the optimal factor combination.*

E. C. Bryant, Iowa State, June, minor in economics, *An analysis of some two-way stratifications.*

D. L. Burkholder, North Carolina, August, *On a certain class of stochastic approximation processes.*

A. T. Butson, Michigan State, June, *Systems of linear congruences and left-associativity of matrices, whose elements are integers from an algebra.*

L. L. Campbell, Toronto, May, minor in quantum mechanics, *Mixed boundary value problems for hyperbolic differential equations.*

Thelma M. Chaney, University of Washington, March, *Invariant means and measures on semigroups.*

C. C. Chang, California, Berkeley, June, *Cardinal and ordinal factorization of relation types.*

Clive Chester, New York, June, *Uniform asymptotic expansions.*

T. S. Chihara, Purdue, June, *Generalized Hermite polynomials.*

Solomon Ciolkowski, New York, June, *Swept-back wing.*

E. J. Cogan, Pennsylvania State, January, minor in philosophy, *A formalization of the theory of sets from the point of view of combinatory logic.*

C. S. Coleman, Princeton, January, *Homogeneous and quadratic differential equations.*

Raymond Collier, Minnesota, *Experimental designs in which the observations are assumed to be correlated.*

G. E. Collins, Cornell, June, *The modelling of Zermelo set theories in new foundations.*

P. E. Conner, Jr., Princeton, June, *The Green's and Neumann's problems for differential forms on Riemannian manifolds.*

Erben Cook, Jr., Boston, June, *The fundamental solution of a parabolic differential equation of arbitrary even order.*

J. W. Coy, Michigan, February, *A differential calculus for functions of matrices.*

T. H. M. Crampton, Indiana, June, minor in chemistry, *Explicit formulas for norm groups in generalized local class field theory.*

E. H. Crisler, Michigan, June, *On partition systems, chordal systems and their application to regular curve-families filling the plane.*

Lonnie Cross, Cornell, September, *On the distribution of eigenvalues of* $\int_{-\infty}^{\infty} a(s-t)\phi(t)dt = \rho\left[\int_{-\infty}^{\infty} b(s-t)\phi(t)dt + \eta\phi(s)\right]$.

R. H. Crowell, Princeton, June, *Invariants of alternating link types.*

C. H. Cunkle, Missouri, August, *Equicontinuity in dynamical systems.*

P. C. Curtis, Jr., Yale, June, *Representations of* AW^* *algebras.*

R. H. Davis, California, Berkeley, June, *Singular normal differential operators.*

Ruth M. Davis, Maryland, June, *On a regular Cauchy problem for the Euler-Poisson-Darboux equation.*

F. W. Donaldson, Texas, August, minor in physics, *Integration formulae and boundary conditions for the elliptic parabolic partial differential equation with three independent variables.*

W. S. Dorn, Carnegie Institute of Technology, June, *On plastic collapse of structures and linear programming.*

L. E. Dubins, Chicago, September, *Generalized random variables.*

R. D. Edwards, Pittsburgh, February, *Almost-continuous mappings of Hausdorff spaces.*

Kathryn P. Ellis, Iowa, August, *On the structure of topological rings.*

Lillian Elveback, Minnesota, *Some aspects of estimation problems in follow-up studies in chronic disease.*

Walter Feit, Michigan, June, *Topics in the theory of group characters.*

Leopold Flatto, Massachusetts Institute of Technology, June, minor in languages, *Periodic solutions associated with singular perturbations of a system of non-linear differential equations.*

H. K. L. W. Flesch, Illinois, October, minor in physics, *On the classification of finite elementary nilpotent groups of class 2.*

N. P. Fofonoff, Brown, June, minor in oceanography, *A theoretical study of zonally uniform oceanic flow.*

J. E. Folkert, Michigan State, August, *The distribution of the number of components of a random mapping function.*

W. C. Fox, Michigan, June, *The critical points of real functions defined on 2-manifolds.*

T. T. Frankel, California, Berkeley, September, *Harmonic integrals.*

R. J. Fruend, North Carolina State, *Introduction of risk into a linear programming model.*

B. A. Galler, Chicago, September, *Some results in algebraic logic.*

Seymour Geisser, North Carolina, June, *On the exact distributions of certain statistics related to the mean square successive difference.*

L. S. Gephart, Florida, January, minor in education, *A study of the effect of non-normality on sampling distributions of the range.*

E. F. Gillette, Syracuse, May, *On the four-vertex theorem and some of its generalisations.*

Malcolm Goldman, Chicago, June, *The structure of AW^*-algebras.*

A. J. Goldstein, Massachusetts Institute of Technology, June, minor in physics, *On some operations in homotopy groups.*

A. V. Granato, Brown, June, *Internal friction and modulus changes due to dislocation damping in undeformed crystals.*

R. T. Gregory, Illinois, June, minor in astronomy, *A numerical solution to a class of Sturm-Liouville systems.*

J. S. Griffin, Jr., Tulane, August, *A notion of parallelism in general fibre bundles.*

R. C. Gunning, Princeton, June, *A classification of factors of automorphy.*

Irwin Guttman, Toronto, May, minor in genetics, *Characterization of tolerance regions.*

C. J. A. Halberg, Jr., California, Los Angeles, May, *Spectral theory of linked operators in the spaces l_p.*

Dilla Hall, St. Louis, June, *A relationship of Reiss with extensions to multiple points, tangential points, inflectional points and simple cusps.*

W. J. Hall, North Carolina, June, *Most economical multiple-decision rules.*

J. W. Hamblen, Purdue, June, *Distributions of roots of algebraic equations with variable coefficients.*

N. T. Hamilton, Chicago, June, *Foundations of the theory of complex analytic varieties.*

J. W. Hardy, Jr., Stanford, June, *On axially-symmetric subsonic flow of a compressible fluid.*

Juris Hartmanis, California Institute of Technology, June, minor in physics, *Some embedding theorems for lattices.*

W. C. Healy, Jr., Illinois, June, minor in mechanical engineering, *Optimum invariant estimation.*

R. E. Heaton, Indiana, June, minor in physics, *Polynomial cocycles.*

Robert Hermann, Princeton, October, *On the differential geometry of homogeneous spaces.*

Smith Higgins, Jr., Notre Dame, August, *On minima of indefinite ternary quadratic forms.*

T. P. Higgins, Texas, June, minors in electrical engineering and physics, *On the Hankel transform.*

J. H. Hodges, Duke, June, minor in physics, *Number of solutions of some matric equations in a finite field.*

J. F. Hofmann, Iowa State, June, minor in economics, *Life testing in controlled environmental conditions.*

L. I. Holder, Purdue, August, *On generalizations of absolute and strong Cesàro summability.*

B. J. Hollingsworth, Kansas, June, minor in physics, *A study of conjugate nets on n-ply systems of hypersurfaces in n-dimensional projective space.*

C. V. Holmes, Kansas, October, minor in astronomy, *Automorphisms of monomial groups.*

J. L. Howland, Harvard, June, *Induced potentials.*

H. T. Hsü, Yale, June, *Semi-groups and the problems of Cauchy for linear partial differential equations.*

D. R. Hughes, Wisconsin, January, *Planar division neo-rings.*

J. A. Hummel, Rice, June, *Doubly orthogonal functions on Riemann surfaces.*

J. P. Jans, Michigan, June, *On the indecomposable representations of algebras.*

George Jeromson, California, Berkeley, September, *Propagation of electromagnetic waves in ferrites.*

J. W. Jewett, Michigan, February, *Differentiable approximations to light interior transformations.*

Guy Johnson, Jr., Rice, June, I. *Collective singularities of families of* *analytic functions.* II. *Regions of flatness for analytic functions and their derivatives.*

J. B. Johnston, California Institute of Technology, June, minor in physics, *Universal infinite partial orders.*

J. A. Kalman, Harvard, June, *Some inequalities related to Hölder inequality and some contributions to lattice theory.*

D. A. Kearns, Boston, June, *Initial value problem for a parabolic equation of arbitrary even order.*

R. W. Kennard, Carnegie Institute of Technology, June, *On the determination of significance levels of control charts for continuous variates.*

Jean Kestens, Brown, June, *The normal self-adjoint eigenvalue problem and upper and lower bounds for the eigenvalues with the iteration method.*

John Killeen, California, Berkeley, September, *Asymptotic perturbation of differential equations.*

E. G. Kimme, Minnesota, *On the convergence of sequences of stochastic processes with independent increments.*

R. F. King, Illinois, June, minor in electrical engineering, *Periodic and reducible nonlinear differential systems.*

Manfred Kochen, Columbia, December, *Organized systems with discrete information transfer.*

V. A. Kramer, California, Berkeley, January, *Investigations in asymptotic perturbation series.*

Lester Kraus, New York, June, *Diffraction by a plane angular sector.*

I. S. Krule, Tulane, August, *Structs on compact spaces.*

W. H. Kruskal, Columbia, *On the problem of non-normality in relation to hypothesis testing.*

R. G. Kuller, Michigan, February, *Locally A-convex topological vector lattices and their representations.*

J. J. J. Lavallee, McGill, October, *Asymptotic theorems in the theory of normal correlation.*

Anneli Lax, New York, June, *Cauchy's problem for a partial differential equation with real multiple characteristics.*

Eugene Levin, California, Los Angeles, June, *Reinforced openings in plane structural members.*

Frank Levin, Cincinnati, June, *On the algebraic theory of linear multi-differential polynomials.*

Gene Levy, Oklahoma, June, *Structure theory for generalized p-rings.*

S. D. Liao, Chicago, March, *On the theory of obstructions of fiber bundles.*

A. H. Lightstone, Toronto, November, minors in methodology and philosophy of science, *Contribution to the theory of quantification.*

C. S. Lin, Ohio State, December, *On sums of sets of integers.*

C. T. Long, Oregon, June, *Minimal conditions for normality of numbers.*

F. W. Lott, Jr., Michigan, February, *The use of a certain linear order statistic, related to the mean difference, as an unbiased estimate of the standard deviation in finite and infinite populations.*

R. A. Lytle, Georgia, June, *Properties of the overlap topology.*

P. J. McCarthy, Notre Dame, June, *On genera of class number one.*

F. S. McFeely, Virginia Polytechnic Institute, *Decision procedures for the comparison of exponential and geometric populations.*

H. P. McKean, Jr., Princeton, April, *Sample functions of stable processes.*

E. K. McLachlan, Kansas, October, minor in astronomy, *Extremal elements of certain convex cones of functions.*

R. M. McLeod, Rice, June, *On the zeros of the derivatives of some entire functions.*

T. J. McMinn, California, Berkeley, January, *Topics in measure theory*: I. *Restricted measurability.* II. *Measure splitting and average measurability.*

Vincent McRae, Catholic, June, minors in physics and education, *On the unitary similarity of matrices.*

R. A. Macauley, Illinois, June, minor in logic, *Analytic group kernels and Lie algebra kernels.*

J. G. MacCarthy, Pittsburgh, August, *The homographic aspect of the Euler-Knopp transformation.*

Mark Mahowald, Minnesota, July, minor in physics, *Measure in groups.*

W. E. Malbon, Virginia, June, *Invariants for quasi-compact mappings.*

M. A. Martino, Illinois, June, minor in aeronautics, *Concerning a measure of non-analyticity for functions of a complex variable.*

H. F. Mattson, Jr., Massachusetts Institute of Technology, June, minor in humanities, *Generalization of the Riemann-Roch theorem.*

C. N. Maxwell, Illinois, October, minor in physics, *Fixed point and coincidence theorems for multi-valued functions.*

J. P. Mayberry, Princeton, October, *Abelian branched covering of knots.*

Elliott Mendelson, Cornell, June, *The independence of the axiom of choice.*

W. E. Mientka, Colorado, June, minor in philosophy, *A method of* *approach towards the solution of the problem concerning the existence of at least one prime between consecutive perfect squares.*

V. J. Mizel, Massachusetts Institute of Technology, September, minor in physics, *Boundary layer problem for an elliptic equation in the neighborhood of a singular point.*

J. T. Mohat, Texas, June, *Concerning spirals in the plane.*

T. O. Moore, Missouri, August, *Metric foundations of elliptic 2-space.*

Dietrich Morgenstern, Indiana, June, minor in physics, *Analytical studies related to the Maxwell-Boltzmann equation.*

W. W. Murray, Brown, June, *Interaction of a spherical acoustic wave with a cylindrical beam.*

J. E. Nelson, Chicago, September, *On the operator theory of Markoff processes.*

W. L. Nicholson, Illinois, June, minor in psychology, *Uniform admissibility and its application to classification problems.*

J. A. Novarro, Purdue, August, *Studies in statistical ecology.*

R. J. Nunke, Chicago, September, *Modules of extensions over Dedekind rings.*

F. C. Ogg, Jr., Johns Hopkins, June, minor in physics, *The domain of existence of the isoperiodic families of the first kind in the restricted problem of three bodies.*

Katsumi Okashimo, Toronto, November, minor in quantum mechanics, *Numerical integration of integro-differential equations of convolution type.*

H. A. Osborn, Stanford, January, *The existence of conservation laws.*

J. M. Osborn, Michigan, February, *Sets of radial discontinuity of entire and meromorphic functions.*

Robert Osserman, Harvard, June, *Contributions to the problem of type.*

J. B. O'Toole, Pittsburgh, August, *Ideals of null sets of a topological space.*

F. P. Palermo, Brown, June, *Cohomology rings of product complexes.*

Prom Panitchpakdi, Kansas, June, minor in physics, *Extension of uniformly continuous transformations and hyperconvex metric spaces.*

H. C. Parrish, Ohio State, December, *On the symmetric structure of unconditioned point sets and real functions.*

M. H. Pearl, Wisconsin, June, *Some extensions of Cayley's theorem.*

B. J. Pearson, Texas, June, *A connected point set in the plane which spirals down on each of its points.*

D. K. Pease, Pittsburgh, August, *A study of affinities and anti-affinities of the plane of two complex variables and their representation in a space of four dimensions.*

W. M. Perel, Indiana, June, minor in physics, *The co-ideal theory of commutative semigroups.*

F. P. Peterson, Princeton, June, *Generalized cohomotopy groups.*

T. J. Pignani, North Carolina, August, *A system of cyclicly related functional equations.*

Roger Pinkham, Harvard, June, *Inversion of the Laplace and Stieltjes transforms utilizing difference operators.*

John Pratt, Stanford, *Some results in the decision theory of one-parameter multivariate Pólya type distributions.*

R. T. Prosser, California, Berkeley, June, *Structure of operator algebras.*

John Raleigh, Pennsylvania, June, *The Fourier coefficients of the invariants* $j(2^{1/2}; \tau)$ *and* $j(3^{1/2}; \tau)$.

K. V. Ramachandran, North Carolina, June, *On certain tests and the monotonicity of their powers.*

Paul Randolph, Minnesota, *Multivariate acceptance sampling.*

Rimhak Ree, British Columbia, May, minor in physics, *On generalized Witt algebras.*

George Resnikoff, Stanford, *Contributions to sampling inspection by variables.*

R. A. Restrepo, California Institute of Technology, June, minor in physics, *Optimal decisions with applications to tactical problems.*

D. E. Richmond, Iowa State, June, minor in physics, *Boundedness of the solutions of some general nonlinear second order differential equations.*

W. J. Riordan, Chicago, December, *On the interpolation of operations.*

Murray Ritterman, New York, February, *Integration of boundary value problems for parabolic equations by difference methods.*

J. B. Roberts, Minnesota, December, minor in philosophy, *Summability methods in valuation fields.*

A. C. Rohloff, Purdue, January, *Sequential tests of composite hypotheses.*

Marvin Rosenblum, California, Berkeley, January, *Perturbation of the continuous spectrum.*

Jean E. H. Rubin, Stanford, January, *Bi-modal logic and double-closure algebras.*

E. E. Said, North Carolina State, *A comparison between alternate techniques using supplementary information in sample survey design.*

Parfeny Saworotnow, Harvard, June, *On the generalization of the notion of* H^**-algebras.*

E. C. Schlesinger, Harvard, June, *Conformal invariants and prime ends.*

S. I. Schlesinger, Illinois Institute of Technology, June, *Some eigenvalue problems in the transport theory of neutrons.*

Morris Schreiber, Chicago, December, *Unitary dilations of operators.*

E. J. Schweppe, Illinois, June, minor in physics, *A projective generalization of convexity.*

R. E. Seall, Illinois Institute of Technology, January, *Characterization of genera of quadratic forms with Gaussian integer coefficients.*

N. C. Severo, Carnegie Institute of Technology, June, *A comparison of tests on the mean of a logarithmico-normal distribution with known variance.*

Norman Shapiro, Princeton, June, *Degrees of computability.*

E. M. Shoemaker, Carnegie Institute of Technology, June, *A plasticity theory of pre-sintered powdered metals.*

R. M. Simons, Massachusetts Institute of Technology, June, minor in physics, *On the nonlinear theory of thin spherical shells.*

R. W. Sloan, Illinois, June, minor in philosophy, *On analytic functions of an infinite harmonic hyper-complex variable.*

E. C. Smith, Jr., Brown, June, *Higher degrees of distributivity and completeness in Boolean algebras.*

O. K. Smith, Princeton, April, *Discontinuous oscillations.*

R. F. Smith, Syracuse, September, *An experimental comparison of two liberal arts courses in general mathematics at Syracuse University.*

W. R. Smythe, Jr., Duke, June, minor in physics, *A theorem on upper semi-continuous decompositions.*

James Snover, Syracuse, September, *Approximation by rational functions which have their zeros and poles in preassigned regions.*

Jerome Spanier, Chicago, September, *Contributions to the theory of almost complex manifolds.*

Clifford Spector, Wisconsin, June, *On degrees of recursive unsolvability and recursive well-orderings.*

R. F. Spring, Illinois, June, minor in chemistry, *On metabelian p-groups of exponent p.*

D. A. Sprott, Toronto, May, minor in genetics, *Balanced incomplete block designs.*

D. P. Squier, Stanford, September, *On some convex cones and associated monotone functions.*

F. W. Stallard, North Carolina, June, *Differential systems with interface conditions.*

G. P. Steck, California, Berkeley, *Limit theorems for conditional distributions.*

E. M. Stein, Chicago, December, *Linear operators on L_p spaces.*

F. M. Stein, Iowa, August, *Approximate solution of integro-differential equations.*

H. A. Steinberg, Yale, June, *Diffusion processes with absorption.*

J. A. Steketee, Toronto, May, minor in gas dynamics, *Some problems in boundary-layer transition.*

W. B. Stenberg, California, Berkeley, June, *On sequences with divergent total variation and on separation of convex sets in linear spaces.*

Ruth Struik, New York, June, *On associative products of groups.*

B. V. Sukhatme, California, Berkeley, *Testing the hypothesis that two populations differ only in location.*

W. J. Swartz, Iowa State, March, minor in physics, *Iterative procedures for obtaining the periodic solution of certain nonlinear differential equations.*

Peter Swerling, California, Los Angeles, June, *Families of transformations in the function spaces H^p.*

W. C. Swift, Kentucky, June, minor in physics, *A generalized matrix method of summability.*

J. C. Swihart, Purdue, January, *A theory of gravitation and its quantization.*

Morris Tenenbaum, Cornell, September, *Transforms of Tauberian series by Riesz methods of different orders.*

P. E. Thomas, Princeton, October, *A generalization of the Pontrjagin square cohomology operation.*

G. T. Thompson, Oregon State, June, minor in college teaching, *On Bateman's method for solving linear integral equations.*

B. D. Tikkiwal, North Carolina State, *Multiphase sampling on successive occasions.*

J. A. Tischendorf, Purdue, June, *Linear estimation techniques using order statistics.*

Sister M. Torline, St. Louis, June, *On Waring's theorem modulo p.*

Donald Truax, Stamford, *Multi-decision problems for the multivariate exponential family.*

H. G. Tucker, California, Berkeley, June, *Contributions to the mathematical theory of accident proneness, with particular reference to multiple exposure.*

Anna E. Vaughan, Stanford, June, *Estimation of a biological population which is subject to a biased mortality.*

R. Z. Vause, North Carolina, June, *On the distribution of the Jacobian symbols.*

M. J. Vitousek, Stanford, January, *Some flows in a gravity field satisfying the exact free surface condition.*

D. T. Walker, Georgia, August, *On the diaphantine equation* $a^x - b^y = 1$.

E. A. Walker, Kansas, October, minor in physics, *Cancellation in direct sums of groups.*

Seth Warner, Harvard, June, *On locally convex spaces and locally multiplicatively-convex algebras.*

L. J. Warren, Oregon, June, *An ideal in the ring of polynomials over a ring.*

A. D. Wasel, Stanford, June, *On integral operators in compressible fluid flow.*

G. P. Weeg, Iowa State, June, minor in education, *Completely primary algebras which are direct products modulo the radical.*

J. W. Weihe, California, Berkeley, *On a Stieltjes integral.*

Robert Weiller, Indiana, June, minor in physics, Part I. *On the differentiability of solutions of two-dimensional regular variational problems.* Part II. *The Harnack-inequality for solutions of linear partial differential equations of elliptic type.*

Eleanor Weiss, Harvard, *Factor analysis of mathematical ability.*

Irving Weiss, Stanford, *Limiting distributions in some occupancy problems.*

Patricia J. Wells, Michigan State, August, *The solution of the integral equation for the prolate spheroidal transmitting antenna.*

D. V. V. Wend, Michigan, June, *Branched regular curve families and finite asymptotic paths of analytic functions.*

Oscar Wesler, Stanford, *A modified minimax principle.*

J. S. White, Minnesota, *Some problems of point estimation in stochastic difference equations.*

Eoin Whitney, Harvard, June, *Variability regions for the univalent functions and their derivatives.*

Harold Widom, Chicago, September, *Embedding of* AW^**-algebras.*

C. H. Wilcox, Harvard, June, *The scattering of electro-magnetic radiation by a cylindrical shell of finite length.*

M. B. Wilk, Iowa State, June, minor in chemical engineering, *Linear models and randomized experiments.*

R. E. Williamson, Pennsylvania, June, *On multiply monotone functions and their Laplace transforms.*

R. L. Wine, Virginia Polytechnic Institute, June, *A power study of multiple range and multiple F tests.*

Izaak Wirszup, Chicago, December, *On an extension of the Cesàro method of summability to the logarithmic scale.*

F. L. Wolf, Minnesota, March, minor in philosophy, *Machine computability in game theory.*

Ralph Wormleighton, Princeton, June, *Some extensions of the sign test.*

L. G. Worthington, Texas, August, *Integration formulae and boundary conditions for the hyperbolic equation with three independent variables and regions exterior to the cone.*

A. W. Wymore, Wisconsin, June, *On weak compactness in functional analysis.*

A. M. Yaqub, California, Berkeley, June, *On the theory of ring-logics and universal algebras.*

J. E. Yarnelle, Pittsburgh, February, *"Heredity" theorems of certain product transformations of the line.*

F. R. Yett, Iowa State, July, minor in general engineering, *Periodic solutions of Duffing's equation with forcing term containing first and third harmonics.*

W. M. Zaring, Kentucky, June, minor in physics, *Multiply monotone complex sequences.*

The following doctorates were conferred in 1954, but were not included in the list in the preceding volume of this Bulletin (vol. 61, pp. 262–274):

R. E. von Holdt, Northwestern, August, *An iterative procedure for the calculation of the eigenvalues and eigenvectors of a real symmetric matrix.*

E. M. Jacobs, Iowa State, June, minor in economics, *A statistical technique for estimating the characteristics of consumer behavior.*

Melvin Shader, Syracuse, August, *Some inequalities for axially-symmetric potential functions.*

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AN OUTLINE OF THE THEORY OF PSEUDOANALYTIC FUNCTIONS[1]

LIPMAN BERS

1. Introduction.

Pseudoanalytic functions are, roughly speaking, solutions of generalized Cauchy-Riemann equations. Such functions were considered by Picard [67; 68] and by Beltrami [3; 4], but the first significant result was obtained by Carleman [28] in 1933, and a systematic theory was formulated only recently. No comprehensive account is available in print and the main results have been announced only episodically and, in most cases, under unnecessarily restrictive hypotheses. In this paper we shall outline the main features of this theory.

The theory of analytic functions of a complex variable occupies a central place in analysis and it is not surprising that mathematical literature abounds in generalizations. In some generalizations one extends the domain of the functions considered, or their range, or both (functions of several complex variables, analytic functions with values in a vector space or an algebra, analytic functions of hypercomplex variables, analytic operators, etc.). If we restrict ourselves to functions from plane domains to plane domains, or, more generally, from Riemann surfaces to Riemann surfaces, we encounter two well known and very useful generalizations of analytic functions: interior functions and quasi-conformal functions. Interior functions [47; 60; 61; 62; 63; 78; 86] have all topological properties of analytic functions and no others. As a matter of fact, they may be defined as functions which can be made analytic by a homeomorphism of the domain of definition. Quasi-conformal functions [2; 27; 46; 66; 75; 79] are interior functions subject to an additional metric condition.[2] If the functions are assumed to be continuously differentiable mappings, this additional condition requires that infinitesimal circles be taken into infinitesimal ellipses of uniformly bounded eccentricity.

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[1] Some of the results presented have been obtained during work under contract DA-30-069-ORD-835 with the Office of Ordnance Research, United States Army. Reproduction in whole or in part permitted for any purpose of the United States Government.

[2] Some authors, to the present writer's regret, use the term pseudoanalytic for quasi-conformal functions.

The class of quasi-conformal functions contains all analytic functions as well as many others. For this very reason the theory of quasi-conformal functions can not have the inner rigidity and harmony of classical function theory. For example, quasi-conformal functions do not have the unique continuation property which Riemann considered to be the most characteristic feature of analytic functions. Pseudoanalytic functions, on the other hand, do possess the unique continuation property, and each class of pseudoanalytic functions has almost as much structure as the class of analytic functions. In particular, the operations of complex differentiation and complex integration have meaningful counterparts in the theory of pseudoanalytic functions, and this theory generalizes not only the Cauchy-Riemann approach to function theory but also that of Weierstrass.

The theory of pseudoanalytic functions was developed from the point of view of partial differential equations, much of the motivation being provided by problems in mechanics of continua. But in this paper we present pseudoanalytic functions axiomatically, as a generalization of classical function theory.

We hope to show that this generalization is not empty (that is, that it preserves essential features of the original theory), not trivial (that is, not a direct consequence of the original theory) and not isolated (that is, that it has applications which are of independent interest).

REMARK. Analogues of complex differentiation and integration for an elliptic partial differential equation were used by Beltrami [3; 4] in the special case of axially symmetric potentials. In a systematic way these concepts, as well as formal powers and power series, have been introduced in 1943 by Gelbart and the author [21; 22; 23] for equations of a special form (cf. §13). Markuševitch [65] observed that this formalism can be extended to general equations. The author's original version of a general theory of pseudoanalytic functions [11; 12] was influenced by the work of Položii [69; 70; 71; 72] and made essential use of Carleman's unique continuation theorem [28]. This theory applied to equations with Hölder continuously differentiable coefficients. It contained, in particular, a description of zeros and singularities of solutions. In this direction similar and somewhat stronger results have been obtained by Vekua [81; 82; 84], whose work contains also many other important results, by Hartman and Wintner [39; 40; 41; 42] and, in connection with mapping problems, by Gergen and Dressel [36; 37; 38]. Two gaps in the theory were filled by Agmon and the author [1] and by Protter [73]. A more general theory, applicable to equations with merely Hölder continuous coefficients was formulated in [18]. The proofs of some of the results described below are not yet published. A monograph on pseudoanalytic functions is in preparation.

I. DEFINITION AND LOCAL PROPERTIES OF PSEUDOANALYTIC FUNCTIONS

2. Definition of pseudoanalytic functions. The class A_D of functions $w(z)$ which are analytic in a fixed domain D can be characterized by four properties. (1) A_D is a real vector space. (2) If $w(z)$ belongs to A_D and $w(z_0)=0$ for some z_0 in D, then the quotient

$$w(z)/(z-z_0)$$

is continuous at z_0. (3) The functions 1 and i belong to A_D. (4) A_D is maximal with respect to properties (1), (2), (3).

In fact, the first three properties imply that the existence of the complex derivative $w'(z)$ at every point of D is necessary in order that a function w should belong to A_D, and property (4) implies that this condition is sufficient.

Pseudoanalytic functions are obtained by a modification of the preceding definition [11; 12; 18].

Let $F(z)$, $G(z)$ be two continuous functions defined in a domain D_0 and satisfying the inequality

$$\operatorname{Im}\{\overline{F(z)}G(z)\} > 0. \tag{2.1}$$

(We write functions of (x, y) as functions of $z=x+iy$, without implying analyticity. The bar over a complex number denotes the complex conjugate.) Let $A_D(F, G)$ be the maximal class of functions (defined in a domain $D\subset D_0$) such that properties (1) and (2) hold, and the "*generators*" F and G belong to $A_D(F, G)$. If $w(z)$ belongs to $A_D(F, G)$ we call w an (F, G) *pseudoanalytic function of the first kind*. It is clear that pseudoanalyticity of w in D implies its pseudoanalyticity in every subdomain of D. Pseudoanalyticity at a point is defined as pseudoanalyticity in some neighborhood of the point.

It follows from (2.1) that every function $w(z)$ defined in a subdomain of D, whether or not it belongs to $A_D(F, G)$, admits the unique representation

$$w(z) = \phi(z)F(z) + \psi(z)G(z) \tag{2.2}$$

where the functions ϕ, ψ are real-valued. It is convenient to associate with the function w the function

$$\omega(z) = \phi(z) + i\psi(z). \tag{2.3}$$

The correspondence between w and ω is one-to-one. We denote it by writing

$$\omega = {}_*w, \qquad w = \omega^* \qquad (\text{mod } F, G). \tag{2.4}$$

If w is (F, G) pseudoanalytic of the first kind, we call ω pseudoanalytic *of the second kind*. In this case w and ω should be considered as two representations of the same mathematical entity.

In general, pseudoanalytic functions do not form a multiplicative semi-group. Neither is a pseudoanalytic function of a pseudoanalytic function in general pseudoanalytic. As a matter of fact, there exists *no* nontrivial generalization of the analyticity concept for complex-valued functions of a complex variable in which unique continuation is preserved and the property of closure under multiplication and substitution is not lost. On the other hand, the definition of pseudoanalyticity is *conformally covariant*.[3] If $\zeta = f(z)$, $z = g(\zeta)$ is a conformal mapping of the domain D onto Δ, then a function $w(z)$ is (F, G) pseudoanalytic if and only if the function $w[g(\zeta)]$ is $(\tilde{F}, \tilde{G})$ pseudoanalytic, where $\tilde{F}(\zeta)$, $\tilde{G}(\zeta)$ are the transplanted generators: $\tilde{F} = F[g(\zeta)]$, $\tilde{G} = G[g(\zeta)]$. This fact leads to a natural definition of pseudoanalyticity on a Riemann surface. In particular, $w(z)$ is (F, G) pseudoanalytic at $z = \infty$ if and only if $\tilde{w}(\zeta) = w(1/\zeta)$ is $(F(1/\zeta), G(1/\zeta))$ pseudoanalytic at the origin.

Two pairs of generators, (F, G) and $(\hat{F}, \hat{G})$, will be called ***equivalent*** if $\hat{F}$ and $\hat{G}$ are linear combinations, with real constant coefficients, of F and G. The two pairs will be called ***equipotent*** if $\hat{F}$ and $\hat{G}$ are (F, G) pseudoanalytic of the first kind. Equivalent generating pairs are always equipotent, and two equipotent pairs determine the same class of pseudoanalytic functions of the first kind. On the other hand, if (F, G) is replaced by an equivalent (equipotent) pair, the functions of the second class undergo a fixed affine transformation with real constant (variable) coefficients. If there exists a function $H(z)$ such that $\hat{F} = HF$, $\hat{G} = HG$, (F, G) and $(\hat{F}, \hat{G})$ are called ***similar***. In this case the (F, G) pseudoanalytic functions of the first kind are transformed into $(\hat{F}, \hat{G})$ pseudoanalytic functions by multiplication by the fixed function H; the class of pseudoanalytic functions of the second kind, however, is insensitive to a replacement of the generating pair by a similar one.

3. **Differentiation and integration.** [10; 11; 18]. The (F, G) derivative of a function (2.2) is defined by the relation

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[3] This shows that the use of complex numbers is here natural. Cf. however the comments of Krikzten [48].

$$(3.1)\qquad \dot{w}(z) = \frac{d_{(F,G)}w(z)}{dz} = \lim_{h\to 0}\frac{[\phi(z+h)-\phi(z)]F(z+h)+[\psi(z+h)-\psi(z)]G(z+h)}{h}$$

provided that the limit to the right exists and is finite. It is easy to see that *a function $w(z)$ is (F, G) pseudoanalytic of the first kind in a domain D if and only if it possesses an (F, G) derivative at every point of D.* The generators, which play the role of constants, have, of course, identically vanishing (F, G) derivatives. It is clear that (F, G) differentiability implies continuity. By letting h in (3.1) approach zero first through real and then through purely imaginary values, one obtains the existence of the partial derivatives of ϕ and ψ, and the relations

$$(3.2)\qquad \phi_{\bar z}F + \psi_{\bar z}G = 0,$$

$$(3.3)\qquad \phi_z F + \psi_z G = \dot{w}$$

where the formal differential operators $\partial/\partial z$, $\partial/\partial \bar z$ are defined by the relations

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Those familiar with the theory of elliptic partial differential equations will not be surprised to learn that continuity of the generators is not sufficient to insure the continuity of the (F, G) derivatives. We shall therefore assume from now on that the generators F and G are *Hölder continuous* (that is, that they satisfy a uniform Hölder condition[4] on every compact subset of their domain of definition). Under this hypothesis it can be shown that the *(F, G) derivative of an (F, G) pseudoanalytic function is continuous.* Thus the class of (F, G) pseudoanalytic functions of the second kind is the class of solutions (ϕ, ψ) of the (elliptic) system (3.2). It might be worthwhile to state explicitly that equation (3.2) can not be transformed, by a change of dependent and independent variables, into the Cauchy-Riemann system.

The proof of the continuity of the (F, G) derivative is rather long and involves some delicate results from the theory of differential equations. A comparatively simpler proof can be given under the additional hypothesis that the generators are Hölder continuously differentiable.

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[4] The function $f(z)$ is said to satisfy a uniform Hölder condition if $|f(z_1)-f(z_2)| \leq K|z_1-z_2|^{\alpha}$, for some $\alpha>0$ and K.

Together with the generating pair (F, G) we consider the ***dual pair*** $(F, G)^* = (F^*, G^*)$ defined by the formulas

$$(3.4) \qquad F^* = \frac{2\bar{G}}{F\bar{G} - \bar{F}G}, \qquad G^* = \frac{2\bar{F}}{F\bar{G} - \bar{F}G}$$

(so that $F^{**} = F$, $G^{**} = G$). Equations (3.2), (3.3) imply the formulas

$$(3.5) \qquad 2\phi_z = F^*\dot{w}, \qquad 2\psi_z = -G^*\dot{w}$$

which suggest the following definition of integration. Let $W(z)$ be a continuous function defined on a rectifiable curve Γ. The (F, G) *integral* of W on Γ is, by definition,

$$(3.6) \qquad \int_\Gamma W d_{(F,G)}z = \mathrm{Re}\int_\Gamma F^*W dz - i\,\mathrm{Re}\int_\Gamma G^*W dz.$$

(F, G) derivatives may be characterized by the property that their (F, G) integrals vanish along every closed curve contractible to a point in the domain considered. If $w(z)$ is (F, G) pseudoanalytic of the first kind, then

$$(3.7) \qquad \omega(z_1) - \omega(z_0) = \int_{z_0}^{z_1} \dot{w}\, d_{(F,G)}z$$

where $\omega = {}_*w$ is the function of the second kind corresponding to w.

The most striking similarity between pseudoanalytic and analytic functions is revealed by the following theorem.

Let (F, G) be a generating pair in a domain D_0. There exist in D_0 two other generating pairs, (F_1, G_1) and (F_{-1}, G_{-1}), called successor and predecessor of (F, G), respectively, such that (i) *(F, G) derivatives of (F, G) pseudoanalytic functions are (F_1, G_1) pseudoanalytic of the first kind, and* (ii) *(F, G) pseudoanalytic functions of the first kind are (F_{-1}, G_{-1}) derivatives of (F_{-1}, G_{-1}) pseudoanalytic functions.* Conversely, (F, G) is a successor of (F_{-1}, G_{-1}) and a predecessor of (F_1, G_1).

The actual construction of a successor pair in the large is a difficult task. In some cases it can be reduced to solving linear integral equations (for instance, if the domain D_0 is bounded and F, G possess bounded derivatives with respect to x and y). In the general case a more complicated limiting process is involved. Also, the successor pair (F_1, G_1) is not determined uniquely; it may be replaced by any other equipotent pair. The construction of predecessors is reduced to the construction of successors, since it is not difficult to show that if (F_1, G_1) is a successor of (F, G), $(F_1, G_1)^*$ is a predecessor of $(F, G)^*$.

The theorem stated above implies that ***every given generating pair** (F, G) **can be embedded in a generating sequence***

$$\cdots, (F_{-1}, G_{-1}), (F_0, G_0), (F_1, G_1), (F_2, G_2), \cdots \tag{3.8}$$

that is in a sequence of generating pairs such that $(F_0, G_0) = (F, G)$ and $(F_{\nu+1}, G_{\nu+1})$ is a successor of (F_ν, G_ν) for all ν.

A generating sequence (F_ν, G_ν) is called periodic with period N if $(F_{\nu+N}, G_{\nu+N}) = (F_\nu, G_\nu)$ for all ν. The ***minimum period*** of a generating pair is the smallest period of a periodic generating sequence into which the pair can be embedded, if such a sequence exists, and is otherwise ∞. Recently Protter [74] succeeded in proving the ***existence of generating pairs with a prescribed minimum period** N* (including the case $N = \infty$).

It is easy to see that a generating pair (F, G) is its own successor, i.e. has minimum period 1, if and only if $(F/G)_z \equiv 0$. If $(F/G)_z \neq 0$ but $(F/G)_y \equiv 0$, the minimum period of (F, G) is 2.[5]

Now let $w(z)$ again be an (F, G) pseudoanalytic function. Using a generating sequence (3.8) in which (F, G) is embedded we can define the higher derivatives of w by the recursion formulas

$$w^{[0]} = w; \qquad w^{[n+1]} = \frac{d_{(F_n, G_n)} w^{[n]}}{dz}, \qquad n = 1, 2, \cdots. \tag{3.9}$$

Thus a pseudoanalytic function of the first kind is, in a certain sense, ***infinitely differentiable***, though as a function of the real variables x and y it need not have partial derivatives even of the first order.

4. Behavior at a point. [11; 18]. In describing the analytic properties of (F, G) pseudoanalytic functions it is best to work with functions of the first kind. Let $w(z) = \phi F + \psi G$ be such a function. Then, in the neighborhood of every point of its domain of regularity we have

$$w(z) = \phi(z_0)F(z) + \psi(z_0)G(z) + \alpha(z - z_0)^n + o(|z - z_0|^n), \quad \alpha \neq 0 \tag{4.1}$$

where α is a complex constant, and n a positive integer, except if $w \equiv 0$ in which case $w \equiv \phi(z_0)F(z) + \psi(z_0)G(z)$. In particular, ***the zeros of a pseudoanalytic function are isolated***, unless the function vanishes identically. This implies the ***unique continuation property***:[6] a pseudoanalytic function is determined by its values on every open set, or even on every infinite set which has an accumulation point within the domain of regularity.

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[5] Cf., in this connection, Lukomskaya [53; 54; 55].

[6] For differentiable F, G this follows also from Carleman's theorem. Concerning various proofs of this theorem cf. [12; 15; 36; 56; 81; 82]. For a more general result, see §15.

Now let z_0 be an isolated singular point of a single-valued pseudoanalytic function w. If $w(z)\to\infty$ as $z\to z_0$, z_0 is called a pole. In this case there exists an integer $n>0$ such that

$$w(z) = \frac{\alpha}{(z-z_0)^n} + o(|z-z_0|^{-n}), \qquad \alpha \neq 0. \tag{4.2}$$

We call z_0 an essential singular point of w if $w(z)$ comes arbitrarily close to every complex number as $z\to z_0$. *An isolated singularity of a single-valued pseudoanalytic function is either removable, or a pole, or essential.*

In view of the conformal covariance of pseudoanalyticity we can extend these results to finitely-many-valued functions. In fact, if $w(z)$ is (F, G) pseudoanalytic for, say, $0<|z|<r$, and is m-valued, then $w(z^m)$ is a single-valued pseudoanalytic function with respect to the generators $F(z^m)$, $G(z^m)$. Hence, if $w(z)$ remains bounded as $z\to 0$ we have

$$w(z)=\phi(z_0)F(z)+\psi(z_0)G(z)+\alpha(z-z_0)^{n/m}+o(|z-z_0|^{n/m}), \qquad \alpha\neq 0 \tag{4.3}$$

where n is a positive integer. Similarly, if $|w(z)|\to\infty$ for $z\to z_0$, we have the asymptotic formula

$$w(z) = \frac{\alpha}{(z-z_0)^{n/m}} + o(|z-z_0|^{-n/m}), \qquad \alpha \neq 0. \tag{4.4}$$

The asymptotic relations stated above can be "differentiated," as in the case of analytic functions. More precisely, if the (single-valued) (F, G) pseudoanalytic function $w(z)$ has an essential singularity at z_0 so does the (F, G) derivative $\dot{w}(z)$. If $w(z)$ is regular at z_0 and (4.1) holds, then

$$\dot{w}(z) = n\alpha(z-z_0)^{n-1} + o(|z-z_0|^{n-1}). \tag{4.1a}$$

If z_0 is a pole of $w(z)$ and (4.2) holds, we have

$$\dot{w}(z) = -\frac{n\alpha}{(z-z_0)^{n+1}} + o(|z-z_0|^{-n-1}). \tag{4.2a}$$

Similar conclusions can be drawn in the case of "algebraic" singularities: relation (4.3) implies that

$$\dot{w}(z) = (n\alpha/m)(z-z_0)^{(n/m)-1} + o(|z-z_0|^{(n/m)-1}) \tag{4.3a}$$

and relation (4.4) that

$$(4.4a)\qquad \dot{w}(z) = -\frac{n\alpha}{m(z-z_0)^{(n/m)+1}} + o(|z-z_0|^{-(n/m)-1}).$$

These results are easily extendable to pseudoanalytic functions defined in a neighborhood of the point at infinity.

In view of relations (4.1) and (4.2) we may speak of the order of a zero or a pole of a pseudoanalytic function, and the so called argument principle holds for pseudoanalytic functions of the first kind. In general, however, functions of the first kind do not share the geometric properties of analytic functions. This is clear from the fact that the generators themselves, which are essentially arbitrary functions, are pseudoanalytic. Geometric properties of pseudoanalytic functions are therefore to be discussed in terms of functions of the second kind.

The basic fact here is the following corollary of the asymptotic formulas (4.1), (4.1a). *The mapping $\omega = \omega(z)$ by a pseudoanalytic function of the second kind $\omega = {}_*w$ is a homeomorphism of a neighborhood of a point z_0 if and only if $\dot{w}(z_0) \neq 0$.* If $\dot{w}(z)$ has at z_0 a zero of order n, then $\omega(z)$ maps a suitably chosen neighborhood of z_0 onto an $(n+1)$ times covered neighborhood of $\omega(z_0)$. A similar statement can be made concerning points at which the function has poles.

Since $\dot{w}(z)$ is (F_1, G_1) pseudoanalytic its zeros are isolated, and the pseudoanalytic function of the second kind $\omega(z)$ is a local homeomorphism everywhere except at isolated points. This local statement can be transformed into a global one via the general uniformization theorem: *every pseudoanalytic function of the second kind is an interior transformation*, that is it can be transformed into an analytic function by a homeomorphism of its domain of definition. It follows that the maximum principle holds for the modulus of a pseudoanalytic function $\omega = \phi + i\psi$ and also for the functions ϕ, ψ.

The eccentricity of a differentiable mapping at a point at which the Jacobian is different from zero is defined as the eccentricity of the infinitesimal ellipse into which the mapping takes an infinitesimal circle. A direct computation shows that the eccentricity of a pseudoanalytic function can be estimated in terms of a bound for $|F|$, $|G|$ and $1/\text{Im}\,(\bar{F}G)$. Thus pseudoanalytic functions of the second kind are not only interior but also, at least on every compact subset of the domain in which the generators are defined, of bounded eccentricity (or *quasi-conformal*). This implies, among other things, that Picard's theorem on essential singularities holds.

II. PSEUDOANALYTIC FUNCTIONS IN THE LARGE

5. Univalent functions. Local formal powers. The Riemann mapping theorem holds for pseudoanalytic functions of the second kind. This follows from a more general theorem (due to Miss Z. Šapiro)[7] which asserts that the Riemann mapping theorem is true even for solutions of certain quasi-linear systems of elliptic partial differential equations. In proving the Riemann mapping theorem, however, one must make some assumptions, albeit very mild ones, on the behavior of the generators near the boundary of the domain considered (uniform boundedness of $|F|$, $|G|$ and $1/\mathrm{Im}\,(\bar{F}G)$ is sufficient). On the other hand, if we do not want to solve a definite mapping problem, but merely find some univalent pseudoanalytic function, no hypotheses whatsoever on the domain or on the generators are required. As a matter of fact, the following is true [13; 18].

Let the generators (F, G) *be defined in a domain* D_0. *Then there exist in* D_0 *two* (F, G) *pseudoanalytic functions of the first kind,* $w_1(z)$ *and* $w_2(z)$, *such that for every choice of real constants* λ_1, λ_2 *with* $\lambda_1^2+\lambda_2^2>0$, *the function* $\omega(z) = {}_*(\lambda_1 w_1+\lambda_2 w_2)$ *is univalent in* D_0.

For the case of analytic functions (that is, for the generators 1, i) the assertion is completely trivial if D_0 is a domain in the plane, since it suffices to set $w_1 = z$, $w_2 = iz$. But the assertion is true even if D_0 is a Riemann surface of genus zero, and in this case it is the (distinctly nontrivial) general uniformization theorem. The theorem stated above (for any pair of generators) is therefore a natural extension of the general uniformization theorem to pseudoanalytic functions.

This theorem is used in order to prove the existence of successors. Indeed, since ${}_*(\lambda_1 w_1+\lambda_2 w_2)$ is univalent if $\lambda_1^2+\lambda_2^2\neq 0$, we have, noting the local univalency criterion stated in §4, that $\mathrm{Im}\,([\dot{w}_1]^-\dot{w}_2)\neq 0$, everywhere.[8] Hence we may assume that $\mathrm{Im}\,([\dot{w}_1]^-\dot{w}_2)>0$ renaming if necessary the functions w_1, w_2. It can be shown that (F, G) derivatives are Hölder continuous (if the generators are). Hence we may define a pair of generators by setting $F_1 = \dot{w}_1$, $G_1 = \dot{w}_2$. It turns out that the pair (F_1, G_1) is a successor of (F, G).

In classical function theory the variable z plays a double role: it is the independent variable and also the simplest nonconstant analytic function defined everywhere. In the case of an arbitrary pair of generators the independent variable z is not pseudoanalytic, but we have just seen how one can find pseudoanalytic functions which have an important property (univalency) in common with the analytic functions z, iz. This suggests a way of constructing other pseudoanalytic functions which play the part of powers of the complex variable.

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[7] [76]. An independent proof for linear systems was given by Gergen and Dressel [36; 37; 38], a strengthened version of Šapiro's result by Nirenberg and the author [24]. Lavrent'ev proved a Riemann mapping theorem for general nonlinear systems [49; 50].

[8] $[\dot{w}]^-$ stands for the complex conjugate of $\dot{w}$.

We assume now that the domain D_0 is simply connected and that a generating sequence $\{(F_\nu, G_\nu)\}$ is defined in D_0. We define the *local formal power* (with center at z_0 in D_0, coefficient α and exponent 0), $Z_\nu^{(0)}(\alpha, z_0; z)$, as the linear combination of the generators $F_\nu(z)$, $G_\nu(z)$ with real constant coefficients λ, μ chosen so that $\lambda F(z_0)+\mu G(z_0)=\alpha$. Thus $Z_\nu^{(0)}$ is determined by the conditions

$$(5.1)\quad Z_\nu^{(0)}(\alpha, z_0; z_0) = \alpha, \quad {}_*Z_\nu^{(0)}(\alpha, z_0; z) = \text{const.} \pmod{F_\nu, G_\nu}.$$

The *local formal powers with exponents* $n=1, 2, \cdots$ are defined by the recursion formulas

$$(5.2)\qquad {}_*Z_\nu^{(n)}(\alpha, z_0; z) = n\int_{z_0}^{z} Z_{\nu+1}^{(n-1)}(\alpha, z_0; \zeta)d_{(F_\nu, G_\nu)}\zeta$$

where we agree once and for all that ${}_*Z_\nu$ is formed modulo F_ν, G_ν. We shall often write F, G, Z instead of F_0, G_0, Z_0.

This definition implies the following properties. (i) $Z_\nu^{(n)}(\alpha, z_0; z)$ is an (F_ν, G_ν) pseudoanalytic function (of the first kind) of z. (ii) If α' and α'' are real constants, then

$$(5.3)\quad Z_\nu^{(n)}(\alpha' + i\alpha'', z_0; z) = \alpha' Z_\nu^{(n)}(1, z_0; z) + \alpha'' Z_\nu^{(n)}(i, z_0; z).$$

(iii) The formal powers satisfy the differential relations

$$(5.4)\qquad \frac{d_{(F_\nu, G_\nu)}}{dz} Z_\nu^{(n)}(\alpha, z_0; z) = nZ_{\nu+1}^{(n-1)}(\alpha, z_0; z).$$

(iv) The asymptotic formulas

$$(5.5)\qquad Z_\nu^{(n)}(\alpha, z_0; z) \sim \alpha(z - z_0)^n, \qquad z \to z_0$$

hold.

Assume now that

$$(5.6)\qquad w(z) = \sum_{n=0}^{\infty} Z^{(n)}(\alpha_n, z_0; z).$$

where the series converges uniformly in some neighborhood of z_0.

It can be shown that the uniform limit of pseudoanalytic functions is pseudoanalytic, and that a uniformly convergent series of (F, G) pseudoanalytic functions can be (F, G) differentiated term by term. Hence the function w in (5.6) is (F, G) pseudoanalytic and its rth derivative admits the expansion

$$w^{[r]}(z) = \sum_{n=r}^{\infty} n(n-1)\cdots(n-r+1)Z_r^{(n-r)}(\alpha_n, z_0; z). \tag{5.7}$$

From this we obtain the Taylor formulas for the coefficients

$$\alpha_n = \frac{w^{[n]}(z_0)}{n!}\,. \tag{5.8}$$

Now let $w(z)$ be a given (F, G) pseudoanalytic function defined for small values of $|z-z_0|$. The series

$$\sum_{n=0}^{\infty} Z^{(n)}(\alpha_n, z_0; z) \tag{5.9}$$

with the coefficients given by (5.8) is called the *Taylor series* of w at z_0, formed with local formal powers. Noting formulas (4.1), (4.1a), (5.5) we conclude that the Taylor series always represents the function asymptotically:

$$w(z) - \sum_{n=0}^{N} Z^{(n)}(\alpha_n, z_0; z) = O(|z - z_0|^{N+1}), \qquad z \to z_0, \tag{5.10}$$

for all N. This implies (since a pseudoanalytic function can not have a zero of arbitrarily high order without vanishing identically) that the *sequence of derivatives* $\{w^{[n]}(z_0)\}$ *determines the function* w *uniquely.*

By the same token, if the series (5.9) converges uniformly in a neighborhood of z_0, it converges to the function w. Unfortunately the necessary and sufficient conditions for convergence are not known. A general expansion theorem holds for the global formal powers which are considered below.

The convergence question does not arise and the expansion (5.6) is certainly valid if all but a finite number of Taylor coefficients (5.8) vanish. This is the case if w is a formal power with center z_1. Thus we have the formula

$$Z^{(n)}(\alpha, z_1; z) = \sum_{\mu=0}^{n} \binom{n}{\mu} Z^{(\mu)}[Z_{\mu}^{(n-\mu)}(\alpha, z_1; z_0), z_0; z] \tag{5.11}$$

which generalizes the binomial theorem.

6. Functions on a sphere. Global formal powers. [11; 12]. In this section we consider a generating pair (F, G) *defined on the full function-theoretical plane* (Riemann sphere). By this we mean that the functions $F(z)$, $G(z)$ are defined, are Hölder continuous and satisfy inequality (2.1) for all values of z, that the limits $F(\infty)$, $G(\infty)$ exist, and that the generating pair $F(1/z)$, $G(1/z)$ has similar properties. It now makes sense to talk about (F, G) pseudoanalytic functions which are regular at $z=\infty$ or have a pole there. A pseudo-analytic function having no singularities, except perhaps poles, in the whole function theoretical plane will be called *rational*. The interiority of functions of the second kind implies that every rational pseudoanalytic function of the second kind can be transformed into a rational analytic function by a homeomorphism of the finite plane onto itself, the homeomorphism being dependent on the function considered.

A regular pseudoanalytic function defined in the whole finite plane is called entire. A bounded entire pseudoanalytic function of the first (second) kind is a linear combination of the generators with constant coefficients (a constant). This extension of *Liouville's theorem* shows that a rational pseudoanalytic function w must have poles, except in the trivial case $w=$ const. *A rational pseudoanalytic function has as many zeros as it has poles* (zeros and poles being counted with their proper multiplicities).

A generating pair (F, G) defined on the Riemann sphere has a successor (F_1, G_1) and a predecessor (F_{-1}, G_{-1}) defined on the Riemann sphere, and these pairs are determined uniquely (except, of course, for equivalences). It follows that (F, G) can be embedded in an essentially unique generating sequence $\{(F_\nu, G_\nu)\}$ defined on the Riemann sphere. I conjecture that the sequence $\{(F_\nu, G_\nu)\}$ is nonperiodic unless (F, G) is equivalent to $(1, i)$.

We are now in a position to define global *formal powers with positive and negative exponents*. The local formal powers formed with respect to the generating sequence described above are called *global formal powers*. The global formal power $Z_\nu^{(n)}(\alpha, z_0; z)$, $n>0$, is a rational (F_ν, G_ν) pseudoanalytic function of z which has a zero of order n at $z=z_0$, a pole of order n at $z=\infty$ and satisfies relation (5.5). An (F_ν, G_ν) pseudoanalytic rational function $w(z)$ which has a pole of order n at $z=z_0\neq\infty$, satisfies the asymptotic relation $w\sim\alpha(z-z_0)^{-n}$, $z\to z_0$, has a zero of order n at $z=\infty$, and has no other zeros or poles, we call a global formal power with exponent $(-n)$, and denote by $Z_\nu^{(-n)}(\alpha, z_0; z)$. The uniqueness of $Z^{(-n)}$ is easily established. In order to prove its existence it suffices to recall the conformal covariance of pseudoanalyticity. Define the generating pair (Φ_ν, Γ_ν), on the Riemann sphere, by setting

$$(6.1) \qquad \Phi_\nu(z) = F_\nu(z_0 + 1/z), \qquad \Gamma_\nu(z) = G_\nu(z_0 + 1/z)$$

and denote the (Φ_ν, Γ_ν) global formal powers (with positive exponents) by $\widehat{Z}_\nu^{(n)}$. For every $n > 0$ and α one can find a number β such that $\widehat{Z}_\nu^{(n)}(\beta, 0; z) \sim \alpha z^n$, $z \to \infty$. Then

$$(6.2) \qquad Z_\nu^{(-n)}(\alpha, z_0; z) = \widehat{Z}_\nu^{(n)}\left(\beta, 0; \frac{1}{z - z_0}\right).$$

The properties (i)–(iv) of formal powers noted in §5 remain valid for global formal powers with negative exponents.

Every rational pseudoanalytic function is a sum of formal powers. There exist rational pseudoanalytic functions with prescribed zeros and poles. Every pseudoanalytic function $w(z)$ which has a pole at $z_0 \neq \infty$ is of the form

$$(6.3) \qquad w(z) = \sum_{n=1}^{N} Z^{(-n)}(\alpha_n, z_0; z) + w_1(z), \qquad \alpha_N \neq 0,$$

where w_1 is pseudoanalytic and regular at z_0. An analogous decomposition into a "critical" and a regular part holds for functions which have a pole at infinity. Finally, there exist rational pseudoanalytic functions which have prescribed poles with prescribed critical parts.

The most important application of formal powers with negative exponents is the *Cauchy integral formula*. We first define a new kind of line integral for formal powers. Let C be a continuously differentiable curve given by the equation $\zeta = \zeta(s)$, $0 \leqq s \leqq l$, s being the arc length. If $g(\zeta)$ is a continuous function defined on C, we set

$$\int_C {}^{*}Z^{(n)}[g(\zeta)d\zeta, \zeta; z] = \int_0^l Z^{(n)}\{g[\zeta(s)]\zeta'(s), \zeta(s); z\}\, ds.$$

This integral as a function of z is pseudoanalytic, everywhere if $n \geqq 0$, and everywhere except on C if $n < 0$. The definition is legitimate since $Z^{(n)}(\alpha, z_0; z)$ can be shown to depend continuously on z_0.

Now if $w(z)$ is single-valued (F, G) pseudoanalytic of the first kind in a domain D interior to a sufficiently smooth curve C, and is continuous on C, then

$$(6.4) \qquad w(z) = \frac{1}{2\pi}\int_C Z^{-1}[iw(\zeta)d\zeta, \zeta; z], \qquad z \text{ in } D,$$

and

$$(6.5)\qquad \frac{1}{2\pi}\int_C Z^{-1}[iw(\zeta)d\zeta,\zeta;z] = 0, \qquad z \text{ exterior to } D.$$

A continuous function satisfying either of these relations for all C is (F, G) pseudoanalytic.

The Cauchy integral formula (6.5) can be used to derive *reciprocity relations* for formal powers. In the case of analytic functions it is trivial that the function $\alpha/(z-z_0)^n$ is an analytic function of z_0 as well as of z, but there exists a similar nontrivial statement about analytic functions on Riemann surfaces of higher genus (the theorem on interchanging parameter and argument in an Abelian integral of the second kind). In the case of pseudoanalytic functions the situation is nontrivial already on the sphere. Together with the generating sequence $\{(F_\nu, G_\nu)\}$ we consider the sequence of dual pairs $(F_\nu, G_\nu)^*$. (We noted before that $(F_\nu, G_\nu)^*$ is a successor of $(F_{\nu+1}, G_{\nu+1})^*$.) Let us denote the $(F_\nu, G_\nu)^*$ pseudoanalytic global formal powers by $Z_\nu^{*(n)}$. Then

$$(6.6)\qquad \begin{aligned}
\operatorname{Re} Z_\nu^{(-n)}(1, z_0; z) &= (-1)^n \operatorname{Im} Z_{\nu-n}^{*(-n)}(i, z; z_0),\\
\operatorname{Re} Z_\nu^{(-n)}(i, z_0; z) &= (-1)^n \operatorname{Re} Z_{\nu-n}^{*(-n)}(i, z; z_0),\\
\operatorname{Im} Z_\nu^{(-n)}(1, z_0; z) &= (-1)^n \operatorname{Im} Z_{\nu-n}^{*(-n)}(1, z; z_0),\\
\operatorname{Im} Z_\nu^{(-n)}(i, z_0; z) &= (-1)^n \operatorname{Re} Z_{\nu-n}^{*(-n)}(1, z; z_0),
\end{aligned}$$

for $n = 1, 2, \cdots$. The corresponding reciprocity relations for powers with positive exponents read

$$(6.6\text{a})\qquad \begin{aligned}
\operatorname{Re} Z_\nu^{(n)}(1, z_0; z) &= (-1)^n \operatorname{Re} Z_{\nu+n}^{*(n)}(1, z; z_0),\\
\operatorname{Re} Z_\nu^{(n)}(i, z_0; z) &= -(-1)^n \operatorname{Im} Z_{\nu+n}^{*(n)}(1, z; z_0),\\
\operatorname{Im} Z_\nu^{(n)}(1, z_0; z) &= -(-1)^n \operatorname{Re} Z_{\nu+n}^{*(n)}(i, z; z_0),\\
\operatorname{Im} Z_\nu^{(n)}(i, z_0; z) &= (-1)^n \operatorname{Im} Z_{\nu+n}^{*(n)}(i, z; z_0)
\end{aligned}$$

for $n = 0, 1, 2, \cdots$. These relations hold also for local formal powers.

The global formal powers with integral exponents correspond to Abelian integrals of the second kind. In order to construct *logarithmic pseudoanalytic functions*, corresponding to Abelian integrals of the third kind on the sphere, we may use either of the two integration processes at our disposal. More precisely, set

$$(6.7)\quad L_\nu(\alpha, z_0; z_1, z_2; z) = \frac{1}{2\pi}\int_{z_1}^{z_2} Z_\nu^{(-1)}\{iZ_\nu^{(0)}(\alpha, z_0; \zeta)d\zeta, \zeta; z)\}$$

where $z_1 \neq z_2$ and the integration is performed along some curve joining these two points. Considered as a function of z, L_ν is a multiple-valued (F_ν, G_ν) pseudoanalytic function, which increases by

$$(r - s)Z_\nu^{(0)}(\alpha, z_0; z)$$

when z goes r times around z_1 and s times around z_2. On the other hand, the (F_ν, G_ν) pseudoanalytic function (of the first kind) of z, $\Lambda_\nu(\alpha, z_0; z_1; z)$ which is defined by setting

$$(6.8) \qquad {}_*\Lambda_\nu(\alpha, z_0; z_1; z) = \frac{1}{2\pi}\int_{1+z_1}^{z} Z_{\nu+1}^{(-1)}\{Z_\nu^{(0)}(\alpha, z_0; z_1), z_1; \zeta\}\, d_{(F_\nu, G_\nu)}\zeta,$$

possesses logarithmic singularities at z_1 and at $z = \infty$ and increases by $Z_\nu^{(0)}(\alpha, z_0; z)$ when z goes once around z_1. These two functions are connected by the relation

$$(6.9) \qquad \dot{L}_\nu(\alpha, z_0; z_1, z_2; z) = \dot{\Lambda}_\nu(\alpha, z_0; z_1; z) - \dot{\Lambda}_\nu(\alpha, z_0; z_2; z)$$

where the dot indicates (F_ν, G_ν) differentiation.

The simplest finitely-many-valued pseudoanalytic functions are the *global formal powers with fractional exponents*. These are defined with the aid of the generating pairs (Φ_ν, Γ_ν),

$$(6.10) \qquad \Phi_\nu(z) = F_\nu(z_0 + z^m), \qquad \Gamma_\nu(z) = G_\nu(z_0 + z^m)$$

where m is an integer, $m > 1$. We denote the (Φ_ν, Γ_ν) pseudoanalytic global formal powers with integral exponents by $\widehat{Z}_\nu^{(n)}$. If r is a positive or negative integer, relatively prime to m, then the (F_ν, G_ν) pseudoanalytic formal power with exponent r/m is defined by setting

$$(6.11) \qquad Z_\nu^{(r/m)}(\alpha, z_0; z) = Z_\nu^{(r)}(\alpha, 0; (z - z_0)^{1/m}).$$

The properties (i)–(iv) of formal powers noted in §5 remain valid also for powers with fractional exponents.

The global formal powers with fractional exponents are the simplest examples of "algebraic" pseudoanalytic functions. In general, a multiple-valued (F, G) pseudoanalytic function will be called *algebraic* if it is finitely-many-valued and can be continued along every path in the finite plane avoiding a finite number of excluded "critical" points, provided that at every critical point as well as at the point at infinity the function becomes infinite of at most a finite order. An analytic function satisfying these conditions is algebraic in the ordinary sense, that is, a solution of an algebraic equation the coefficients of which are rational functions. For pseudoanalytic algebraic functions only a weakened form of this statement is true. At all but a finite number of points a pseudoanalytic algebraic function takes on the same finite number of distinct values. The sum of these values is called the trace of the function. It is not difficult to prove that *the trace of an algebraic pseudoanalytic function is a rational pseudoanalytic function.* In particular, the trace of a formal power $Z^{(n)}(\alpha, z_0; z)$, with n not an integer, is zero.

7. **Doubly-periodic pseudoanalytic functions.** [20]. A generating pair (F, G) is called doubly periodic if the functions $F(z)$ and $G(z)$ possess two periods Ω_1 and Ω_2, with $\operatorname{Im}(\bar{\Omega}_1\Omega_2) > 0$. An (F, G) pseudoanalytic function which has the same periods and is regular everywhere except for poles may be called an (F, G) elliptic function. An (F, G) elliptic function of the second kind can be transformed into an ordinary (analytic) elliptic function by a homeomorphism of the plane. Different (F, G) elliptic functions, however, will, in general, be transformed into analytic elliptic functions with different periods. The classical existence theorems of the theory of elliptic functions remain valid, when properly restated, in the pseudoanalytic case.

In particular, there exist, in analogy to the Weierstrass ζ-function, two linearly independent everywhere regular (F, G) pseudoanalytic functions, $\zeta_1(z)$ and $\zeta_2(z)$, such that for all integers m, n

$$(7.1)\quad \zeta_j(z + m\Omega_1 + n\Omega_2) - \zeta_j(z) = m\lambda_j F(z) + n\mu_j G(z), \qquad j = 1, 2,$$

the constants λ_j, μ_j being real. The derivatives $\dot{\zeta}_1(z)$, $\dot{\zeta}_2(z)$ have, therefore, the periods Ω_1, Ω_2, and it can be shown that if we set $F_1 = \dot{\zeta}_1$, $G_1 = \dot{\zeta}_2$ (interchanging, if necessary, the functions ζ_1, ζ_2), we obtain a generating pair (F_1, G_1) which is a successor to (F, G). It follows that *(F, G) can be embedded in a generating sequence $\{(F_\nu, G_\nu)\}$ such that all functions $F_\nu(z)$, $G_\nu(z)$ have the periods* Ω_1, Ω_2. The pairs (F_ν, G_ν) are determined uniquely, except for equivalences.

I conjecture that if such a sequence satisfies the relation $(F_n, G_n) = (F_0, G_0)$, then either $n = 1$ or $n = 2$ (that these two cases are indeed possible can be shown by simple examples).

8. **Pseudoanalytic functions and differentials on Riemann surfaces.** [20]. Let S be an abstract Riemann surface. Two functions F, G defined on S form a generating pair if they are Hölder continuous in terms of local parameters and satisfy the relation $\operatorname{Im}(\bar{F}G) > 0$. A function w defined on S will be called an (F, G) function if in the neighborhood of each point p of S and for every local parameter z defined in such a neighborhood, w is an (F, G) pseudoanalytic function of z which has no singularities except poles. A differential dW defined on S will be called an (F, G) differential if for every local parameter z defined in a sufficiently small neighborhood of a point

p of S there exists an (F, G) function $w(z)$ defined in the neighborhood such that $dW = w dz$. Every (F, G) function gives rise to an (F, G) differential, but an (F, G) differential is the differential of an (F, G) function only if all its (F, G) periods vanish, the (F, G) period of dW over a closed curve Γ being defined as the integral

$$\int_\Gamma \operatorname{Re}(F^* dW) - i \operatorname{Re}(G^* dW). \tag{8.1}$$

The (F, G) period of an (F, G) differential dW over a small simple closed curve enclosing a singular point p of dW is called the residue of dW at p.

A multiple-valued $(\tilde{F}, \tilde{G})$ pseudoanalytic function $w(z)$ defined in a subdomain Δ of the full function-theoretical plane may be considered as a single-valued (F, G) pseudoanalytic function defined on a suitably chosen covering surface S of Δ. The generators F, G are obtained from $\tilde{F}$, $\tilde{G}$ by transplantation: if the point p of S covers the point z of Δ, then $F(p) = \tilde{F}(z)$, $G(p) = \tilde{G}(z)$. In particular, if Δ is the whole Riemann sphere and the function w is algebraic, the surface S is closed (compact). The converse statements, however, are not generally true, unlike the classical situation. In fact, while an abstract Riemann surface S carrying a generating pair (F, G) can be represented, in infinitely many ways, as a covering surface of a domain Δ on the Riemann sphere, one can not, in general, choose this representation so that the generators F, G are obtained by transplantation from a generating pair defined in Δ.

Uniformization of multiple-valued pseudoanalytic functions (or, more generally, of functions on a Riemann surface) is accomplished by the usual procedure: construction of the universal covering surface, and conformal mapping on a standard domain Σ which is either the full function-theoretical plane, or the finite plane, or the unit disc. The result reads: every multiple-valued pseudoanalytic function $w(z)$ admits the parametric representation

$$w = \hat{w}(\tau), \qquad z = \hat{z}(\tau) \tag{8.2}$$

where τ ranges over Σ, $\hat{z}(\tau)$ is analytic and $\hat{w}(\tau)$ is $(\hat{F}, \hat{G})$ pseudoanalytic, the functions $\hat{F}(\tau)$, $\hat{G}(\tau)$, $\hat{z}(\tau)$, $\hat{w}(\tau)$ being automorphic with respect to the same properly discontinuous group of fixed point free conformal mappings of Σ onto itself.

We consider next (F, G) functions and differentials on a closed Riemann surface S of genus g. The cases $g = 0$ and $g = 1$ have already been considered in the preceding two sections. If $g > 1$, then there exists no preferred parametrization and it makes no sense to talk about the successor of (F, G) on S.

Using the interior character of pseudoanalytic functions of the second kind it is easy to show that an (F, G) function (which is not a linear combination of the generators with real constant coefficients) has *as many zeros as it has poles*, and that an (F, G) differential (which does not vanish identically) has $2(g-1)$ more zeros than poles, zeros and poles being counted with their multiplicities. A similar argument shows that the sum of the residues of an (F, G) differential vanishes, and that an everywhere regular (F, G) differential which does not vanish identically has a nonreal (F, G) period.

The *existence theorem for* (F, G) *differentials* states that there are exactly $2g$ linearly independent, everywhere regular (F, G) differentials, and that the number of linearly independent (F, G) differentials which have poles of order not exceeding $\beta_1, \cdots, \beta_r$ at $r>1$ distinct points $p_1, \cdots, p_r$ is $2(g-1+\beta_1+\cdots+\beta_r)$. This implies that one may prescribe the real parts of the periods of an (F, G) differential dw along $2g$ homologically independent closed curves as well as the singularities of the differential, provided one chooses these singularities in such a way that the sum of the residues vanishes.

In order that an (F, G) differential which has poles at $r>1$ distinct points be the differential of an (F, G) function, its r residues and its periods on $2g$ homologically independent closed curves must vanish. This amounts to not more than $2(g+r-1)$ linear (real) conditions. Hence the existence theorem for (F, G) differentials also implies the *existence of nontrivial* (F, G) *functions*.

The *Riemann-Roch theorem* for pseudoanalytic functions involves the dual generating pair. Let A be the number of linearly independent (F, G) functions which have zeros at r distinct points $p_1, \cdots, p_r$ of orders at least $\alpha_1, \cdots, \alpha_r$, and poles at most at s distinct points $q_1, \cdots, q_s$, of orders not exceeding $\beta_1, \cdots, \beta_s$, and let B be the number of linearly independent $(F, G)^*$ differentials which have no poles except perhaps at the points p_j, of orders not exceeding α_j, and which have zeros at the points q_k of orders at least β_k. Set $m=\beta_1+\cdots+\beta_s-\alpha_1-\cdots-\alpha_r$. Then

$$A - B = 2(m + 1 - g). \tag{8.3}$$

In particular, if $m>2g-2$ then $B=0$ and therefore $A=2(m+1-g)$.

While the Riemann-Roch theorem can be stated in terms of divisors it should be remembered that the (F, G) functions on S do not form a field. As for the proof of the theorem, it depends on reciprocity relations between (F, G) differentials and $(F, G)^*$ differentials, of which relations (6.6) are a typical though very special sample.

III. DIFFERENTIABLE GENERATORS AND THE SIMILARITY PRINCIPLE

9. Differentiable generators. [11; 12; 15; 81; 82]. We return to single-valued pseudoanalytic functions defined in a plane domain D_0. The theory acquires new features if we assume that in D_0 the generators possess partial derivatives with respect to the real variables x and y. In this case we can define the "characteristic coefficients" $a(z), b(z), A(z), B(z)$ of the pair (F, G). These are determined by the formulas

$$(9.1) \qquad F_{\bar z} = aF + b\bar F, \qquad G_{\bar z} = aG + b\bar G,$$

$$(9.2) \qquad F_z = AF + B\bar F, \qquad G_z = AG + B\bar G.$$

A simple computation shows that the existence of the (F, G) derivative $\dot w$ implies the equations

$$(9.3) \qquad w_{\bar z} = aw + b\bar w,$$

$$(9.4) \qquad \dot w = w_z - Aw - B\bar w.$$

Another generating pair (F_1, G_1) having the characteristic coefficients a_1, b_1, A_1, B_1 will be a successor of (F, G) if and only if

$$(9.5) \qquad a_1 = a, \qquad b_1 = -B.$$

Equivalent generating pairs have the same characteristic coefficients; for equipotent generating pairs the first two characteristic coefficients (a and b) coincide.

Assume now, for the sake of orientation, that the domain D_0 is bounded and that the characteristic coefficients are Hölder continuous and bounded. Then the following theorems hold.

(i) *A function $w(z)$ is (F, G) pseudoanalytic of the first kind if and only if it satisfies equation* (9.3).

(ii) *A bounded continuous function $w(z)$ defined in D_0 is (F, G) pseudoanalytic of the first kind if and only if the function $h(z)$ defined by the equation*

$$(9.6) \qquad w(z) = -\frac{1}{\pi}\iint_{D_0} \frac{a(\zeta)w(\zeta) + b(\zeta)\overline{w(\zeta)}}{\zeta - z}\, d\xi d\eta + h(z)$$

is analytic.

We note that if the boundary C_0 of D_0 is sufficiently smooth and w is continuous on C_0 the function h in (9.6) admits the representation

$$(9.7) \qquad h(z) = \frac{1}{2\pi i}\int_{C_0} \frac{h(\zeta)d\zeta}{\zeta - z}.$$

Equation (9.6) may be considered as an integral equation for the unknown function $w(z)$.

(iii) *Equation* (9.6) *has a unique solution w for every bounded continuous h.*

This last theorem is due to Vekua [81] who also investigated in detail the resolvent of equation (9.6).[9]

The hypotheses under which we stated these theorems can be considerably relaxed. First of all, there is no need to assume that the partial derivatives of F and G (or, what is the same, the characteristic coefficients) are Hölder continuous, though if one drops this assumption the derivatives occurring in equations (9.1)–(9.4) must be understood in some generalized sense. Vekua considered the case of continuous a, b. The continuity assumption, however, is still too restrictive, and has the disadvantage of being nonhereditary. If the generators F, G have continuous partial derivatives, the same is not necessarily true for other (F, G) pseudoanalytic functions of the first kind; and the continuity of a, b does not insure the continuity of A, B. A sufficient and hereditary smoothness condition is the following: the characteristic coefficients a, b are measurable and are absolutely integrable to a power $p>2$ in the neighborhood of every point of D_0. An example (due to Soboleff and published by Vekua [83]) shows that the condition $p>2$ can not be weakened.

The assumption that the domain and the characteristic coefficients be bounded is also too severe. Let $\rho(z)$ be a measurable function defined in the whole plane such that $|\rho(z)|^p$ is locally integrable for some $p>1$. We call ρ *admissible* if there exist constants $M>0$ and ϵ, $0<\epsilon<1$ such that

$$\iint \frac{|\rho(\zeta)|\,d\xi d\eta}{|\zeta - z|} \leqq \frac{M}{1+|z|^{1+\epsilon}},$$

$$\iint \frac{|\rho(\zeta)|\,d\xi d\eta}{|\zeta - z_1|\,|\zeta - z_2|} \leqq \frac{M}{|z_1 - z_2|^{1-\epsilon}},$$

the integration being performed over the whole plane. A function $\rho(z)$ defined in a domain D_0 is called admissible if it becomes so upon setting $\rho=0$ in the complement of D_0.

If $\rho(z)$ vanishes outside a large circle, it is admissible if and only if there exists a $K>0$ such that

$$\iint_{|\zeta-z|<r<1} |\rho(\zeta)|\,d\xi d\eta \leqq K r^{2-\epsilon}.$$

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[9] Vekua assumed some regularity of the boundary of D. For the general case see I. Polonsky's forthcoming thesis.

A sufficient condition (Vekua) is that ρ belong to L_p, $p>2$.

Theorems (i)–(iii) *are valid if the characteristic coefficients* a, b *are admissible.*

The following result [15] shows that we could have defined pseudoanalyticity by means of equation (9.3).

(iv) *If the functions* $a(z)$, $b(z)$ *are admissible in* D_0, *there exists a generating pair* (F, G) *in* D_0 *for which* a *and* b *are the first two characteristic coefficients.*

The proof of (iv) depends on the similarity principle stated below.

10. **Similarity principle.** Let (F, G) be a generating pair defined in a domain D_0. Under certain conditions the following theorem [10; 11; 81; 84] holds for every subdomain $D \subset D_0$.

(v) (*Similarity principle for a general domain*). (a) *Let* $w(z)$ *be a single-valued* (F, G) *pseudoanalytic function of the first kind defined in a domain* D. *Then there exists an analytic function* $f(z)$, *and a complex-valued function* $s(z)$ *defined on the closure of* D *and having a bound and a Hölder modulus of continuity depending only on* (F, G), *such that*

$$w(z) = e^{s(z)}f(z). \tag{10.1}$$

(b) *Let* $f(z)$ *be a single-valued analytic function defined in a domain* D. *There exists a function* $s(z)$ *satisfying the conditions stated above such that the function* $w(z)$ *defined by* (10.1) *is* (F, G) *pseudoanalytic.*

A sufficient condition for the validity of the similarity principle is the existence of admissible characteristic coefficients a, b. The proof of statement (a) is then almost trivial, since the function s can be written down explicitly (modulo an analytic function):

$$s(z) = \frac{1}{\pi}\iint_D \left\{a(\zeta) + b(\zeta)\frac{\overline{w(\zeta)}}{w(\zeta)}\right\}\frac{d\xi d\eta}{\zeta - z}. \tag{10.2}$$

To prove statement (b) one may consider the nonlinear integral equation which must be satisfied (modulo an analytic function) by the function s. It reads

$$s(z) = \frac{1}{\pi}\iint_D \left\{a(\zeta) + b(\zeta)e^{\overline{s(\zeta)} - s(\zeta)}\frac{\overline{f(\zeta)}}{f(\zeta)}\right\}\frac{d\xi d\eta}{\zeta - z}. \tag{10.3}$$

Since s enters into the right hand side only in the form $e^{\bar{s}-s}$ it is easy to show, using the Schauder fixed point theorem, that this equation is solvable. Another way to prove statement (b) is to consider the linear integral equation, similar to equation (9.6), which must be satisfied by the function e^s.

Under the same conditions we can prove:

(vi) (*Similarity principle with a boundary condition*) [15]. *Assume that the boundary of D contains a simple closed Lyapounoff curve*[10] *C. Then theorem* (iv) *remains valid if one requires that $s(z)$ be real on C and vanish at a given point on C.*

An extension of this theorem to the case of several contours C is possible if one considers not single-valued functions but functions with a single-valued argument (Lee [51]).

The similarity principle may fail for a trivial reason, since it may happen that one of the generators, say $F(z)$, can not be represented in the form $e^s f$, with a bounded s and an analytic f. It is an important *open problem* to find necessary and sufficient conditions for the validity of the similarity principle.

Partial extensions of part (a) of the similarity principle to the case of nonadmissible coefficients have been considered by Vekua [84] and by C. Morawetz.[11] A refinement of the similarity principle in the case in which the behaviour of the characteristic coefficients is known at a point is due to N. Newman.[11] A quite different and far-reaching extension of statement (a) occurs in the theory of elliptic equations with discontinuous coefficients developed by Nirenberg and the writer (cf. §15 below).

Assume now that for the generators considered the similarity principle holds. Then we can at once transfer a large number of classical function-theoretical results to the case of pseudoanalytic functions. Thus, according to statement (b), there exist in every domain pseudoanalytic functions with prescribed zeros and poles, and every domain is an existence domain for a pseudoanalytic function which can not be continued into any larger domain. Using statement (a), on the other hand, we can extend to pseudoanalytic functions such theorems on bounded analytic functions as Jensen's inequality, Blashke's theorem on the location of zeros, Fatou's theorem on the existence of radial limits almost everywhere, the uniqueness theorems of F. and M. Riesz and of Privaloff and Lusin, etc. The similarity principle with a boundary condition may be used for the solution of boundary value problems, for the construction of Green and Neumann functions, etc.

The most striking application of the similarity principle, however, is to global formal powers and power series.

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[10] A Lyapounoff curve is a curve $z=z(t)$ with $z'(t)\neq 0$ and $z'(t)$ satisfying a uniform Hölder condition.

[11] To appear.

11. Expansion and approximation theorems. In this section we consider a generating pair (F, G) defined on the Riemann sphere for which the similarity principle holds. This condition will be satisfied, for instance, if the functions $F(z)$, $G(z)$, $F(1/z)$, $G(1/z)$ have continuous partial derivatives for $|z| < +\infty$. We consider the global formal powers[12] and apply to them part (a) of the similarity principle. Noting relation (5.5) and the fact that $Z^{(n)}(\alpha, z_0; z)$ is $O(|z|^n)$ at infinity we conclude that $Z^{(n)}(\alpha, z_0; z) = e^{s(z)}\beta(z-z_0)^n$ where the constant β is such that $e^{s(\infty)}\beta = \alpha$. Hence

$$e^{-2K} \leq \left| \frac{Z^{(n)}(\alpha, z_0; z)}{\alpha(z-z_0)^n} \right| \leq e^{2K} \tag{11.1}$$

where K is the bound on $|s|$ which depends only on (F, G). This inequality shows that the global formal power series

$$\sum_{n=0}^{\infty} Z^{(n)}(\alpha_n, z_0; z) \tag{11.2}$$

converges within the circle of convergence of the ordinary power series $\sum \alpha_n(z-z_0)^n$ and diverges outside of this circle. Moreover, we have the following theorem (due to Agmon and the author [1]).

If $w(z)$ is an (F, G) pseudoanalytic function defined for $|z-z_0| < r \leq +\infty$, then the Taylor series of w formed with global formal powers $Z^{(n)}(\alpha, z_0; z)$ converges (and hence represents the function) for $|z-z_0| < r$.

We shall sketch the proof as an example of the application of the similarity principle. Consider first the special function

$$w(z) = Z^{(-1)}(\beta, \zeta; z) \tag{11.3}$$

and denote its Taylor coefficients at a point $z_0 \neq \zeta$ by α_n. According to (5.4) and (5.8) we have

$$\alpha_n = (-1)^n Z_n^{(-1-n)}(\beta, \zeta; z_0), \tag{11.4}$$

but here we make no use of these relations. For every n the function

$$W_n(z) = Z^{(-1)}(\beta, \zeta; z) - \sum_{\nu=0}^{n-1} Z^{(\nu)}(\alpha_\nu, z_0; z) \tag{11.5}$$

is a rational (F, G) pseudoanalytic function which has no poles except at $z = \zeta$ and $z = \infty$, and satisfies the relations

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[12] Under the hypotheses stated above global formal powers can be constructed independently of the procedure described in §§5, 6, by means of part (b) of the similarity principle.

$$(11.6)\quad W_n(z) \sim \beta(z-\zeta)^{-1},\ z\to\zeta;\qquad W_n(z)\sim \alpha_n(z-z_0)^n,\qquad z\to z_0;$$
$$W_n(z) = O(|z|^n),\qquad z\to\infty.$$

By the similarity principle we have

$$(11.7)\qquad W_n(z) = e^{s(z)}\gamma\frac{(z-z_0)^n}{z-\zeta},\qquad \gamma = \text{const.}$$

where $|s| \leq K$, and using the first two relations (11.6) we conclude that $\beta = \gamma e^{s(\zeta)}(\zeta-z_0)^n$, $\alpha_n = \gamma e^{s(z_0)}(z_0-\zeta)^{-1}$, so that

$$(11.8)\qquad |\alpha_n| \leq e^{2K}|\beta|\,|\zeta - z_0|^{-n-1},\qquad n = 0, 1, \cdots.$$

Thse inequalities imply that the expansion

$$(11.9)\qquad Z^{(-1)}(\beta,\zeta;z) = \sum_{n=0}^{\infty} Z^{(n)}(\alpha_n, z_0; z)$$

converges for $|z-z_0|<|\zeta-z_0|$. The expansion theorem being established for the case of the special function (11.3), the transition to arbitrary functions is accomplished, just as in the classical case, by means of the Cauchy integral formula (6.4) applied to a circle.

Similar arguments yield the more general Laurent expansion.

Let $w(z)$ *be a single-valued* (F, G) *pseudoanalytic function defined in the domain*

$$(11.10)\qquad 0\leq r<|z-z_0|<R\leq\infty.$$

Then

$$(11.11)\qquad w(z) = \sum_{n=-\infty}^{+\infty} Z^{(n)}(\alpha_n, z_0; z)$$

the series being uniformly and absolutely convergent in every annulus

$$r<r'\leq|z-z_0|\leq R'<R.$$

As a corollary we obtain an expansion theorem for multiple-valued functions.

If $w(z)$ *is an* m*-valued* (F, G) *pseudoanalytic function defined in the domain* (11.10), *then it admits in* (11.10) *the unique expansion*

$$(11.12)\qquad w(z) = \sum_{n=-\infty}^{+\infty} Z^{(n/m)}(\alpha_n, z_0; z).$$

Finally, the expansion theorem in conjunction with the Cauchy formula yields the analogue of Runge's theorem [12].

A single-valued (F, G) *pseudoanalytic function defined in a domain* D

can be expanded into a series of rational pseudoanalytic functions which converges uniformly and absolutely on every compact subset of D. If D is simply connected, w can be expanded into a series of formal polynomials (entire rational pseudoanalytic functions).

Summarizing we see that in the case of sufficiently smooth generators (F, G), defined over the Riemann sphere, (F, G) pseudoanalytic functions can be characterized by each of the following properties: (i) having (F, G) derivatives, (ii) being (F_{-1}, G_{-1}) derivatives, (iii) being (F, G) integrals of (F_1, G_1) pseudoanalytic functions, (iv) satisfying the generalized Cauchy-Riemann equation (3.2), (v) satisfying the generalized Cauchy-Riemann equation (9.3) (vi) obeying the generalized Cauchy formula (6.4) or (6.5), (vii) admitting an expansion in a formal power series in every disc of regularity, (viii) being the uniform limit of a sequence of formal polynomials in every simply connected domain of regularity. The analogy with classical function theory is now complete.

(*Added in proof* (August 2, 1956). For generators (F, G) defined on the Riemann sphere one can prove, without assuming differentiability or the similarity principle, that the expansion (11.2) converges and represents the function for $|z-z_0| < \theta r$, where $\theta > 0$ is a constant depending only on (F, G). The proof [88] depends on the reciprocity relations (6.6) and on the Calderón-Zygmund theory of singular integrals [89]. Instead of the inequalities (11.1) one has now the weaker inequalities

$$e^{-|n|K} \leqq \left| \frac{Z^{(n)}(\alpha, z_0 : z)}{\alpha(z - z_0)^n} \right| \leqq e^{|n|K} \tag{11.13}$$

where the constants K, depend on (F, G) and on ν.)

IV. APPLICATIONS

12. Reduction of linear elliptic equations to pseudoanalytic functions. In two dimensions every harmonic function can be expressed in terms of analytic functions of a complex variable, in two ways. If $h(x, y)$ satisfies Laplace's equation, then it is the real part of an analytic function $f(z)$. Also, the complex gradient $g(z) = h_x - ih_y$ is analytic and $g(z) = f'(z)$. A similar relation exists between pseudoanalytic functions and solutions of linear elliptic equations of second order, subject to *very mild* smoothness assumptions [11; 12; 15; 19].

We assume at first that all equations considered have Hölder continuous coefficients.

(a) The simplest linear elliptic equation is of the form

$$\phi_{xx} + \phi_{yy} + A_1\phi_x + A_2\phi_y = 0. \tag{12.1}$$

Assume that the coefficients A_1, A_2 are admissible in the domain considered (cf. §9). Using Theorem (iv) we can find functions $\sigma > 0$ and τ such that (12.1) can be obtained by elimination from the elliptic system

$$\phi_x = \tau\psi_x + \sigma\psi_y, \qquad \phi_y = -\sigma\psi_x + \tau\psi_y. \tag{12.2}$$

This system may be written in the form (3.2) with

$$F = 1, \qquad G = -\tau + i\sigma. \tag{12.3}$$

Thus every solution ϕ of (12.1) is the real part of an (F, G) pseudoanalytic function of the second kind, and vice versa. On the other hand, the complex gradient of ϕ, $W = \phi_x - i\phi_y$ satisfies the equation

$$W_{\bar z} = a_1 W + \bar{a}_1 \bar{W}, \qquad a_1 = -(A_1 + iA_2)/4.$$

By Theorem (iv) there exists a generating pair (F_1, G_1) such that every solution of this equation is (F_1, G_1) pseudoanalytic of the first kind. Finally, the (F, G) derivative of the (F, G) function $w = \phi + \psi G$ is

$$\dot{w} = \phi_x + \psi_x G = W.$$

Hence (F_1, G_1) is a successor of (F, G).

For further reference we note that system (12.1) may also be written in the form

$$\omega_{\bar z} = \nu\bar{\omega}_{\bar z} \tag{12.4}$$

where $\omega = \phi + i\psi$ and

$$\nu = -(F + iG)/(F - iG)$$

so that $|\nu| < 1$.

(b) If the equation

$$A_{11}\phi_{xx} + 2A_{12}\phi_{xy} + A_{22}\phi_{yy} = 0 \tag{12.5}$$

is elliptic, we may assume without loss of generality that

$$A_{11}A_{22} - A_{12}^2 = 1, \qquad A_{11} > 0. \tag{12.6}$$

Consider the Riemann metric

$$A_{22}dx^2 - 2A_{12}dxdy + A_{11}dy^2. \tag{12.7}$$

There exists a homeomorphism $\zeta = \zeta(z) = \xi + i\eta$ of the domain of definition of (12.5) which is conformal with respect to the metric (12.7), that is a solution of the Beltrami equation

$$\zeta_{\bar{z}} = \hat{\mu}\zeta_z, \tag{12.8}$$

where

$$\mu = \frac{A_{22} - A_{11} - 2iA_{12}}{A_{22} + A_{11} + 2}. \tag{12.9}$$

If the coefficients A_{ij} are Hölder continuously differentiable, the functions ξ and η have Hölder continuous second derivatives and introducing them as new independent variables we transform equation (12.5) into the canonical form (12.1).

If the A_{ij} are only Hölder continuous, the reduction to pseudoanalytic functions is still possible. On the one hand, we can find an elliptic system of the form

$$\phi_x = b_{11}\psi_x + b_{12}\psi_y, \qquad \phi_y = b_{21}\psi_x + b_{22}\psi_y \tag{12.10}$$

which is equivalent to the single equation (12.5). On the other hand, the complex gradients $\phi_x - i\phi_y = U + iV$ of a solution of (12.5) satisfy the elliptic system

$$A_{11}U_x = 2A_{12}V_x + A_{22}V_y, \qquad U_y = -V_x. \tag{12.11}$$

The transformation $\zeta = \zeta(z)$ takes both systems into equations of the form (12.4). Thus $\phi + i\psi$ and $\phi_x - i\phi_y$ are pseudoanalytic functions of ζ (with different generators).

(c) Solutions of an elliptic equation

$$A_{11}\phi_{xx} + 2A_{12}\phi_{xy} + A_{22}\phi_{yy} + A_1\phi_x + A_2\phi_y = 0, \tag{12.12}$$

with A_{ij} satisfying (12.6), can be expressed by pseudoanalytic functions if the coefficients A_{ij} are Hölder continuously differentiable, for in this case the mapping $\zeta(z)$ described above takes (12.12) into an equation of the form (12.1).

If the coefficients are only Hölder continuous, the reduction is possible locally. In fact, in the neighborhood of any point we can transform equation (12.12) to the form (12.5) by introducing as new independent variables two solutions of (12.12).

(d) The most general linear homogeneous elliptic equation

$$A_{11}\phi_{xx} + 2A_{12}\phi_{xy} + A_{22}\phi_{yy} + A_1\phi_x + A_2\phi_y + A_0\phi = 0 \tag{12.13}$$

can be transformed into an equation of the form (12.12) in every domain in which it has a positive solution ϕ_0. This is accomplished by introducing the new unknown function ϕ/ϕ_0. A solution $\phi_0 > 0$ will exist in every small domain, and in every large domain if $A_0 \leqq 0$.

(e) We consider next a first order elliptic system in normal form

(12.14) $u_x = \tau v_x + \sigma v_y + b_{11}u + b_{12}v, \qquad -u_y = \sigma v_x - \tau v_y + b_{21}u + b_{22}v$

with $\sigma > 0$. Setting

(12.15) $$u + iv = w,$$

we write this system in the form

(12.14′) $$w_{\bar z} = \nu \bar w_{\bar z} + \alpha w + \beta \bar w$$

where $|\nu| < 1$.

If $\alpha = \beta = 0$, this equation shows that w is a pseudoanalytic function of the second kind (cf. (a)). If α and β do not vanish the reduction to pseudoanalytic functions is possible in the large if ν is Hölder continuously differentiable, and locally if ν is Hölder continuous.

In the first case set $W = w - \nu \bar w$. Equation (12.14′) takes the form

(12.16) $$W_{\bar z} = aW + b\bar W.$$

Hence (cf. Theorem (iv) of §9) W is a pseudoanalytic function of the first kind.

In the second case we can find near any given point two solutions w_1 and w_2 of (12.14′) such that $\operatorname{Im}(\bar w_1 w_2) > 0$. The new unknown function W defined by the relation

$$w = W(w_1 - iw_2) + \bar W(w_1 + iw_2)$$

satisfies an equation of the form (12.16) and is therefore pseudoanalytic.

The general linear homogeneous system

(12.17) $$\begin{aligned} u_x &= a_{11}v_x + a_{12}v_y + b_{11}u + b_{12}v, \\ -u_y &= a_{21}v_x + a_{22}v_y + b_{21}u + b_{22}v \end{aligned}$$

is called elliptic if

(12.18) $$4a_{12}a_{21} - (a_{11} + a_{22})^2 > 0, \qquad a_{12} > 0.$$

In this case it is equivalent to a single complex equation for the function (12.15),

(12.17′) $$w_{\bar z} = \mu w_z + \nu \bar w_{\bar z} + \alpha w + \beta \bar w$$

with

(12.18) $$|\mu| + |\nu| < 1.$$

Let $\zeta = \zeta(z)$ be a homeomorphism conformal with respect to the metric of (12.17), that is, a univalent solution of the Beltrami equation (12.8) with

$$(12.19)\qquad \hat{\mu} = \frac{a_{12} - a_{21} - i(a_{11} + a_{22})}{a_{12} + a_{21} + [4a_{11}a_{22} - (a_{11} + a_{22})^2]^{1/2}}\,.$$

The mapping $z \to \zeta$ takes equation (12.17′) into the canonical form (12.14′).

The preceding results may be extended by noting that there was no need to assume the coefficients A_1, A_2, A_0, b_{ij}, α, β to be Hölder continuous. It would suffice to require that they be measurable functions belonging to L_p for some $p > 2$. In this case the second derivatives in equations (12.1), (12.12) and (12.13) and the first derivatives in equations (12.14), (12.16) and (12.17) are to be understood in a generalized sense.

13. **An example. Equations of mixed type.** The concepts of (F, G) differentiation and integration were first formulated, by Gelbart and the author [21; 22], for systems of the special form

$$(13.1)\qquad \sigma(x)\phi_x = \tau(y)\psi_y, \qquad \sigma(x)\phi_y = -\tau(y)\psi_x.$$

Systems of this form occur frequently in mechanics of continua. For $\sigma = 1$, $\tau = y^p$, one obtains the equations of Weinstein's generalized axially symmetric potential theory [85].

System (13.1) is equivalent to equation (3.2) for the generating pair

$$(13.2)\qquad F = \sigma(x)^{1/2}\tau(y)^{-1/2}, \qquad G = i\sigma(x)^{-1/2}\tau(y)^{1/2}$$

where we assume that $\sigma > 0$, $\tau > 0$. If (ϕ, ψ) is a solution of (13.1) we may form the functions

$$(13.3)\qquad \phi' = \sigma\phi_x, \qquad \psi' = -\phi_y/\sigma$$

and

$$(13.4)\qquad \Phi = \int \sigma\phi dx - \tau\psi dy, \qquad \Psi = \int (\psi/\sigma)dx + (\phi/\tau)dy,$$

the path independence of the integrals being a consequence of (13.1). Clearly, (ϕ', ψ') and (Φ, Ψ) are solutions of the system

$$(13.5)\qquad \phi_x/\sigma(x) = \tau(y)\psi_y, \qquad \phi_y/\sigma(x) = -\tau(y)\psi_x.$$

In other words: if (13.1) holds, the function

$$w = (\sigma/\tau)^{1/2}\phi + i(\tau/\sigma)^{1/2}\psi = \phi F + \psi G$$

is (F, G) pseudoanalytic, and its (F, G) derivative

$$\dot{w} = (\sigma/\tau)^{1/2}(\phi_x - i\phi_y) = \phi' F_1 + \psi' G_1$$

is (F_1, G_1) pseudoanalytic, where

$$F_1 = \sigma(x)^{-1/2}\tau(y)^{-1/2}, \qquad G_1 = i\sigma(x)^{1/2}\tau(y)^{1/2}.$$

The function

$$W = (\sigma\tau)^{-1/2}\Phi + i(\sigma\tau)^{1/2}\Psi = \Phi F_1 + \Psi G_1$$

is (F_1, G_1) pseudoanalytic, and its (F_1, G_1) derivative is w. Thus (F_1, G_1) is a successor and a predecessor of (F, G) and

$$\cdots, (F_1, G_1), \qquad (F, G), \qquad (F_1, G_1), \qquad (F, G), \cdots$$

is a generating sequence.

The local formal powers determined by this sequence are easily found. For the sake of simplicity we put the center at the origin and set

$$_*Z^{(n)}(a, 0; z) = {}_*Z^{(n)}(a, z), \qquad (\mathrm{mod}\ F, G).$$

Define the functions $X_j^{(n)}(x)$, $Y_j^{(n)}(y)$, $j=1, 2$; $n=0, 1, \cdots$ by the recurrence relations:

$$\begin{gathered} X_j^{(0)} = Y_j^{(0)} = 1, \qquad X_j^{(n)}(0) = Y_j^{(n)}(0) = 0 \\ \frac{dX_j^{(n)}}{dx} = \begin{cases} n\sigma X_j^{(n-1)}, & n+j \text{ even}, \\ nX_j^{(n-1)}/\sigma, & n+j \text{ odd}, \end{cases} \\ \frac{dY^{(n)}}{dy} = \begin{cases} n\tau Y_j^{(n-1)}, & n+j \text{ even}, \\ nY_j^{(n-1)}/\tau, & n+j \text{ odd}. \end{cases} \end{gathered} \tag{13.6}$$

Then

$$_*Z^{(n)}(\alpha + i\beta, z) = \sum_{\nu=0}^{n} i^{\nu}\binom{n}{\nu}\left[\alpha X_k^{(n-\nu)}Y_l^{(\nu)} + i\beta X_l^{(n-\nu)}Y_k^{(\nu)}\right] \tag{13.7}$$

where $k=2$, $l=1$ if n is odd and $k=1$, $l=2$ if n is even.

Since equations (13.1) may be solved by separation of variables, functions pseudoanalytic with respect to the generators (13.2) have many properties not encountered in the general theory [23; 9]. Of principal importance is the fact that the formalism described above remains valid also for a system of the form

$$\sigma_1(x)\phi_x = \tau_1(y)\psi_y, \qquad \sigma_2(x)\phi_y = -\tau_2(y)\psi_x \tag{13.8}$$

provided one replaces, in (13.3)–(13.6)

$$\sigma \text{ by } \sigma_2, \qquad \frac{1}{\sigma} \text{ by } \frac{1}{\sigma_1}, \qquad \tau \text{ by } \tau_2, \qquad \frac{1}{\tau} \text{ by } \frac{1}{\tau_1}.$$

If the coefficients σ_1, σ_2, τ_1, τ_2 are positive, system (13.8) may be reduced to the form (13.1) by changing independent variables. But the formalism works also for *hyperbolic* and *elliptico-hyperbolic* systems (13.8), that is, if some of the coefficients are negative or change sign. In the latter case, of course, some of the integrals defining the functions (13.6) may diverge.

We note in particular the system

$$\phi_x = \psi_y, \qquad \phi_y = -K(y)\psi_x, \qquad yK(y) > 0. \tag{13.9}$$

Elimination of ϕ leads to the elliptico-hyperbolic equation of Tricomi-Chaplygin type

$$K(y)\psi_{xx} + \psi_{yy} = 0 \tag{13.10}$$

which is of paramount importance in transonic gas dynamics. The particular solutions (13.7) have been used in studying initial [10] and boundary value problems (Morawetz [58]) for equation (13.10). In the Tricomi case ($K(y) = y$) these particular solutions are ordinary polynomials in x and y.

14. Applications of pseudoanalytic functions. These applications are all based on the fact noted in §12. In two dimensions the theory of second order elliptic equations with, say, Hölder continuous coefficients (and of systems of two first order equations), is essentially *identical* with that of pseudoanalytic functions.[13] Hence every result described in §§2–11 can be reformulated as a theorem on differential equations, in which the concept of pseudoanalyticity is not even mentioned. One obtains in this way very precise information on the nature of solutions defined in the whole plane, on zeros and singularities, on boundary behavior of solutions (Chang [30]) and on the uniqueness of solution of the Cauchy problem [15]. The similarity principle of §10, which is a general structure and existence theorem, plays a decisive part in this and other applications; in particular, it leads to the construction of Green's functions [15] and analogues of harmonic measures (Lee [51]). The similarity principle and the integral equation (9.6) have been applied also to boundary value problems (Vekua [81], and others).

Even when the same result can be proved directly, the use of pseudoanalytic functions may be advantageous. For instance, it is known that solutions of elliptic equations may be expressed, using the so-called fundamental solution, in terms of Cauchy data on the boundary (Bergman [8], Položii [70], Šabat [75]). But our derivation of the "Cauchy formula" (6.4) gives this representation under very weak hypotheses and, instead of relying on the existence of the fundamental solutions, yields a simple existence proof.

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[13] On the other hand, the theory of pseudoanalytic functions has few points of contact with the methods of Bergman [7] and Vekua [80] who use integral operators to represent solutions of analytic elliptic equations.

As another example we mention a lemma of H. Lewy which has several applications in differential geometry in the large. Let $\phi(x, y)$ satisfy a nonlinear elliptic differential equation

$$\Omega(x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy}) = 0. \tag{14.1}$$

Assume that $\Omega(x, y, 0, 0, 0, 0, 0, 0) = 0$ and that ϕ vanishes at the origin together with its second derivatives. Then either ϕ is a linear function, or $\phi_{xy}^2 - \phi_{xx}\phi_{yy} > 0$ for $0 < x^2 + y^2 < \epsilon$.

Lewy [52] proved this for an analytic equation, Hartman and Wintner [41] for Ω of class C^2. An application of pseudoanalytic functions yields the lemma under essentially sharp conditions: Hölder continuous differentiability of Ω [19].

There also exist other applications to nonlinear elliptic equations [14]. But the most interesting of these refer to the equations of gas dynamics.

The velocity potential ϕ, stream-function ψ and density ρ of a two-dimensional flow are connected by the equations

$$\rho\phi_x = \psi_y, \qquad \rho\phi_y = -\psi_x \tag{14.2}$$

which show that $\phi + i(\psi/\rho)$ is an $(1, i/\rho)$ pseudoanalytic function of the first kind; the $(1, i/\rho)$ derivative of this function is the complex velocity $\phi_x - i\phi_y$. In gas dynamics one assumes ρ to be a given function of $q = |\phi_x - i\phi_y|$. In this case equations (14.2) are nonlinear, but the hodograph transformation leads to a linear system of the form (13.1). Pseudoanalytic function techniques have been used in establishing the existence and uniqueness of subsonic flows past given profiles [16], the existence of compressible Helmholtz flows (Berg, [5]), the Kutta-Joukowsky theorem for gas flows (Newman[11]), and the infinitesimal nonexistence theorem for transonic flows (Morawetz [57]).

V. EXTENSIONS

15. Elliptic equations with discontinuous coefficients. While the theory of pseudoanalytic functions yields a complete description of elliptic equations (12.13) and elliptic systems (12.1) with smooth, say Hölder continuous, coefficients it is important, especially in view of nonlinear problems, to consider also equations with merely measurable coefficients.

System (12.17′) is called *uniformly elliptic* if the coefficients are measurable functions and there exists a constant k such that

$$|\mu| + |\nu| \leqq k < 1. \tag{15.1}$$

We also assume that

$$|\alpha| + |\beta| \leqq k' < +\infty,$$

though this condition can be weakened. In general there will be no continuously differentiable solution. A solution $w = u + iv$ of (12.17′) is required only to be continuous and to have strong L_2 derivatives in the sense of Soboleff and Friedrichs. The basic result is the following representation theorem due to Nirenberg and the author [24].

Let $w(z)$ *be a solution of* (12.17′) *defined in a subdomain* D *of the unit disc. Then* $w(z)$ *may be written in the form*

$$w(z) = e^{s(z)} f[\chi(z)] \tag{15.2}$$

where $s(z)$ *is continuous for* $|z| \leqq 1$, *real on* $|z| = 1$ *and vanishes at* $z = 0$, $\zeta = \chi(z)$ *is a homeomorphism of* $|z| \leqq 1$ *onto* $|\zeta| \leqq 1$ *such that* $\chi(0) = 0$, $\chi(1) = 1$, *and* $f(\zeta)$ *is analytic on* $\Delta = \chi(D)$. *The functions* $s(z)$, $\chi(z)$ *and* $z = \chi^{-1}(\zeta)$ *satisfy uniform Hölder conditions depending only on* k *and* k'.

If $\alpha \equiv \beta \equiv 0$, (15.3) holds with $s \equiv 0$. This result is contained in the work of Morrey [59]. If $\mu \equiv \nu \equiv 0$, (15.2) holds with $\chi(z) \equiv z$, and we obtain the similarity principle of §9.

The representation theorem implies that unless $w \equiv 0$, the zeros of $w(z)$ are isolated and not of infinite order. Thus solutions of (12.17) always have the unique continuation property. The representation (15.3) also permits us to classify isolated singularities into poles and essential singularities. But such precise asymptotic formulas as the ones given in §4, let alone generalized differentiation, are out of the question.

Equation (12.13) is called uniformly elliptic if the coefficients are measurable, satisfy (12.6), and are uniformly bounded (the latter condition can be weakened for the coefficients A_1, A_2 and A_0). If ϕ satisfies a uniformly elliptic equation (12.12), $W = \phi_x - i\phi_y$ satisfies a uniformly elliptic equation (12.17′) and obeys the representation theorem. This leads, in particular, to the strong maximum principle for equation (12.12), and also for equation (12.13) with $A_0 \geqq 0$. (E. Hopf's well known proof [44] of the maximum principle does not work for discontinuous coefficients.)

The most important consequences of the representation theorem, however, are strong a priori estimates which permit one to treat boundary value problems for nonlinear elliptic equations [25].

16. Elliptic systems of order $2m>2$ in two dimensions. Sobrero [77] observed that solutions of the biharmonic equation $\Delta^2\phi(x, y)=0$ can be expressed in terms of analytic functions on a certain algebra. More generally, solutions of $\Delta^m\phi(x, y)=0$ can be expressed in terms of analytic functions on an algebra A_m, an algebra over the real field generated by a unit j such that $(1+j^2)^m=0$, which is isomorphic to the algebra over the complex field generated by a unit ω with $\omega^m=0$.

Diaz [31] considered pseudoanalytic functions of $x+jy$, with values in A_m, defined by certain special systems of $2m$ first order equations for $2m$ unknown functions. This system yields, for each function, an elliptic equation of the form

$$L^m\phi = 0,$$

L being an elliptic operator which may be reduced to the form

$$L\phi = \Delta\phi + a(x)\phi_x + b(y)\phi_y.$$

To these functions Diaz extended the formalism of §13 (differentiation, integration, and formal powers and power series).

The algebra A_m also plays a part in the more general investigations of Douglis [32; 34]. Douglis obtained a normal form for the general linear elliptic system of $2m$ first order equations and pointed out a special case of these systems, called generalized Beltrami systems, solutions of which can be expressed in terms of "hyper-analytic" functions. The theory of hyper-analytic functions bears a very close resemblance to classical function theory, since these functions themselves form an algebra. For $m=1$ one obtains ordinary Beltrami systems which can be transformed into the Cauchy-Riemann equations. It is remarkable that for $m>1$ a generalized Beltrami system can not, in general, be transformed into a system with constant coefficients.

Douglis also stated the problem of developing a theory of pseudoanalytic functions applicable to the general case (that is, to non-Beltrami systems). But this problem is still open. The chief obstacle is the lack of a Carleman type theorem asserting the nonexistence of zeros of infinite order. (The closely related but weaker theorem on the unique solvability of Cauchy's problem has been proved; cf. Carleman [29], Douglis [33], Hartman and Wintner [40].)

17. The higher-dimensional case. During the last years spectacular progress has been achieved in the theory of elliptic equations and systems of order $2m \geqq 2$ in a space of $n>2$ dimensions. But as far as the deeper function-theoretical properties of solutions are concerned little is known, even in the classical case $2m=2$.

It is very likely that a solution of any homogeneous elliptic equation or system cannot vanish of infinite order at a point without vanishing identically. The statement is obvious for equations with analytic coefficients, since such equations have only analytic solutions (John [45]); the difficulty lies in the nonanalytic case. For $n=2$, $2m=2$ the answer is known, as we have seen above. For $n>2$ this unique continuation theorem has been proved thus far (by Müller [64] and Heinz [43]) for equations of the form

$$\Delta\phi+\sum_{i=1}^{n}a_i\frac{\partial\phi}{\partial x_i}+a_0\phi=0 \tag{17.1}$$

where Δ is the Euclidean Laplace operator.

(*Added in proof* (August 2, 1956). Aronszajn [87] proved the unique continuation theorem for second order elliptic equations with sufficiently smooth coefficients. Another proof is due to Cordes (to appear). The weaker form of the unique continuation theorem, i.e. the unique solvability of the Cauchy problem, was also proved (for $2m=2$, $n>2$) by Landis [91] and by Pedersen (to appear). New proofs of the Müller-Heinz result were given by Hartman and Wintner [90] and by Lax (to appear). Nirenberg proved the uniqueness theorem for the Cauchy problem for certain equations with $2m>2$ and $n\geqq 2$, under restrictive assumptions on the shape of the initial surface (to appear).)

About zeros of finite order there exists a rather general result [17]. If $\phi(x)=\phi(x_1,\cdots,x_n)$ satisfies an elliptic equation

$$L\phi=\sum_{0\leqq i_1+\cdots+i_n\leqq 2m}a_{i_1\cdots i_n}(x)\frac{\partial^{i_1+\cdots+i_n}\phi(x)}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}=0 \tag{17.2}$$

with Hölder continuous coefficients, and vanishes at $x=0$, but not of infinite order, then

$$\phi(x)=p(x)+o(|x|^N) \tag{17.3}$$

where $p(x)$ is a homogeneous polynomial of degree N and $p(x)\neq 0$. This polynomial satisfies the "osculating" equation with constant coefficients, $L_0p=0$, L_0 being obtained from L by replacing $a_{i_1}\cdots_{i_n}(x)$ by $a_{i_1}\cdots_{i_n}(0)$ if $i_1+\cdots+i_n=2m$ and by 0 otherwise.

If $\phi(x)$ is defined in a deleted neighborhood of the origin and $\phi(x)=O(|x|^{2m-n})$, then

$$\phi(x)=cJ(x)+o(|x|^{2m-n}), \tag{17.4}$$

$J(x)$ being a fundamental solution of the osculating equation. If $\phi(x)=o(|J(x)|)$, the singularity at $x=0$ is removable. Also, the asymptotic relations (17.3), (17.4) may be formally differentiated $2m$ times. (Much more is known about singularities in the analytic case [45].)

The parallelism between these results and those described in §4 suggests strongly that analogues of formal powers and corresponding expansion theorems could be found for general linear elliptic equations. For second order equations of the special form

$$\Delta\phi + \sum_{i=1}^{n} a_i(x_i) \frac{\partial \phi}{\partial x_i} = 0$$

this was accomplished by Protter [74] several years ago. The general problem is open.

While the results described in this and the preceding sections are still rather episodic, they all point in the same direction—toward a general function theory of elliptic equations which will relegate pseudoanalytic functions to the position of a special case.

BIBLIOGRAPHY

1. S. Agmon and L. Bers, *The expansion theorem for pseudoanalytic functions*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 757–764.

2. L. Ahlfors, *On quasi-conformal mappings*, Journal d'Analyse Mathématique vol. 4 (1954) pp. 1–58.

3. E. Beltrami, *Sulle funzioni potenziali di sistemi simmetrici intorno ad un asse*, Opere matematice, Vol. 3, Milano, 1911, pp. 115–128.

4. ———, *Sulla teoria delle funzioni potenziali simmetriche*, ibid., pp. 349–377.

5. P. W. Berg, *On the existence of Helmholtz flows of a compressible fluid*, Dissertation, New York University, 1953.

6. S. Bergman, *Functions satisfying certain partial differential equations of elliptic type and their representation*, Duke Math. J. vol. 14 (1947) pp. 349–366.

7. ———, *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155.

8. ———, *A formula for the streamfunction of certain flows*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1943) pp. 276–281.

9. L. Bers, *The expansion theorem for sigma-monogenic functions*, Amer. J. Math. vol. 72 (1950) pp. 705–712.

10. ———, *On the continuation of a potential gas flow across the sonic line*, N.A.C.A. Technical Note No. 2058, 1950.

11. ———, *Partial differential equations and generalized analytic functions*, Proc. Nat. Acad. Sci. U.S.A. vol. 36, (1950) pp. 130–136; *Second note*, Ibid., vol. 37 (1951) pp. 42–47.

12. ———, *Theory of pseudo-analytic functions*, Lecture notes (mimeographed), New York University, 1953.

13. ———, *Univalent solutions of linear elliptic systems*, Comm. Pure Appl. Math. vol. 6 (1953) pp. 513–526.

14. ———, *Nonlinear elliptic equations without nonlinear entire solutions*, Journal of Rational Mechanics and Analysis vol. 3 (1954) pp. 767–787.

15. ———, *Function-theoretical properties of solutions of partial differential equations of elliptic type*, Annals of Mathematics Studies, no. 33, 1954, pp. 69–94.

16. ———, *Existence and uniqueness of a subsonic flow past a given profile*, Comm. Pure Appl. Math. vol. 7 no. 3 (1954) pp. 441–504.

17. ———, *Local behavior of solutions of general linear elliptic equations*, Comm. Pure Appl. Math. vol. 8, no. 4 (1955).

18. ———, *Local theory of pseudoanalytic functions*, Lectures on functions of a complex variable, Michigan University Press, 1955, pp. 213–244.

19. ———, *Remark on an application of pseudoanalytic functions*, Amer. J. Math. vol. 78 (1956) pp. 486–496.

20. ———, *Partial differential equations and pseudoanalytic functions on Riemann surfaces*, Annals of Mathematics Studies, no. 30, 1953, pp. 157–165.

21. L. Bers and A. Gelbart, *On a class of differential equations in mechanics of continua*, Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188.

22. ———, *On a class of functions defined by partial differential equations*, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 67–93.

23. ———, *On generalized Laplace transformations*, Ann. of Math. vol. 48 (1947) pp. 342–357.

24. L. Bers, and L. Nirenberg, *On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications*, Convegno Internazionale sulle Equazioni Derivate e Parziali, Agosto, 1954, pp. 111–140.

25. ———, *On linear and nonlinear elliptic boundary value problems in the plane*, Convegno Internazionale sulle Equazioni Derivate e Parziali, Agosto, 1954, pp. 141–167.

26. W. M. Boothby, *The topology of the level curves of harmonic functions with critical points*, Amer. J. Math. vol. 73 no. 3 (1951) pp. 512–538.

27. R. Cacciopoli, *Fondamenti per una teoria generale delle funzioni pseudoanalitiche di una variabile complessa*, Academia nazionale dei Lincei, Rendiconti vol. 13 (1952) pp. 197–204 and 321–329.

28. T. Carleman, *Sur les systèmes linéaires aux dérivées partielles du premier ordre à deux variables*, C. R. Acad. Sci. Paris vol. 197 (1933) pp. 471–474.

29. ———, *Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes*, Arkiv för Matematik, Astronomi och Fysik vol. 26B No. 17 (1939) pp. 1–9.

30. H. H. Chang, *Approximately analytic functions of bounded type and boundary behavior of solutions of elliptic partial differential equations*, Acta Math. Sinica vol. 3 (1953) pp. 101–132.

31. J. B. Diaz, *On a class of partial differential equations of even order*, Amer. J. Math. vol. 68 (1946) pp. 611–659.

32. A. Douglis, *A function-theoretic approach to elliptic systems of equations in two variables*, Comm. Pure Appl. Math. vol. 6 (1953) pp. 259–289.

33. ———, *Uniqueness in Cauchy problems for elliptic systems of equations*, Comm. Pure Appl. Math. vol. 6 (1953) pp. 291–298.

34. ———, *Function-theoretic properties of certain elliptic systems of first order equations*, Lectures on Functions of a Complex Variable, Michigan University Press, 1953, pp. 335–340.

35. A. Gelbart, *On a function-theory method for obtaining potential flow patterns of a compressible fluid*, N.A.C.A. ARR No. 3G27, 1943.

36. J. J. Gergen and F. G. Dressel, *Mapping by p-regular functions*, Duke Math. J. vol. 18 no. 1 (1951) pp. 185–210.

37. ———, *Mapping for elliptic equations*, Trans. Amer. Math. Soc. vol. 77 (1954) pp. 151–178.

38. ———, *Uniqueness for p-regular mapping*, Duke Math. J. vol. 19 (1952) pp. 435–444.

39. P. Hartman and A. Wintner, *On the third fundamental form of a surface*, Amer. J. Math. vol. 75 (1953) pp. 298–334.

40. ———, *On the local behavior of solutions of non-parabolic partial differential equations*, ibid. (1953) pp. 449–476.

41. ———, *Umbilical points and W-surfaces*, Amer. J. Math. vol. 76 (1954) pp. 502–508.

42. ———, *On the local behavior of solutions of non-parabolic partial differential equations*, II; *The uniqueness of the Green singularity*, Amer. J. Math. vol. 76 (1954) pp. 351–361.

43. E. Heinz, *Über die Eindeutigkeit beim Cauchyschen Anfangswertproblem einer elliptischen Differentialgleichung zweiter Ordnung*, Nachrichten Goettingen, 1955, No. 1, pp. 1–12.

44. E. Hopf, *Elementare Bemerkungen über Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Preuss. Akad. Wiss. Sitzungsber. vol. 19 (1927) pp. 147–152.

45. F. John, *The fundamental solution of linear elliptic differential equations with analytic coefficients*, Comm. Pure Appl. Math. vol. 3 (1950) pp. 273–304.

46. S. Kakutani, *On the family of pseudo-regular functions*, Tôhoku Math. J. vol. 43 (1937) pp. 211–215.

47. W. Kaplan, *Topology of level curves of harmonic functions*, Trans. Amer. Math. Soc. vol. 63 no. 3 (1948) pp. 514–522.

48. A. Kriszten, *Hyperkomplexe und pseudo-analytische Funktionen*, Comment. Math. Helv. vol. 26 (1952) pp. 6–35.

49. M. A. Lavrent'ev, *A general problem of the theory of quasi-conformal representation of plane regions*, Rec. Math. (Mat. Sbornik) N.S. vol. 21 (1947) pp. 285–320.

50. ———, *A fundamental theorem of the theory of quasi-conformal mapping of plane regions*, Izvestya Akademii Nauk SSSR. vol. 12 (1948) pp. 513–554.

51. C. Y. Lee, *Similarity principle with boundary conditions for pseudo-analytic functions*, Duke Math. J. vol. 23 (1956) pp. 157–163.

52. H. Lewy, *On differential geometry in the large*, I, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 258–270.

53. M. A. Lukomskaya, *On one generalization of the class of analytic functions*, C.R. (Doklady) Acad. Sci. URSS. vol. 73 (1950) pp. 885–888 (Russian).

54. ———, *On cycles of systems of linear homogeneous differential equations*, Rec. Math. (Mat. Sbornik) N.S. vol. 29 (1951) pp. 551–558 (Russian).

55. ———, *Solution of some systems of partial differential equations by means of inclusion in a cycle*, Akad. Nauk SSSR. Prikl. Mat. Meh. vol. 17 (1953) pp. 745–747 (Russian).

56. A. Milkis, *Uniqueness of the solution of Cauchy's problem*, Uspehi Matematičeskih Nauk vol. 3 (1948) pp. 3–46 (Russian).

57. C. S. Morawetz, *On the non-existence of continuous transonic flows past arbitrary profiles*, I. Comm. Pure Appl. Math. vol. 9 (1956) pp. 45–68.

58. ———, *Uniqueness for the analogue of the Neumann problem for mixed equations*, Michigan Math. J. (to appear).

59. C. B. Morrey, *On the solutions of quasilinear elliptic partial differential equations*, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 126–166.

60. M. Morse, *Topological methods in the theory of functions of a complex variable*, Annals of Mathematics Studies, no. 15, Princeton University Press, 1947.

61. M. Morse and M. Heins, *Topological methods in the theory of functions of a complex variable*, Ann. of Math. vol. 46 (1945) pp. 600–624, 625–666; vol. 47 (1946) pp. 233–274.

62. ———, *Deformation classes of meromorphic functions and their extension to interior transformations*, Acta. Math. vol. 78–79 (1947) pp. 51–103.

63. M. Morse, and J. A. Jenkins, *Multiply-valued harmonic functions and their conjugates on arbitrary Riemann surfaces*, Lectures on Functions of a Complex Variable, Michigan University Press, 1955, pp. 123–186.

64. C. Müller, *On the behavior of solutions of the differential equation* $\Delta U = F(x, U)$ *in the neighborhood of a point*, Comm. Pure Appl. Math. vol. 7 (1954) pp. 505–515.

65. I. G. Petrovsky, *On some problems in the theory of partial differential equations*, Uspehi Matematičeskih Nauk vol. 1 (1946) pp. 44–70 (Russian).

66. A. Pfluger, *Quasikonforme Abbildungen und logarithmische Kapazität*, Annales de l'Institute Fourier, vol. 2 (1950) pp. 69–80.

67. E. Picard, *Sur une système des équations aux dérivées partielles*, C. R. Acad. Sci. Paris vol. 112 (1891) pp. 685–688.

68. ———, *Sur une généralisation des équations de la théorie des functions d'une variable complexe*, ibid., pp. 1399–1403.

69. G. N. Položii, *On p-analytic functions of a complex variable*, C. R. (Doklady) Acad. Sci. URSS. vol. 58 (1947) pp. 1275–1278 (Russian).

70. ———, *A generalization of Cauchy's integral formula*, Math. Sbornik vol. 24 (1949) pp. 375–384 (Russian).

71. ———, *Singular points and residues of p-analytic functions of a complex variable*, C. R. (Doklady) Acad. Sci. URSS. vol. 60 (1949) pp. 769–772 (Russian).

72. ———, *The theorem on preservation of domain for some elliptic systems of differential equations and its applications*, Rec. Math. (Mat. Sbornik) N.S. vol. 32 (1953) pp. 485–492 (Russian).

73. M. H. Protter, *Generalized spherical harmonics*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 314–341.

74. ———, *The periodicity problem in the theory of pseudoanalytic functions*, to appear in Ann. of Math.

75. B. V. Šabat, *Cauchy's theorem and formula for linear classes of quasi-conformal mappings*, C. R. (Doklady) Acad. Sci. URSS. vol. 69, (1949) pp. 305–308 (Russian).

76. Z. Schapiro, *Sur l'existence des représentations quasiconformes*, C. R. (Doklady) Acad. Sci. URSS. vol. 30 (1941) pp. 690–692.

77. L. Sobrero, *Theorie der ebenen Elastizitaet unter Benutzung eines Systems hyperkomplexer Zahlen*, Hamburger Mathematische Einzelschriften, Leipzig, 1934.

78. S. Stoïlow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Paris, Gauthier-Villars, 1938.

79. O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildungen*, Deutsche Mathematik vol. 3 (1938) pp. 621–678.

80. I. N. Vekua, *New methods for solving elliptic equations*, Moscow-Leningrad, 1948 (Russian).

81. ———, *Systems of partial differential equations of first order of elliptic type and boundary value problems with applications to the theory of shells*, Rec. Math. (Mat. Sbornik) N.S. vol. 31 (1952) pp. 217–314 (Russian).

82. ———, *On a property of solutions of the system of generalized Cauchy-Riemann equations*, Bull. Acad. Sci. Georgian SSR. vol. 14 (1953) pp. 449–453 (Russian).

83. ———, *General representation of functions of two independent variables admitting derivatives in the sense of S. L. Sobolev, and the problem of primitives*, C. R. (Doklady) Acad. Sci. URSS. vol. 89 (1953) pp. 773–775 (Russian).

84. ———, *On some properties of solutions of systems of equations of elliptic type*, C. R. (Doklady) Acad. Sci. URSS. vol. 98 (1954) pp. 181–184 (Russian).

85. A. Weinstein, *Generalized axially symmetric potential theory*, Bull. Amer. Math. Soc. vol. 59 (1953) pp. 20–38.

86. G. T. Whyburn, *Introductory topological analsyis*, Lectures on Functions of a Complex Variable, Michigan University Press, 1953, pp. 1–14.

REFERENCES ADDED IN PROOF.

87. Nachman Aronszajn, *Sur l'unicité du prolongement des solutions des équations aux dérivées partielles elliptiques du second ordre*, Comptes rendus vol. 242 (1956) pp. 723–725.

88. Lipman Bers, *Formal powers and power series*, Comm. Pure Appl. Math. vol. 9 (1956).

89. H. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. vol. 88 (1952) pp. 85–139.

90. P. Hartman and A. Wintner, *On the local behavior of solutions of non-parabolic partial differential equations*. III: *Approximations by spherical harmonics*, Amer. J. Math. vol. 77 (1955) pp. 453–483.

91. E. M. Landis, *On some properties of solutions of elliptic equations*, Doklady Acad. Nauk SSSR. vol. 107 (1956) pp. 640–643 (Russian).

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STRUCTURE OF SIMPLE FLUIDS

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One of the most important of the many achievements of Willard Gibbs was the derivation of a single equation of universal validity, by which the properties of a macroscopic system in equilibrium could be expressed in terms of the submicroscopic mechanical properties of the molecules composing it. I wish to discuss the present status of the methods of numerical evaluation of one problem using this equation.

It is known experimentally, with a simple extrapolation from thermodynamic theory, that any system composed of real molecules at any finite nonzero temperature, T, and at infinite dilution, for which the number density, $\rho = N/V$, of molecules approaches zero, exists as a perfect gas. For the perfect gas the pressure, P, is given by the equation, $P = \rho kT$, with k equal to Boltzmann's constant, $k = 1.38 \times 10^{-16}$ ergs/deg °K. The energy of the perfect gas depends only on T, and not on the density, ρ. At sufficiently high temperature, and what is high or low depends on the type of molecule, the material remains gaseous even if the pressure is increased to any experimentally attainable value. The perfect gas equation is no longer obeyed exactly at high densities, but P remains a smooth, and presumably analytic, monotonic function of ρ. At low temperatures, below the critical temperature, condensation occurs as the density is increased. If only one species of molecule is present the condensation is abrupt on the pressure plot, that is the density increases discontinuously from that of the gas to that of the condensed phase. Below the triple point temperature the condensed phase is crystalline. (The one exception is helium for which the triple point does not exist above zero.) Above the triple point the condensed phase is a liquid.

The thermodynamic properties of a system are completely determined if one knows the equation for the energy, E_0, as a function of T in the perfect gas state, $E_0(T)$, and the equation of state, that is the equation for the pressure as a function of density and temperature, $P = P(\rho, T)$. The energy, $E_0(T)$, of the perfect gas, can be readily expressed from the Gibbs formulation as $3NkT/2$ plus a sum over the internal quantum states of the molecules, and even for relatively

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complicated molecules the equations can be evaluated numerically, often even with greater precision than calorimetric measurements achieve. For the gas, the pressure, at a given temperature, may be expressed as a power series in the density, ρ. The temperature dependent coefficient of ρ^ν, known as the ν'th virial coefficient, can be expressed as an integral of a function of their mutual potential over the relative position coordinates of ν molecules. Although numerical evaluation for coefficients higher than the second is relatively difficult, no fundamental problem seems to be involved.

For the condensed phase the situation is less satisfactory. At low temperatures, for which the material condenses to a crystal, one proceeds by making use of this empirically known fact, and the known crystal structure. The integrand, which occurs in the Gibbs formulation, is then assumed to have a (computable) maximum when the atoms are all situated on the lattice sites of the known experimental lattice. The energy is then assumed to be quadratic in the small displacements from these equilibrium positions, and by the method of normal coordinates reasonably satisfactory computations are possible.

For the intermediate temperature range between the triple point and the critical point, and for which the condensed phase is a liquid, the situation is in still poorer shape. In the liquid (as also in the very high density gas), each molecule is simultaneously in interaction with a large number of neighbors. The development of the virial expansion, in which one considers first pair interactions, then triples, etc., no longer converges. On the other hand there is no single, regular, periodic configuration of dominant probability, which can be used as a starting point of a development for small displacements, as in the crystal. One is forced to a general consideration of the fundamental equations.

The computation of the thermodynamic properties does not answer all possible questions about a liquid. Among others one would certainly wish to know what may be vaguely called its structure. For some time even a precise question had not been formulated, the answer to which would describe the structure of a liquid unambiguously. At the present time, possibly unwisely, one assumes the description to be answered best in terms of the probability densities, ρ_n, for $n = 1, 2, 3, \cdots$ molecules. The function, $\rho_1(r)$, is defined conceptually by the statement that $\rho_1(r)dr$ is the probability that a molecule will be found at the position r in the volume element dr. For a fluid $\rho_1(r)$ is a trivial constant equal to the average number density ρ. The quantity $\rho_2(r_1, r_2)dr_1dr_2$ is defined as the probability of finding simultaneously a molecule at r_1 in dr_1 and one at r_2 in dr_2. This pair density, ρ_2, is a function only of the distance, $r_{12}=|r_1-r_2|$, in a fluid, and can be found experimentally as the Fourier transform of the X-ray scattering intensity. The higher probability densities, such as $\rho_3(r_1, r_2, r_3)$, proportional to the probability of simultaneously finding molecules at r_1, r_2, and r_3, are experimentally unobservable, at least at present, but are essential concepts in almost any attempt to treat the transport properties of fluids such as viscosity or conductivity. These probability densities for the equilibrium system are also formally derivable from the general Gibbs equation.

The discussion of the status of the attempt to obtain a general treatment by which the formal Gibbs equations can be evaluated numerically is my subject. Any such general treatment would apply equally to crystal, or to dilute gases, although in these cases more special methods lead to simpler solutions.

The mathematical problem may be stated to be the evaluation for small values of n, $n=0, 1, 2$, etc., of a set of functions, $G_n\{n\}$, given by the equation,

$$(1)\qquad G_n\{n\} = \sum_{N\geq 0} \frac{z^N}{N!}\int\int_V \cdots \int G^{(0)}_{n+N}\{\{n\}+\{N\}\}d\{N\},$$

where the symbol $\{n\}$ is used for the $3n$ cartesian coordinates of n molecules in 3-dimensional space,

$$(2)\qquad r_1, r_2, \cdots, r_n,$$

and $d\{n\}$ for the $3n$-dimensional volume element,

$$(3)\qquad d\{n\} = dr_1 \cdots dr_n.$$

The parameter z is real positive, and we are interested in values up to those for which zV may be very large, $zV\cong 10^{23}$. The functions, $G^{(0)}_N$, are real positive symmetric functions of the coordinates of N molecules, such that the function $G^{(0)}_{N+M}$ approaches in value the product of functions $G^{(0)}_N G^{(0)}_M$ when the coordinates, $\{N\}$, of N of the molecules are all very far distant from all the coordinates, $\{M\}$, of M of them,

$$(4)\qquad \begin{gathered} G^{(0)}_{N+M}\{\{N\}+\{M\}\} \to G^{(0)}_N\{N\}G^{(0)}_M\{M\}, \\ \text{if } r_{nm}\to\infty,\quad 1\leqq n\leqq N,\quad N+1\leqq m\leqq N+M. \end{gathered}$$

The function $G^{(0)}_1(r_i)$ is unity, so that,

$$(5)\qquad G^{(0)}_N\{N\} \to \prod_{i=1}^{i=N} G^{(0)}_1(r_i) = 1, \qquad r_{ij}\to\infty,\ N\geqq i>j\geqq 1.$$

Furthermore the functions $G_N^{(0)}$ approach zero in value if any of the distances, r_{ij}, approach zero,

$$G_N^{(0)}\{N\} \rightarrow 0, \qquad \text{any } r_{ij} \rightarrow 0. \tag{6}$$

The volume, V, over which the integration is to be extended, is to be taken as very large compared to r_0^3 if r_0 is the significant distance compared to which the relations (4), (5), and (6) become valid.

The physical interpretation of the problem is that of the computation of the equation of state, with the Grand Canonical Partition Function of Gibbs, after summation over the internal molecular quantum states, and integration over the momenta. A macroscopic system of one kind of molecule, in equilibrium, has its thermodynamic state specified by three variables, one of which must determine the over-all size of the system, and may be chosen as the volume, V. The other two may be chosen as intensive variables, one of which determines the concentration of energy in the system, and the other the concentration of molecules. Of these, the first is usually chosen as the absolute temperature, T. The other, that which determines the concentration of molecules or density, we choose as the activity, z. This is related to the chemical potential, μ, of Gibbs, by the equation,

$$z = \exp\left[(\mu - \mu_0)/kT\right], \tag{7}$$

where μ_0 is so chosen that z becomes equal to the number density,

$$\rho = N/V, \tag{8}$$

at the limit that both become zero,

$$\lim_{z\rightarrow 0}\left[\frac{z}{\rho}\right] = 1, \qquad \text{(determines } \mu_0\text{)}, \tag{9}$$

for which density the system always approaches a perfect gas in properties.

If $G_N^{(0)}$ is defined to be

$$G_N^{(0)} = \exp(-U_N\{N\}/kT), \tag{10}$$

with $U_N\{N\}$ the mutual potential energy of the molecules,

$$U_N\{N\} \rightarrow 0 \qquad \text{all } r_{ij} \rightarrow \infty, \qquad N \geqq i > j \geqq 1, \tag{11}$$

$$U_N\{N\} \rightarrow \infty \qquad \text{any } r_{ij} \rightarrow 0, \tag{12}$$

then the function G_0 has the physical significance that

$$G_0 = e^{PV/kT}. \tag{13}$$

with P the pressure of the system. One might point out here that PV/kT is of the order of the number, $\bar{N}$, of molecules in the volume, V, and we wish to choose V large enough that this is something like 10^{23}.

Since P is expressed by the equation (1) for $n=0$ as a function of z and T, the thermodynamic relations,

$$z[\partial(P/kT)/\partial z]_T = \rho, \tag{14}$$

$$[\partial(P/kT)/\partial T]_z = H - H_0, \tag{15}$$

with H the Enthalpie and H_0 the Enthalpie of the perfect gas state, serve to give the difference of all thermodynamic properties of the system from those of the perfect gas.

The computation of the thermodynamic properties of the perfect gas is not only formally given by the equations of Gibbs, but is numerically evaluated with high precision for many even relatively complicated molecules, using spectroscopic data. Given U_N, which is an appropriately averaged mutual potential over the internal molecular states, the evaluation of equation (1) for $n=0$ would complete the computation of all of the thermodynamic properties at all temperatures and densities. Even for systems which are essentially quantum mechanical a modification of the meaning of $G_N^{(0)}$, leads to the correct equations. We might also mention that the extension to systems composed of several kinds of molecules does not introduce essentially new complications.

The functions G_n of Equation (1) for $n=1$, 2, etc., go beyond the thermodynamic specification of the system. In general the physical meaning is that

$$G_n\{n\} = e^{PV/kT} z^{-n} \rho_n\{n\}, \tag{16}$$

where $\rho_n\{n\}$ is the probability density that molecules will be found simultaneously at all of the positions $\{n\}$. Since for a fluid $\rho_1(\mathbf{r}_i)$ is a constant,

$$\rho_1(\mathbf{r}) = \rho, \tag{17}$$

the equation for $n=1$ gives no essentially new information, but $\rho_2(\mathbf{r}_1, \mathbf{r}_2)$ for a fluid, which is a function of the distance,

$$r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \tag{18}$$

$$\rho_2(\mathbf{r}_1, \mathbf{r}_2) = \rho_2(r_{12}), \qquad \text{(fluid)} \tag{19}$$

is the Fourier transform of the X-ray scattering intensity, and hence is known experimentally, (Fig. 1).

For a fluid $\rho_2(r_1, r_2) = \rho_2(|r_1 - r_2|)$

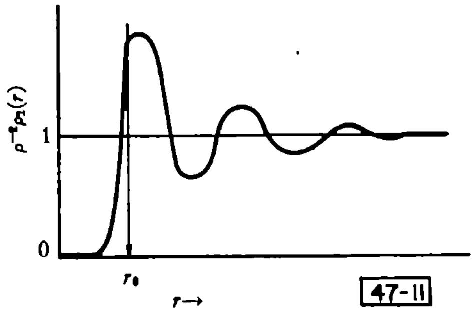

FIG. 1

The theoretical physicist has one great advantage over the pure mathematician as recompense for the difficulty of the problems which he is forced to undertake: he usually knows the correct answers to his problems, answers supplied by the experimentalists. The solutions of Equation (1) for $n = 0$ lead to curves of P versus z that look qualitatively like those of Fig. 2. Starting at the origin, P increases linearly with z with a slope kT. The curves then tend to bend upwards at all low temperatures, and at sufficiently low T values there is a singularity followed by an enormous increase in slope corresponding to the discontinuous density increase at the activity of condensation. For increasing T values the singularity moves toward

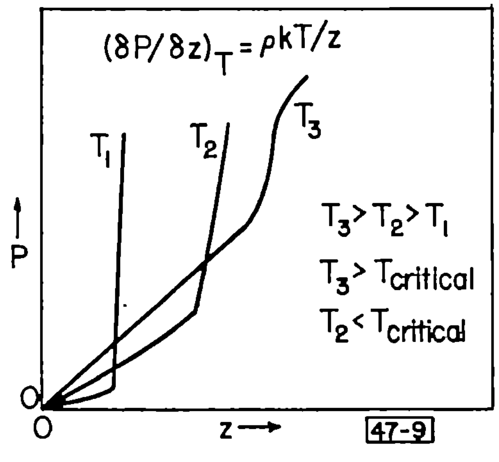

FIG. 2

increasing z-values, becomes less marked, and finally seems to disappear at the critical temperature, T_c. Above T_c the curves look smooth. Translated to plots of P versus ρ^{-1} these are the familiar P versus V plots of Fig. 3 with the shaded two-phase region for all temperatures below T_c, for which both gas and the condensed phase of liquid or crystal are coexistent.

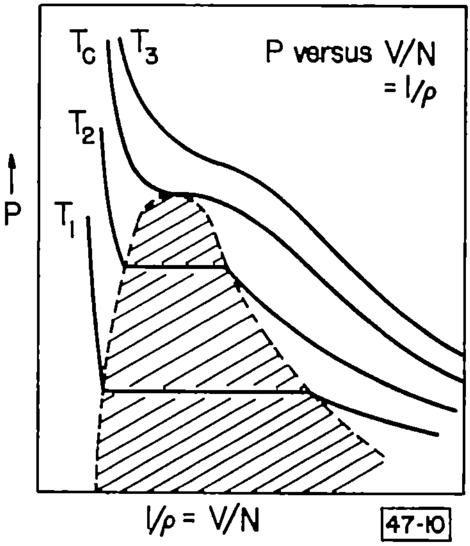

FIG. 3

Finally to make the characteristics of the functions $G_N^{(0)}$ more explicit we may mention that for simple spherical, or nearly spherical, chemically saturated molecules, one generally assumes U_N to be a sum of mutual pair potentials,

$$U_N = \sum_{N \geq i > j \geq 1} \sum u(r_{ij}) \tag{20}$$

with the potential of a single pair given by the Lennard-Jones 6-12 potential,

$$u(r) = u_0\left[(r_0/r)^{12} - 2(r_0/r)^6\right], \tag{21}$$

with a minimum value, $u = -u_0$, at $r = r_0$, (Fig. 4). For such molecules the pressure will be a universal function of kT/u_0 and zr_0^3, a statement equivalent to the "Law of Corresponding States."

Returning now to the mathematical problem of evaluation of equation (1), we see that the assumption of a zero potential,

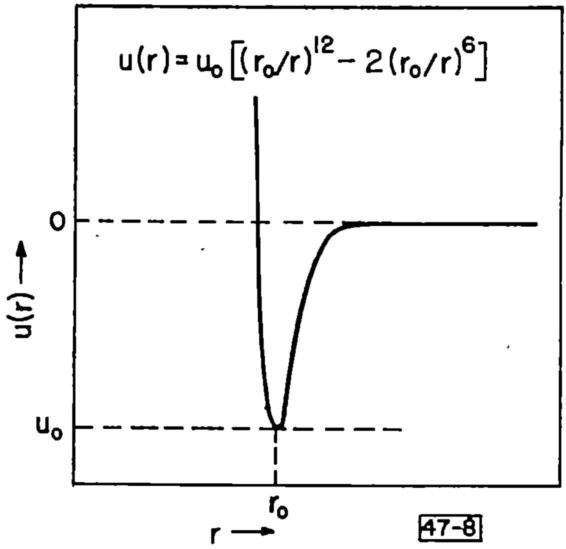

Fig. 4

$$U_N = 0, \qquad G_N^{(0)} = 1, \tag{22}$$

leads to the trivially simple result,

$$G_0 = e^{PV/kT} = \sum_{N \ge 0} \frac{z^N}{N!} V^N = e^{zV}, \tag{23}$$

$$P/kT = z \tag{24}$$

which, with equation (14) gives $z = \rho$, or the perfect gas equation, $\rho = N/V$, that

$$PV/NkT = 1. \tag{25}$$

This is the starting point for the solution at low values of z. We write $G_N^{(0)}$ as unity, the value which it has when all molecules are far apart, plus correction terms which are nonzero only when there are clusters of molecules close together. There will be $N(N-1)/2$ terms which are nonzero when single pairs are close together. The correction terms for two pairs are the products of the terms for the two single pairs. Those for n_2 pairs, n_3 triples, $\cdots$, n_ν ν-fold clusters, are the product of functions of n_2 pair corrections, n_3 triple corrections, $\cdots$, n_ν ν-fold corrections all of which are nonzero only when all the molecules of the cluster are close to each other. That is, one expresses $G_N^{(0)}$ as

$$G_N^{(0)} = 1 + \sum_{n_2} \sum_{n_3} \cdots \sum_{n_\nu} \prod_{p=1}^{p=n_2} (\text{pair corrections})$$
$$\times \prod_{t=1}^{t=n_3} (\text{triple corrections}) \cdots \prod_{\sigma_\nu=1}^{\sigma_\nu=n_\nu} (\nu\text{-fold corrections}).$$

It is not difficult to express the ν-fold cluster correction quite generally in terms of $G_\mu^{(0)}$ for $\mu \leqq \nu$. If an integral,

$$(26) \qquad b_\nu = (1/\nu! V) \int\int_V \cdots \int (\nu\text{-fold corrections})\, d\{\nu\},$$

is defined, one sees that at the limit $V \to \infty$, b_ν is independent of V. The combinatorial problem of the number of ways one can make n_ν clusters of ν each is easily solved, and the simple result,

$$(27) \qquad P/kT = z + \sum_{\nu \geq 2} b_\nu z^\nu,$$

is found. The coefficient, b_2, for instance, is

$$(28) \qquad b_2 = \frac{1}{2} \int 4\pi r^2 \left\{ \left[\exp - \frac{1}{kT} u(r) \right] - 1 \right\} dr.$$

Equation (27) gives a useful expression for P for low z-values, and is valid up to the first singularity, $z = z_0$, on the positive real axis of z, which singularity is at the activity, z_0, of condensation. For the condensed state we are forced to use more erudite methods.

Before discussing the integral equation approach, one should at least mention the cell or free volume method, which does lead, without too great computational difficulties, to expressions for P that are reasonably good in the condensed phase range. The method may be justified in various ways. That which I prefer is as follows. One may readily prove that any thermodynamic potential is always an extremum for the equilibrium distribution of molecules. One may then assume a distribution, namely the functions $G_N^{(0)}$ of equation (1), in some analytical form such that the integrations over the coordinates can actually be carried out, and some thermodynamic potential, say the Helmholtz Free Energy, actually computed for this nonequilibrium assumed distribution. If there are any adjustable parameters of the distribution they should be so determined as to give a minimum Free Energy for fixed V, T, and N.

The simplest integrable distribution is that of placing the N molecules in a close-packed lattice of N-cells, with one molecule per cell constrained to a probability distribution within the cell, say a simple

Gaussian around the center, independently of the positions of the others. The single adjustable parameter is then that determining the Gaussian width. The model may be improved by adjusting the lattice parameter, permitting more cells than molecules, with some cells empty, or further by permitting a correlation which changes the probability of finding a cell empty depending on the number of neighboring empty cells.

Different variations of this method have led to reasonably respectable equations for the pressure in the condensation range. However it is fair to say that none of these various approaches has grown to the stature of distinguishing two condensed phases, crystalline and liquid in different temperature-density ranges. Essentially the method always assumes a long range crystalline order.

The most ambitious method of solution of the problem is that which leads to a system of integral equations.

Consider any linear operator, O, which operates on $G_N^{(0)}$ to give,

$$\begin{aligned} OG_N^{(0)} = G_N^{(0)}\Bigg[& \sum_{i=1}^{i=N} \psi_1^{(0)}(r_i) + \sum_{N\geq i>j\geq 1}\sum \psi_2^{(0)}(r_i, r_j) \\ & + \sum_{N\geq i\geq j\geq k\geq 1}\sum\sum \psi_3^{(0)}(r_i, r_j, r_k) + \cdots \Bigg], \end{aligned} \tag{29}$$

and on G_n to give a similar sum,

$$OG_n = G_n\left[\sum_{i=1}^{i=n} \psi_1(r_i) + \sum\sum + \cdots \right]. \tag{30}$$

Operate on both sides of equation (1) with O. On the right, under the integral, consider a single term of the sum of equation (29), say the term $\psi_{\nu+\mu}^{(0)}$ with $\{\nu\}$ consisting of some of the molecules of n, and $\{\mu\}$ a subset of the molecules of N. Sum over all values of $N \geq \mu$, which will contain such a term, and integrate over the coordinates $d\{N-\mu\}$. One obtains just $G_{n+\mu}\psi_{\nu+\mu}^{(0)}$ to be integrated over $d\{\mu\}$. One has, then, on the left the functions on the right hand side of equation (30), and on the left a sum of integrals of $\int G_{n+\mu}\psi_{\nu+\mu}^{(0)} d\{\mu\}$. With a little algebraic manipulation this can be brought into the form of a generalized vector-matrix equation,

$$\Psi = L\Psi^{(0)}, \tag{31}$$

i.e.

$$\psi_n = \sum_m L_{n,m}\psi_m^{(0)}, \tag{31'}$$

in which Ψ is a vector whose components, ψ_n, are themselves functional vectors in the continuous $3n$-dimensional coordinate space of n molecules, and L is a matrix, whose elements, $L_{n,m}$, are themselves rectangular continuous matrices of $3n$- and $3m$-dimensional coordinates, that is, $L_{n,m}\psi_m^{(0)}$ means matrix multiplication,

$$(32) \qquad L_{n,m}\psi_m^{(0)} = \int\int \cdots \int L(\{n\}, \{m\})\psi_m^{(0)}\{m\}d\{m\}.$$

The elements $L_{n,m}$ are functions that can be expressed in terms of $\rho_{n+m}\{n+m\}$, $\rho_{n+m-1}\{n+m-1\}$, etc., that is in terms of the probability densities, $\rho_\mu\{\mu\}$, for $1 \leqq \mu \leqq n+m$.

$$(33) \qquad L_{n,m} = L_{n,m}(\rho_{n+m}\{n+m\}, \rho_{n+m-1}\{n+m-1\}, \cdots).$$

If the elements $L_{n,m}$ were known this could be used for a perturbation calculation. For instance, if the potential U_N were written as

$$(34) \quad U_N = U_N^{(0)} + \lambda \sum_{i=1}^{i=N} \psi_1^{(0)}(r_i) + \lambda \sum_{N \geqq i > j \geqq 1}\sum \psi_2^{(0)}(r_i, r_j) + \cdots,$$

and the operator O as $O = -kT(d/d\lambda)_{\lambda=0}$, then the functions ψ_ν would give the effect, in first order, of the perturbations due to $\psi_\nu^{(0)}$ on the functions G_n.

Actually the functions $G_n\{n\}$ composing the elements $L_{n,m}$ are exactly the unknowns that we wish to determine. Equation (31), then, is only of value if the operator O is so chosen that there is a functional relationship between the functions ψ_n and the ρ_n, and this is the trick which has been used by Yvon, and by Kirkwood. I will discuss this in more detail later.

One remark may be inserted here. The determination of $\rho_2(r_1, r_2)$ as a function of z and T is adequate also for a determination of P, and hence all other thermodynamic functions, since P is given by the virial theorem as an integral involving the forces between pairs of molecules multiplied by the pair density.

However, equation (1) can be solved for the functions $G_N^{(0)}$ in terms of the functions G_N, and the solution is,

$$(35) \qquad G_n^{(0)} = \sum_{N \geqq 0} \frac{(-z)^N}{N!} \int\int_V \cdots \int_V G_{n+N}\{n+N\}d\{N\},$$

which can be readily checked by using one expression in the other and obtaining an identity, and which is adequate proof since the condition (6) requires that both series (1) and (35) are absolutely convergent for finite V. The operation O on both sides of (35) results in the expression,

$$\Psi^{(0)} = L^{(0)}\Psi, \tag{36}$$

where the elements, $L^{(0)}_{n,m}$, of $L^{(0)}$ are expressible in terms of $(-z)^{\mu}e^{U_{\mu}/kT}$ for $1 \leqq \mu \leqq n+m$, in the same way that $L_{n,m}$ depends on ρ_{μ}. Thus the elements $L^{(0)}_{n,m}$ are now known functions.

Combining equations (31) and (36) one sees that

$$LL^{(0)} = L^{(0)}L = 1 \tag{37}$$

with 1 the unit matrix. The sums

$$\sum_{m} L_{k,m}L^{(0)}_{m,k} = \sum L^{(0)}_{k,m}L_{m,k} = 1, \tag{37'}$$

can now also be seen to converge for relatively small m values, for instance if $k=1$ and $m>13$ there is no range of the coordinates $\{m\}$ for which both $L_{1,m}(\mathbf{r}_1, \{m\})$ and $L^{(0)}_{m,1}(\{m\}, \mathbf{r}_1)$ differ significantly from zero.

A specialized form of equation (31) was first derived by Yvon in France, and a similar one by Kirkwood. Later Born and Green independently derived the Yvon form. In all these cases the operation O involved one unique molecule, whose coordinates could be placed at the origin. Some renumbering of the matrix elements is necessary, and one then has,

$$L_{n,m} = L_{n,m}[\rho_{n+m+1}\{n+m+1\}, \rho_{n+m}, \cdots], \tag{38}$$

instead of (33), in particular $L_{1,1}$ depends on $\rho_3(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k)$. However the operator was so chosen that the functions $\psi_{\nu}^{(0)}$ and ψ_{ν} bore known relations to $G_n^{(0)}$ and G_n, respectively. In the Yvon (and Born-Green) case,

$$O = -kT\nabla_0 \tag{39}$$

with ∇_0 operating on the coordinate of the unique molecule whose coordinate can be taken at the origin, so that,

$$\psi_1^{(0)} = \nabla u(|\mathbf{r}|), \qquad \psi_1 = -kT\nabla \ln \rho_2(|\mathbf{r}|). \tag{40}$$

If the total potential, U_N, is purely a sum of pair terms, equation (20), one has

$$\psi_{\nu}^{(0)} = 0, \qquad \nu > 1. \tag{41}$$

The equations become, in our notation,

(42) $$\psi_n = L_{n,1}\psi_1^{(0)}$$

of which only the first,

(42′) $$\psi_1 = L_{1,1}\psi_1^{(0)},$$

is actually considered. The matrix element, $L_{1,1}$, now depends on ρ_3 as well as ρ_2. Some assumption is necessary. The Kirkwood assumption,

(43) $$\rho_3(r_1, r_2, r_3) = \rho^{-3}\rho_2(|r_1 - r_2|)\rho_2(|r_2 - r_3|)\rho_2(|r_3 - r_1|),$$

that the number density of triples is the product of that for the three pairs, properly normalized, leads to a definite equation in (42′).

The actual equation becomes

(44) $$-kT\nabla \ln \rho_2(r_1) = \nabla u(r_1) + \rho^{-1}\int \rho_2(r_{13})\,[\rho_2(r_3) - \rho^2]\nabla u(r_3)dr_3.$$

The similar equation with the Kirkwood operator has been solved numerically by Kirkwood, and the results are fair, in the sense of looking very much like the experimentally known function, ρ_2.

One would like to believe that this solution represents the first approximation to a method which could be carried out, in principal at least, to any degree of approximation. Some light on this has been shed by the work of Liliane Sarolea. She has examined the reciprocal set of equations, which with $\psi_\nu^{(0)} = 0$, $\nu > 1$ become,

(45) $$\psi_1^{(0)} = \sum_m L_{1,m}^{(0)}\psi_m,$$

(45′) $$0 = \sum_m L_{n,m}^{(0)}\psi_m.$$

The diagonal elements, $L_{n,n}$, contain a diagonal part, the Dirac δ-function, so that it is convenient to define,

(46) $$K = L_1 - 1,$$

in which case we can write,

(47) $$\psi_1 = \psi_1^{(0)} - K_{1,1}^{(0)}\psi_1 - \sum_{m>1} K_{1,m}^{(0)}\psi_m,$$

(47′) $$\psi_m = -K_{m,1}^{(0)}\psi_1 - \sum_{n>1} K_{m,n}^{(0)}\psi_n.$$

Now the Kirkwood approximation of equation (43), if taken literally, also would require that $\psi_n = 0$ for $n > 1$. One might thus consider the sum which is the last term on the right of (47)′ to be at least small. If this sum were neglected entirely equation (47) becomes a linear integral equation of the Fredholm type in the one unknown function ψ_1.

Sarolea has attempted to find the contribution due to this sum. She uses equation (47′) to iterate the sum, and writes:

$$\begin{aligned} \psi_1 = \psi_1^{(0)} - \Big\{ & K_{1,1}^{(0)} - \sum_{m>1} K_{1,m}^{(0)} K_{m,1}^{(0)} \\ & + \sum_{n>1} \sum_{m>1} K_{1,n}^{(0)} K_{n,m}^{(0)} K_{m,1}^{(0)} - \cdots \Big\} \psi_1 = \psi_1^{(0)} - M_{1,1}^{(0)} \psi_1. \end{aligned} \tag{48}$$

One then finds that the matrices $L_{1,1}$ and $1 - M_{1,1}^{(0)}$ are reciprocal,

$$L_{1,1}(1 - M_{1,1}^{(0)}) = (1 - M_{1,1}^{(0)}) L_{1,1} = 1. \tag{49}$$

Dr. Sarolea has completed the cycle by showing that the relation (49) is trivially satisfied, namely that the series represented by $M_{1,1}^{(0)}$ of equation (48) is actually a power series in z whose analytical continuation beyond its first singularity does obey equation (49). However, in showing this, and using arguments similar to those outlined in our discussion of the virial method, equation (27), she has also shown that the series of equation (48) diverges at the values of z corresponding to the condensed phase. This most obvious method of attempting to improve the Kirkwood assumption seems to indicate that it is, at best, the first step in what one may term an asymptotic approximation.

There is little reason to doubt that the Kirkwood solutons are approximately correct solutions to the problem. They are, however, difficult to obtain, and apparently not of very high accuracy to compute the pressure or other thermodynamic properties adequately. In view of the Sarolea conclusions it is not easily apparent how they can be improved. In view of the generality of the matrix-integral equation approach one might hope that other methods of attack might possibly be found. However a new idea seems to be needed. Although the many particle number density functions, $\rho_n\{n\}$, seem to be the most natural functions to choose for the description of the liquid structure they are difficult to visualize, and apparently impossible to measure experimentally for $n > 2$. Although the Kirkwood assumption of equation (43) that ρ_3 is proportional to the product of the three pair functions sounds highly plausible, it seems that all attempts to improve the justification for it lead only to new demonstrations that it is far from being exact. The cell method concept, although simple to visualize, and relatively simple for numerical computation, seems to be inherently limited by its assumption at the outset of a long range order in the liquid.

It may be that some entirely new set of logically defined functions might be more fruitful than the number density functions ρ_n, and might lead to both a more readily visualizable concept of liquid structure, and to a convergent analytical method of computation. At least one attempt of this nature shows some hope in simplification of the equations, although using even less easily visualized functions. This is to employ the Fourier transforms of the number density functions. An operation similar to that used in obtaining the integral equations leads to a set of purely algebraic equations. However it is not clear that the equations obtained have any superiority over the integral equation method.

THE ENRICO FERMI INSTITUTE FOR NUCLEAR STUDIES,
UNIVERSITY OF CHICAGO

THE APRIL MEETING IN CHICAGO

The five hundred twenty-third meeting of the American Mathematical Society was held at the University of Chicago on Thursday, Friday and Saturday, April 12, 13 and 14. There were a total of 360 registrations, including 317 members of the Society.

By decision of the Council of the Society, there was a Symposium on *Calculus of variations and its applications*. The Symposium was supported by contract with the cosponsoring organization, the Office of Ordnance Research. Sessions of the Symposium were held on Thursday and Friday morning and on Thursday afternoon.

In the first session on Thursday morning, Professor Eric Reissner of the Massachusetts Institute of Technology spoke on *Variational methods in linear theory of elasticity*; Professor D. C. Drucker of Brown University on *Variational principles in the mathematical theory of plasticity*; and Professor J. B. Keller, New York University, on *Variational methods in wave propagation*. The Chairman of the session was Professor L. M. Graves of the University of Chicago.

The Chairman for the session on Thursday at 2:00 P.M. was Professor C. A. Truesdell of Indiana University. The three addresses were entitled *Upper and lower bounds for eigenvalues*, *Stationary principles for forced vibrations in elasticity and electromagnetism*, and *Applications of variational methods in the theory of conformal mapping*. The speakers were, respectively, Professor J. B. Diaz, University of Maryland; Professor J. L. Synge, Dublin Institute for Advanced Studies; and Professor M. M. Schiffer, Stanford University.

At the session on Friday at 9:30 A.M., Dr. R. E. Bellman of the RAND Corporation, Professor Subrahmanyan Chandrasekhar, University of Chicago, and Professor E. H. Rothe of the University of Michigan were the speakers. The topics were, respectively, *Dynamic programming and its application to variational problems in mathematical economics*, *Variational principles in stability problems in hydrodynamics and hydromagnetics*, and *Some applications of functional analysis to the calculus of variations*. The Chairman of the session was Professor J. J. Gergen of Duke University.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor R. C. Buck of the University of Wisconsin addressed the Society on the topic *Linear transformations on function spaces*. Professor Buck's lecture was given on Friday afternoon with Professor T. H. Hildebrand presiding.

Sessions for the presentation of contributed papers were held on Friday afternoon, Saturday morning and afternoon. Presiding officers were Professors M. L. Curtis, M. R. Hestenes, D. H. Brunk, M. F. Smiley, and Drs. F. P. Peterson and E. H. Brown, Jr.

The Society takes this opportunity to thank the ladies of the Department of Mathematics for entertaining at tea on Thursday and Friday afternoons.

Abstracts of the papers presented follow. The number of a paper presented by title is followed by "*t*". In the case of joint authorship, the name of the person presenting the paper is followed by (p).

ALGEBRA AND THEORY OF NUMBERS

396. A. A. Albert: *On partially stable algebras.*

Consider a commutative simple power-associative algebra A of degree two over an algebraically closed field F of characteristic $p>5$. Then $A=C+L$ where $C=A_u(1)+A_u(0)$ and $L=A_u(1/2)$, where u is a nontrivial idempotent of A. We call A partially stable if there is an idempotent u such that $CL\subseteq L$. In this paper we shall determine the structure of all such algebras. We shall show that $C=B+B(u-v)$ where B is either a simple algebra of degree one or is associative. It is not known whether or not there exist simple algebras of degree one which are not one-dimensional but, if they do exist, they will yield algebras $A=B+Bz+Bw$ where $(bw)(cw)=bc$, $b(cw)=(bc)w$, $(Bz)(Bw)=0$ for every b and c of B where $B+Bz$ is the direct product of B and the associative algebra $F[z]$ When B is associative we have $(Bz)L=0$ and the multiplicative formulas for the algebra A are essentially those of an earlier paper (Trans. Amer. Math. Soc. vol. 74 (1953) pp. 336–343). (Received February 10, 1956.)

397*t*. R. M. Baer: *Closure operators on partially ordered sets.*

A closure operator ϕ is an idempotent order-homomorphism which carries a partially ordered set X into itself and which satisfies $x\leqq\phi(x)$, all $x\in X$. A nonempty set F in X is called a *partial ordinal* in X if $(x)\cap F$ is a nonempty set having a first element, for every dual principal ideal (x) in X. The set C_X of all ϕ on X is partially ordered in a natural way. Then (*) *A nonempty subset F of X is the fixed-point set of some closure operator ϕ on X if and only if F is a partial ordinal in X.* Using (*), the relation between lattice properties of X and lattice properties of C_X is studied and certain results of Ward [Ann. of Math. vol. 43 (1942) pp. 191–196] and Dwinger [Neder. Akad. Wetensch. vol. A58 (1955) pp. 36–40] are generalized. Further, it is shown that *if X satisfies the finite chain condition, then C_X is a lattice.* (Received April 13, 1956.)

398. S. K. Berberian: *Reduction of the projection geometry of a finite AW^*-algebra.*

Let A be a finite AW^*-algebra. The continuous geometry of projections of A can be reduced into a complete system of irreducible continuous geometries in the following way Take any maximal ideal M, and let I be the ideal generated (algebraically) by the projections of M. Then the projections of A/I form an irreducible continuous geometry. If C is the regular ring attached to A, and J is the ideal of C generated by the projections of M, then C/J is the regular ring attached to A/I. (Received February 6, 1956.)

399. Richard Block: *New simple Lie algebras of order p^n-2.*

Let F be a field of characteristic p and G an n-dimensional vector space ($n>1$) over the prime field which is a direct sum of subspaces $G_0, \cdots, G_m$, with $m=m(G)>1$ in case $p=2$ and $G_0=0$. Let δ_i be a nonzero element of G_i for $i=1, \cdots, m$. Let f_i be a nondegenerate skew-symmetric bilinear function on (G_i, G_i) to F such that for $i>0$ there are linear functions g_i, h_i, on G_i to F with $g_i(\delta_i)=0$ and $f_i(\alpha_i, \gamma_i)=g_i(\alpha_i)h_i(\gamma_i)-g_i(\gamma_i)h_i(\alpha_i)$, and in case $G=G_1$, with (δ_1) the kernel of g_1. Let $L(G, \delta, f)$ be the algebra with basis $\{v(\alpha)\,|\,\alpha\in G,\ \alpha\neq 0, -\delta_1-\cdots-\delta_m\}$ and multiplication $v(\alpha)v(\gamma)=\sum_{i=0}^{m} f_i(\alpha_i, \gamma_i)v(\alpha+\gamma-\delta_i)$, $v(0)=0$. Then L is a simple Lie algebra of dimension p^n-2 (in case $G=G_0$, of dimension p^n-1), not of type A unless $p=3$ and $n=2$. These algebras generalize the algebras L_+, L_0, and V_m of Albert and Frank [Rend. Sem. Mat. Torino vol. 14 (1954–1955) pp. 117–139]. A nondegenerate invariant form is determined by setting $t\{v(\alpha), v(\gamma)\}=1$ if $\alpha+\gamma=-\delta_1-\cdots-\delta_m$, $=0$ otherwise. $L(G, \delta, f)$ is restricted if and only if $G_0=0$ and G_i is 2-dimensional for $i>0$ The derivations are determined and show that $L(G, \delta, f)$, $L(G', \delta', f')$ are isomorphic only if $m(G)=m(G')$. Examples are given showing that an algebra may have Cartan subalgebras of distinct dimensions. (Received February 27, 1956.)

400. S. J. Bryant and J. L. Zemmer (p): *A theorem on topological ring extensions.*

A ring is called a *semi-topological ring* if its additive group is a topological group. A subring of a semi-topological ring is called a *topological subring* if it is a closed subset and its multiplication is continuous in the induced topology. The following theorem and several corollaries are proved: If a compact semi-topological ring A contains an open semi-simple topological subring then A is a topological ring. (Received February 17, 1956.)

401*t*. John DeCicco: *Quadratic extensions of a field.*

A quadratic extension of a field R is a commutative ring Γ with unit. This ring Γ admits an involutorial automorphism for which the self-conjugate elements form a field isomorphic to R. Characterizations within isomorphisms are obtained for such rings Γ. As applications, the systems of complex, dual complex, and hyperbolic complex numbers are characterized. Similarly quadratic extensions of modular number systems and other known fields are studied. (Received February 9, 1956.)

402. D. E. Edmondson: *Modular lattices.*

This is an algebraic study of the arithmetical properties of modular lattices and its application to the study of the congruence relations on a modular lattice. The principal tool is a fundamental characterization of the reducible and irreducible ideals of the lattice. This property characterizes modularity and is used to show that any congruence relation on a modular lattice can be generated by a collection of irreducible ideals Finally it is shown that the congruence relations on the lattice of ideals of a modular lattice can be a Boolean algebra if and only if the lattice is finite dimensional. (Received March 1, 1956.)

403. C. C. Faith: *Extensions of normal bases and completely basic fields.*

Let K/F be normal. $x\in K$ is *basic in* K/F, if x generates a normal basis for K/F. The existence of a normal basis of K/F which is the extension of a normal basis of

K/M, for any inter-field M, is settled by the construction of $u \in K$ which is basic in K over every inter-field, if F is infinite. Any such u is a *completely basic element of* K/F. The class $\mathfrak{C}$ of all *completely basic extensions*, i.e., all normal extensions for which every basic u is completely basic, is shown to properly include the Kummer extensions. Let $\mathfrak{Z}_e/\mathfrak{F}$ be cyclic of deg p^e, p a prime with $g = g(\mathfrak{F}, p) \geqq 1$, where g is the largest integer for which $x^{p^g} - 1$ factors linearly in $\mathfrak{F}$ (if no such g exists, set $g = \infty$). Result: If $g(\mathfrak{F}, p) \geqq e - 1$, then $\mathfrak{Z}_e/\mathfrak{F} \in \mathfrak{C}$. Otherwise, $\mathfrak{Z}_e/\mathfrak{F} \in \mathfrak{C}$ if and only if $g(\mathfrak{Z}_e, p) = g(\mathfrak{F}, p)$. Let P be the root field over $\mathfrak{F}$ of $x^{p^{e-1}} - 1 = 0$, contained in a field $\supseteq \mathfrak{Z}_e$. If p is odd with $e - 1 > g(\mathfrak{F}, p) \geqq 1$, then $\mathfrak{Z}_e/\mathfrak{F} \in \mathfrak{C}$ if and only if $\mathfrak{Z}_e \wedge P = \mathfrak{F}$. Other results: if $\mathfrak{Z}/\mathfrak{F}$ is cyclic of deg n with generating automorphism S, then $u \in \mathfrak{Z}$ is basic in $\mathfrak{Z}/\mathfrak{F}$ if and only if $\sum_{i=0}^{n-1} u^{S^i} x^i = 0$ has no root ζ obeying $\zeta^n = 1$; Kummer fields K/F have a basis $\{\theta_i\}$ such that $u = \sum_{i=1}^{n} \alpha_i \theta_i$, $\alpha_i \in F$, is basic in K/F, and hence completely basic, if and only if each $\alpha_i \neq 0$; if $N/F \in \mathfrak{C}$, and if $N = N_1 X \cdots X N_t$ over F, then each $N_i/F \in \mathfrak{C}$. (Received February 29, 1956.)

404. L. E. Fuller: *Congruence of matrices over a principal ideal ring modulo p^k.*

In a previous paper (*A canonical set for matrices over a principal ideal ring modulo m*, Canadian Journal of Mathematics vol. 7 (1955) pp. 54–59) a definition of the degree of an element in the system was made. By a slight modification of the concept of degree, the Hermite form for a field under row equivalence becomes a special case of that paper. A diagonal form under general equivalence is also shown to include that for the field. Congruent equivalence is considered for both symmetric and skew symmetric matrices. The diagonal forms obtained are similar to those for the case of a field. The major difference is the appearance of the prime of the modulus to various powers. By making a generalized definition of quadratic residue, a further simplification in the symmetric case becomes possible for some systems. The diagonal elements can be reduced to the form $\pm p^t$ where $t \leqq k$. Again the form for a real symmetric matrix is shown to be a special case. (Received February 21, 1956.)

405. Franklin Haimo: *Normal automorphisms of a class of holomorphs.*

The group of normal automorphisms of a group G is the centralizer of the group of inner automorphisms in the group of automorphisms of G. Consider the relative holomorph of G over a group B of automorphisms of G where B includes the group of inner automorphisms and is included in the centralizer of the group of normal automorphisms. It is shown that the group of normal automorphisms of this holomorph is isomorphic to a splitting extension of the (additive) group of homomorphisms of B into the group F of fixed points of G under the mappings from B by the group of automorphisms of G, the members of which induce the identity on G/F. (Received February 28, 1956.)

406. D. R. Hughes: *Regular collineation groups.*

A λ-plane π with parameters v, k, λ (i.e., a (v, k, λ) configuration, or a symmetric balanced incomplete block design) is termed *regular of degree m* if π possesses a collineation group G of order m, no non-identity element of which fixes any point or line of π. Then $t = v/m$ is an integer. If π is a λ-plane with parameters v, k, λ, regular of degree m, then there is a square matrix A of order t, consisting entirely of non-negative integral entries, such that $AA^T=A^TA=B$, where B has $(k-\lambda)+\lambda m$ on the main diagonal and λm elsewhere. Thus, if t is odd, it follows that the equation $x^2=(k-\lambda)y^2+(-1)^s\lambda m z^2$, where $s=(t-1)/2$, has a nontrivial solution in integers. It is thus possible to show that for many choices of v, k, λ, any λ-plane cannot be regular of degree greater than one (nontrivial results for $\lambda=1$ are included). Since any λ-plane is regular of degree one, the above results (with $m=1$) apply to any λ-plane, and yield the well-known incidence matrix equations. (Received February 20, 1956.)

407. Bernard Jacobson: *Sums of distinct divisors of algebraic integers.*

In a recent paper B. M. Stewart discussed sums of positive distinct divisors of rational integers [Amer. J. Math. vol. 76 (1954) pp. 779–785]. In this article these results are generalized. Let $\alpha(M)$ be the number of positive integers n which can be written in the form $n=\sum d$, where the d are distinct positive or negative divisors of M. The author has proved that $\alpha(M)=\sigma(M)$ if and only if n is of the form $n=2^b3^c\prod_{i=1}^{k} p_i^{a_i}$ where b and c are not both zero, $3<p_1<p_2<\cdots<p_k$, $p_1\leqq 2\sigma(2^b3^c)+1$ and $p_{j+1}\leqq 2\sigma(2^b3^c\prod_{i=1}^{j} p_i^{a_i})+1$ for $j=1, 2, \cdots, k-1$. The function $\alpha(M)/\sigma(M)$ is everywhere dense on the interval 0 to 1. In the quadratic fields $x+y(2^{1/2})$ and $x+y(5^{1/2})$ every integer in the field can be written as a finite sum of distinct units, the algorithm used depending upon the representation of each integer of the field as a lattice point in the plane. In any real quadratic field there exist infinitely many integers n, having the property that every integer in the field can be written as a finite sum of distinct divisors of n. Explicitly if $a+b(m^{1/2})$ is the unit of smallest absolute value for which $a>0$ and $b>0$, then any integer $2^{t+1}(m^{1/2})$ where $2^t>a$ satisfies this condition. Analogous results have been found for imaginary quadratic fields. (Received February 27, 1956.)

408. E. D. Nering: *An integral basis theorem for algebraic function fields.*

Let K be an algebraic function field with field of constants k. Let x be an element in K transcendental over k so that K is finite algebraic over $F=k(x)$. Let $\mathfrak{o}=k[x]$ be the ring of polynomials with coefficients in k, and let $\mathfrak{O}\subset K$ be the ring of elements in K integral over $\mathfrak{o}$. Let h be a discrete non-archimedean valuation of k extended to a valuation of F by the Gaussian definition. Let H_i $(i=1, \cdots, s)$ be all the extensions of h to valuations of K. Let F^* denote the ring of elements in F integral with respect to h, and let K^* be the ring of elements in K integral over F^*. For any set $\mathfrak{A}$ in K, let $\mathfrak{A}^*=\mathfrak{A}\cap K^*$. If all the extensions of H on F to H_i on K have radical degree 1, then for every ideal $\mathfrak{A}$ of $\mathfrak{O}$, there exists a basis of K over F which is an integral basis of $\mathfrak{A}^*$ over $\mathfrak{o}^*$. (Received February 27, 1956.)

409t. C. A. Nicol: *On the number of solutions of certain linear congruences.*

Let $F_n(s, x)=\prod_{r=1}^{n-1}(s-x^r)$. If this product is expanded as a polynomial in x, the coefficients are polynomials in s, whose coefficients have combinatorial properties related to partitions. The combinatorial properties of these polynomials in s are shown to be related to the number of solutions of $x_1+\cdots+x_r\equiv t \pmod{n}$. Then the number of solutions of these linear congruences is expressed in terms of the Von Sterneck number [A. Von Sterneck, *Arithmetical function and restricted partitions with respect to a modulus*, C. A. Nicol and H. S. Vandiver, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 825–35]. Some of these results may be found in *Niedere Zahlentheorie, P. Bachmann, Part 2* (*Leipzig, Teubner*, 1909). (Received March 1, 1956.)

410. A. M. Yaqub: *On the identities of certain algebras.*

Let $\mathfrak{A} = (A, \times, \circ, \cdots)$ be a binary algebra with primitive operations $\times, \circ, \cdots$. An *A-function* $f(\zeta, \eta, \cdots)$ is a function from $A, A, \cdots$ to A. An *$\mathfrak{A}$-function* is an A-function which is a primitive composition of some indeterminate-symbols $\zeta, \cdots$ of A and a (possibly empty) set of constants ($=$fixed $\in A$). A *strict $\mathfrak{A}$-function* is an $\mathfrak{A}$-function which involves no constants. An *$\mathfrak{A}$-identity* $f(\zeta, \cdots) = g(\zeta, \cdots)$ is an identity between the $\mathfrak{A}$-functions f, g. When both f and g are *strict*, the identity is called *strict $\mathfrak{A}$-identity*. $\mathfrak{A}$ is *strictly complete* if A is finite and if each A-function may be expressed as some *strict $\mathfrak{A}$-function* (see A. L. Foster, Math. Zeit. vol. 58 (1953) pp. 306–336). The following theorem is proved: A strictly complete algebra with more than one element has a finite basis, i.e., a finite set of identities from which *all* the strict identities of the algebra are logical consequences. The proof uses the concept of free algebra. Finally, an example by Lyndon (Proc. Amer. Math. Soc. vol. 25 (1954) pp. 8–9) is treated in regard to the above finite basis theorem. (Received February 23, 1956.)

ANALYSIS

411*t*. R. W. Bass: *Global continuation of periodic solutions of ordinary differential equations.*

When Poincaré's method of continuation is formulated abstractly, it appears that his classical hypothesis (concerning the nonexistence of nontrivial solutions of the variational equations of the known solution) merely asserts that the Leray-Schauder index does not vanish. Similarly, in the important but difficult degenerate cases, a result of the author supplies the *LS* index as the Kronecker index of the bifurcation equations. These two results permit establishment of the existence of periodic solutions by *non-local* perturbation processes, which was Poincaré's original goal. Moreover, the known results become special cases of a single theory. The integral equation technique of D. C. Lewis, Jr. (based on that of L. Lichtenstein and E. Hölder) is brought to completion in a form which can be shown "best possible of its type." In illustration, a direct perturbation proof is applied to van der Pol's equation. It is often convenient, e.g. for autonomous oscillations, to employ instead the method of undetermined Fourier coefficients (following G. W. Hill, A. Wintner, C. L. Siegel). In this approach, several new techniques are developed. There are applications to nonlinear mechanics. (Received February 29, 1956.)

412*t*. R. W. Bass: *On solution of singular functional equations by the Leray-Schauder theory.*

For each fixed $\lambda \in [0, \Lambda]$ let $F(x; \lambda)$ be a completely continuous map of a Banach space into itself. Let F be uniformly continuous in λ for x in any fixed ball, and suppose that x_0^k ($k = 1, \cdots, n$) satisfy (E): $x - F(x; \lambda) = 0$ at $\lambda = 0$. Suppose that the Fréchet differentials $L^k = F_x(x_0^k; 0)$ exist and are completely continuous linear operators. The "regular" case occurs when $(I - L^k)x = 0$ implies $x = 0$ ($k = 1, \cdots, n$); it is then usually possible to compute the Leray-Schauder total index $i(F)$. If $i(F) \neq 0$, and if a uniform *a priori* bound for the solutions of (E) is known, then (E) has a solution for every $\lambda \in I_\Lambda$. In the singular case, finite dimensional bifurcation equations (V): $g(x, c; \lambda) = 0$ occur; this case has been treated purely *locally* (i.e., for values of λ

sufficiently near to zero) by Liapounoff, Schmidt, Iglisch, Lichtenstein, Friedrichs, Cronin, Bartle, and Graves. However it is frequently possible (by inspection, or Newton's polygon) to replace (V) by modified equations (V′): $g'(x, c; \lambda)=0$ in such a way that $i(F)=\mathrm{KI}(g')$, where KI, the Kronecker Index of g', is just the sum of the signs of the Jacobians of the initial solutions of (V′). In this way (E) can be solved for all $\lambda \in I_\Lambda$ even in the singular case. (Received February 29, 1956.)

413. G. U. Brauer: *Tauberian theorems for general integral transforms.*

Let $\phi(t)$ denote a real-valued function, integrable on each finite interval $[0, T]$, and let $K(x, t)$ be defined for nonnegative values of x and t, and be in L^1 on $0 \leqq t < \infty$, for each non-negative value of x. The K-transform of ϕ is the function $\int_0^\infty K(x, t)\phi(t)dt$. The integral is assumed to converge for $x>0$. If $\phi(t)=o(1/t)$, $t\rightarrow\infty$, and the kernel K satisfies the conditions (i) $|K(x, t)-K(0, t)| \leqq Lxt$, where L is a universal constant, (ii) $\int_0^\infty |K(x, t)|dt=O(1/x)$, $x\rightarrow 0+$, then the existence of $\lim_{x\rightarrow 0+}\int_0^\infty K(x, t)\phi(t)dt$ ensures the convergence of the integral $\int_0^\infty K(0, t)\phi(t)dt$. If the condition on $\phi(t)$ is replaced by $\int_0^\infty t[\phi(t)]^2dt<\infty$ and condition (ii) on K is replaced by $\int_0^\infty [K(x, t)]^2dt =O(1/x)$ as $x\rightarrow 0+$, then the same conclusion holds. These results are analogues of well-known theorems on Taylor series. If x is a complex variable and the kernel is analytic in x, then an analogue of Riesz' theorem can be obtained. (Received March 2, 1956.)

414. T. A. Elkins: *Orthogonal harmonic functions in space.*

In two dimensions it is easily shown by the use of the complex variable that given any nonconstant harmonic function, another nonconstant harmonic function can be found such that the product of the two is harmonic. It is shown by presenting a counter example that this result does not extend to three dimensions. A study is made of some classes of harmonic functions in space for which such a nonconstant harmonic multiplier does exist. This is the case for all harmonic functions depending on only one or two variables. Finally, solutions are given to the following two problems: (1) determining whether or not for a given harmonic function of three variables there exists a nonconstant harmonic function such that the product of the two is harmonic; (2) determining all such functions in case one does exist. When a harmonic multiplier v exists for a given harmonic function u, the two functions are easily shown to be orthogonal in the sense that $u_xv_x+u_yv_y+u_zv_z=0$; which explains the name given to them in the title. (Received February 27, 1956.)

415. Jacob Feldman and J. M. G. Fell (p): *Separable representations of rings of operators.*

Let A be a ring of operators in a Hilbert space H. A *-representation of A (that is, a *-homomorphism of A into the bounded operators on some Hilbert space K) is separable if both H and K are separable. Now any *-representation L is norm-continuous; if in addition it is σ-weakly continuous, then the kernel of L is a direct summand of A, and the range of L is a ring of operators in K It is shown that a separable *-representation L of A is always σ-weakly continuous if one of the following conditions holds: (1) A is purely infinite, i.e., contains no finite direct summand, (2) A is a factor of Type II_1, (3) A is finite of Type II and the range of L is contained in a finite ring of operators in the range space K. An immediate corollary of this result is the nonexistence of separable irreducible *-representations of A when A is purely infinite or a factor of Type II_1. It remains an open question whether, for an arbitrary Type II finite algebra A, all separable *-representations of A are σ-weakly continuous. (Received February 27, 1956.)

416*t*. Casper Goffman: *Sector properties.*

In his work (*Exceptional sets*, Fund. Math. vol. 32, pp. 3–32) on sets derived from interval properties, a simple observation made by H. Blumberg was that the set of points of anti-symmetry of an arbitrary interval property is countable. He then obtained analogues of this fact, in the plane, for segment properties, region properties, etc. The purpose here is to obtain a result regarding circular sectors in the plane of which these facts are special cases. It is to be remarked that while an interval property is a subdivision of the set of all intervals into two disjoint sets, those which have the property, and those which do not have the property, the present result pertains to a subdivision of the set of sectors into three disjoint sets, A, B, and C where, for two mutually piercing sectors, if one belongs to A the other does not belong to B. A set S, for example, leads to such a subdivision if a sector is in set A if it is a subset of S; in set B, if it is a subset of CS; and in set C, otherwise. Certain converse results are also obtained. (Received February 10, 1956.)

417. R. R. Goldberg (p) and R. S. Varga: *Moebius inversion of Fourier transforms.*

Let $F(t)=\int_0^\infty \phi(u)\cos tu\,du$. Let $G_N(t)=(1/t)[F(0)/2+\sum_{k=1}^{N}(-1)^kF(k\pi/t)]$. If (1) $\int_0^\infty|\phi(t)|dt<\infty$ and (2) $\phi(t)$ is of bounded variation on every finite interval, then $G(t)=\lim_{N\to\infty}G_N(t)=\sum_{k=1}^\infty\phi[(2k-1)t]$ almost everywhere $(0<t<\infty)$. Let $\{\mu_n\}_{n=1}^\infty$ be the Moebius numbers defined as $\mu(1)=1$, $\mu(n)=(-1)^s$ if n is the product of s distinct primes, $\mu(n)=0$ if n is divisible by a square. Then, using the well-known property $\sum_{d|m}\mu(d)=\delta_{1,m}$, one can show that $\sum_{n=1}^\infty \mu_{2n-1}G[(2n-1)t]=\phi(t)$ almost everywhere $(0<t<\infty)$, provided that, in addition to conditions (1) and (2) above, the condition (3) $\int_1^\infty|\phi(t)|\log t\,dt<\infty$ holds also. The sine transform may be inverted by a similar procedure. (Received February 20, 1956.)

418. J. H. B. Kemperman: *A functional equation.*

Let A be an open connected subset of the Euclidean space E_m and let Ω be the class of all functions on A equal to a polynomial. Then Ω satisfies the following property Q. Let a_i be real constants, $a_i\neq a_0=0$ $(i=1,\cdots,k)$. Let B be a set of positive measure in E_m and let $f_i(t)$ be defined on $A+a_iB$ $(i=0,\cdots,k)$ such that, for $x\in B$, $\sum_0^k f_i(a_ix+t)\in\Omega$. If, moreover, $f_0(t)$ is bounded on a set $C\subset A$ of positive measure then $f_0(t)\in\Omega$. This yields an easy proof of a result due to V. P. Skitovic (Izv. Akad. Nauk vol. 18 (1954)) concerning the normal distribution. Property Q is shown to hold for many other classes Ω. As a corollary to one of the auxiliary theorems the author obtains (cf. H. Steinhaus, Fund. Math. vol. 1 (1920)): let B_1 and B_2 be subsets of E_m, each of positive measure. Then, after deleting proper subsets of measure 0 from B_1 and B_2, the set of differences B_1-B_2 becomes an open subset of E_m. (Received February 28, 1956.)

419. G. L. Krabbe: *Rings and spectra of factor-sequence operators on L^p.*

Suppose $1\leqq p<\infty$ throughout, and let $\mathfrak{A}_p$ denote the set of all factor-sequences of type (L^p, L^p) (see Hille, *Functional analysis and semi-groups*, pp. 343–344). Any a in $\mathcal{A}_p$ defines a transformation T_a belonging to the set $\mathfrak{E}_p$ of all bounded linear mappings of L^p into itself. *Lemma*: If $a \in \mathcal{A}_p$, then the point-spectrum of T_a is the range of a. If $\mathcal{R}$ is a ring, denote by $\mathcal{R}^*$ the set of all regular elements of $\mathcal{R}$. Recall that $\mathcal{A}_1 \subset \mathcal{A}_p \subset \mathcal{M}$, where $\mathcal{M}$ is the ring of all bounded sequences. *Theorems*: (I). Suppose that R is a subring of $\mathcal{A}_p$ with $\mathcal{R} \cap \mathcal{M}^* \subset \mathcal{R}^*$; if $a \in \mathcal{R}$, then the spectrum of T_a is the closure $E(a)$ of the range of a. (II). Suppose a Banach algebra $\mathcal{R}$ satisfies the conditions of (I). If the topology of $\mathcal{R}$ is finer than the product-topology, then the mapping $a \rightarrow T_a$ of $\mathcal{R}$ into $\mathfrak{E}_p$ is a continuous isomorphism; moreover, if $a = \{a_n\}_n \in \mathcal{R}$ and if f is an analytic function on $E(a)$, then the sequence $\{f(a_n)\}_n$ is in $\mathcal{R}$. This extends a result of Hille [17.2.3 in loc. cit.] concerning the case $\mathcal{R} = \mathcal{A}_1$, $p = 1$. When $p = 2$, then $\mathcal{R} = \mathcal{A}_2$ verifies (II). For $p > 1$, a ring $\mathcal{R} = \mathcal{R}_p$ studied by Marcinkiewicz [Studia Math. vol. 8 (1939) pp. 78–91] satisfies the conditions of (I), and the set of all sequences of bounded variation form a ring $\mathcal{R} \subset \mathcal{R}_p$ satisfying the conditions of (I) and (II). (Received February 23, 1956.)

420. M. Z. Krzywoblocki: *On the generalized Bergman's method of linear integral operators.*

Assume that in the fundamental equation underlying the final form to which Bergman's method is applied, there appears an arbitrary given function of independent variables. Then the author shows that under these assumptions the Bergman method of the linear integral operator can be applied directly to that generalized equation. The proof of existence follows directly; it is shown what kind of changes one has to introduce in the tabular representation of Bergman's functions in order to adjust them to the new assumption. (Received February 17, 1956.)

421. R. G. Kuller: *Representations of locally convex vector lattices.*

The Banach lattice has been defined as a lattice ordered Banach space in which the norm and the ordering are related by the axiom (A) If $|x| \leq |y|$ then $\|x\| \leq \|y\|$. In this paper the locally convex vector lattice is defined as a locally convex Hausdorff linear space, which is lattice ordered, and in which there exists a fundamental system of neighborhoods $[V_\alpha]$ of 0, such that the corresponding semi-norms satisfy the inequality (a). If $|x| \leq |y|$, then $p_\alpha(x) \leq p_\alpha(y)$. A complete locally convex vector lattice is the projective limit of Banach lattices. If, with (a), one postulates (m) If $x \wedge y \geq 0$, then $p_\alpha(x \cup y) = \text{Max}\,[p_\alpha(x), p_\alpha(y)]$, for all α, one obtains the locally m-convex vector lattice, a natural generalization of the abstract M space defined by Kakutani. If, with (a), one postulates (1) If $x \wedge y \geq 0$, then $p_\alpha(x+y) = p_\alpha(x) + p_\alpha(y)$ for all α, one obtains the locally 1-convex vector lattice, the generalization of the abstract L space of Kakutani. Representation theorems for these two types of topological vector lattices are given. (See the Ann. of Math. (2) vol. 42 (1941) for Kakutani's papers.) (Received February 22, 1956.)

422*t*. Karel deLeeuw: *Homogeneous Banach algebras*. II. *Classification of uniformly closed algebras.*

Let G be a compact abelian topological group. A homogeneous Banach algebra on G is a separating, translation invariant Banach algebra of complex valued continuous functions on G that contains 1 and is generated by the characters of G that it contains and also whose topology is stronger than pointwise convergence. *Theorem*: There is a natural $1-1$ correspondence between the class of homogeneous algebras A on G that are uniformly closed on their maximal ideal space and the class of all pairs (S, p)

where S is a subsemigroup with unit of G^* that generates G^* and p is a real valued function on S that satisfies 1. $p(s) \geq 1$ for all s in S; 2. $p(e) = 1$; 3. $p(s_1s_2) \leq p(s_1)p(s_2)$; 4. $p(s^n) = p(s)^n$. The correspondence is set up by taking S to be the subset of characters in G^* that occur in A and defining p by $p(s) = \|s\|$, where $\|\cdot\|$ is the sup norm. The only real difficulty arises in showing that no two distinct algebras can give rise to the same p. This is accomplished by the use of a result in the theory of several complex variables that relates polynomial and monomial convexity. (See Bull. Amer. Math. Soc Abstract 62-1-142.) Those algebras having $p = 1$ have been studied extensively by Arens and Singer in *Generalised analytic functions*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 379–393. (Received February 27, 1956.)

423. Karel deLeeuw: *Silov boundary and Cauchy integral formula for circular subsets of the space of n complex variables.*

For any point z in C^n, z_i will denote its ith coordinate. Let X be a bounded subset of C^n that contains, for each x in X, every y that satisfies $|y_i| = |x_i|$. Let A be the Banach algebra obtained as the uniform closure of polynomials on X. *Theorem*: The maximal ideal space of A is identified with $Y = \{y : |m(y)| \leq \operatorname{Sup}_{x \in X} |m(x)|$, all monomials $m\}$ by associating to each y in Y the multiplicative linear functional $f \to f(y)$. A point y in Y is called a *multiplicative extreme point* if $(y_1^2, \cdots, y_n^2) = (u_1v_1, \cdots, u_nv_n)$ for points u and v in Y implies $|u_i| = |v_i|$. *Theorem*: The Silov boundary of A, that is, the smallest closed subset of Y on which every f in A attains its maximum modulus, is the closure of the multiplicative extreme points in Y. Let $\{u\}$ be a collection of multiplicative extreme points in Y and $\{M_u\}$ a partition of the set of monomials such that each monomial in M_u attains its maximum modulus at u. These can always be found. Let T_u be $\{(u_1t_1, \cdots, u_nt_n) : |t_i| = 1\}$ and μ_u the measure on T_u induced by Haar measure on $\{t : |t_i| = 1\}$. Let $K_u(x, y) = \sum_{m \in M_u} m(x)m^{-1}(y)$. *Theorem*: For any f in A, in particular for any f analytic in a neighborhood of Y, and any x in the interior of Y, $f(x) = \sum_u \int_{T_u} f(y)K_u(x, y)\,d\mu_u(y)$. (Received February 27, 1956.)

424t. Karel deLeeuw: *Subsemigroups of compact groups are equidistributed.*

Let G be a compact topological group with normalized Haar measure μ. Let S be a subsemigroup that generates a dense subgroup of G and $C(S)$ be the space of bounded complex valued functions on S with the uniform topology. *Theorem*: If f in $C(S)$ is the restriction of a continuous function h on G, then the unique constant function in the convex closure of the translates of f on S has the value $\int_G h\,d\mu$. In most special cases the equidistribution can be exhibited more explicitly. Let $\{U_m\}_{m=1}^{\infty}$ be a collection of finite subsets of S that satisfies $N(sU_m \Delta U_m)/N(U_m) \to 0$ for every s in S, where $N(X)$ is the number of elements of X and Δ is symmetric difference. *Theorem*: For f and h as above, $\sum_{s \in U_m} f(sx)/N(U_m) \to \int_G h\,d\mu$ uniformly in x. As an application, if α and β are irrational, the semigroup that consists of all $m\alpha + n\beta$, for all lattice points (m, n) in a sector of the plane, is equidistributed mod 1. Also if γ and δ are positive real with γ/δ irrational, the semigroup of $m\gamma + n\delta$, for m and n as before, is equidistributed in the almost-periodic compactification of the reals. (Received February 27, 1956.)

425. Lee Lorch: *The Gibbs phenomenon for Borel means.*

The following theorem is proved: Let $B_x(t)$ denote the xth Borel exponential or integral mean of the Fourier series $\sum_1^\infty n^{-1}\sin nt$ and let T be given, $0 \leq T \leq \infty$. Then $B_x(t_x) \to \int_0^T v^{-1}\sin v\,dv$ as $x \to \infty$, where $t_x \to 0+$ and $xt_x \to T$. Thus, the Gibbs ratio is the same for Borel means as for convergence, and is achieved for the same value, π, of the parameter T. For the exponential means the derivation is similar to that of the Lebesgue constants for the corresponding case (Duke Math J. vol. 11 (1944) pp. 459–467). For integral means the result then follows from a standard identity connecting the two Borel methods [cf. Hardy, *Divergent series*, 1949, p. 182, formula (8.5.5)]. (Received February 21, 1956.)

426. T. D. Oxley, Jr. (p) and H. K. Hughes: *On a generalised factorial series.*

It was shown (T. Fort, *Infinite series*, Oxford, 1930) that the series $\sum_{n=1}^\infty a_n A_n^{(k)}(s)$, where $A_n^{(k)}(s) = k\Gamma(nk)\Gamma(s+k)/[(s+k-1)\Gamma(s+nk)\Gamma(k)]$, converges in a half-plane $R(s) > \lambda$ (absolutely for $R(s) > \mu$) and has other function theoretic properties common to the simple factorial and Dirichlet series which it includes. (1) Let $\Omega_k(s) = \sum_{n=1}^\infty a_n A_n^{(k)}(s)$, $k \geq 1$ and m a positive number $> \mu$. Then $\Omega_k(s) = \int_0^1 t^{s-1}\phi(t)\,dt$, where $\phi(t) = 1/2\pi i \int_{m-i\infty}^{m+i\infty} t^{-s}\Omega_k(s)\,ds = \sum_{n=1}^\infty a_n k\Gamma(nk)t^{k-1}(1-t)^{(n-1)k}/[\Gamma(k)\Gamma\{(n-1)k+1\}]$, for $t \in [0, 1]$. (2) If $\phi(t)$ is analytic in a region including the interval $0 \leq t \leq 1$, then $\Omega_k(s)$ is analytically continued to the entire s-plane (except at the points where $s-1$ is an integer) by $\Omega_k(s) = 1/[e^{2\pi i(s-1)} - 1]\int_L t^{s-1}\phi(t)\,dt$, where L is a contour proceeding from 1 along the upper side of the positive real t-axis, circling the origin, and returning to 1 along the lower side. (3) If k is complex, $R(k) > 0$, $|\arg k| \leq \pi/2 - \epsilon$, and there exists $g(w)$ for which $(-1)^n g(n) = a_n$, $g(w)$ is single-valued, analytic for $R(w) \geq 1/2$, $|g(x+iy)| \leq Kx^p e^{|y|(\pi-\epsilon)}$, p a positive number, then for all s (except when $s+k = 1, 0, -1, \cdots$) $\Omega_k(s) = -k\Gamma(s+k)/[2(s+k-1)\Gamma(k)]\int_{-\infty}^{+\infty} g(\frac{1}{2}+iy)\Gamma(\frac{1}{2}k+iky)/[\Gamma(s+\frac{1}{2}k+iky)\cosh \pi y]\,dy$. Results (2) and (3) reduce to those of H. K. Hughes (Amer. J Math. vol. 53 (1931) pp. 757–780) for the factorial series. (Received February 27, 1956).

427. W. T. Reid: *A note on adjoint linear differential operators.*

With the aid of the fundamental lemma of the calculus of variations the following result is established for linear differential operators $L(y) = \sum_{\mu=0}^n p_\mu(x)y^{(\mu)}$ $(n \geq 1)$, with coefficients in the space $\mathfrak{L}$ of complex-valued Lebesgue integrable functions on $ab: a \leq x \leq b$. If $\mathfrak{C}_n$ is the class of functions with continuous derivatives of the first n orders on ab, $\mathfrak{C}_n^0 = [y;\ y \in \mathfrak{C}_n : y^{(\alpha)}(a) = 0 = y^{(\alpha)}(b),\ \alpha = 0, \cdots, n-1]$, $\mathfrak{A}_n = [y;\ y \in \mathfrak{C}_{n-1} : y^{(n-1)}$ absolutely continuous on $ab]$, and $\mathfrak{D}^*$ is the set of $z(x)$ with $z \in \mathfrak{L}$, $zp_\mu \in \mathfrak{L}$, $(\mu = 0, \cdots, n)$, for which there exists a corresponding $f_z \in \mathfrak{L}$ such that $\int_a^b [zL(y) - \bar{f}_z y]\,dx = 0$ for arbitrary $y \in \mathfrak{C}_n^0$, then in case $x^\lambda/\lambda! \in \mathfrak{D}^*$ $(\lambda = 0, \cdots, n)$, we have: (i) there exist functions $\pi_0(x) \in \mathfrak{L}$, $\pi_j(x) \in \mathfrak{A}_{k_j}$ $(k_j = [(j+1)/2];\ j = 1, \cdots, n)$, such that $L(y) = \sum_{j=0}^n \Lambda_j(x;\ \pi_j)$, where $\Lambda_0(y;\ \pi) = \pi(x)y$, $\Lambda_{2r}(y;\ \pi) = [\pi(x)y^{(r)}]^{(r)}$, and $\Lambda_{2r-1}(y;\ \pi) = \{[\pi(x)y^{(r-1)}]^{(r)} + [\pi(x)y^{(r)}]^{(r-1)}\}/2$, $(r = 1, 2, \cdots)$; (ii) $\mathfrak{A}_n \subset \mathfrak{D}^*$, and $f_z = \sum_{j=0}^n \Lambda_j(z;\ (-1)^j\bar{\pi}_j)$ for $z \in \mathfrak{A}_n$. Various results related to the above are established; in particular, there are presented relations between the above result and certain theorems of H. L. Hamburger (Proc. London Math. Soc. (3) vol. 3 (1953) pp. 446–463) for Sturm-Liouville differential operators. (Received February 28, 1956.)

428t. E. M. Stein: *Generalization of a theorem of Paley and Wiener.*

The author generalizes a well-known result of Paley and Wiener concerning functions of exponential type to the case of n dimensions. One such generalization has been given by Plancherel and Pólya (Comment. Math. Helv. vol. 9, pp. 224–248) but the following is of different nature. Let Ω be a compact, convex, and symmetric domain in Euclidean n-space. Let Ω^* denote its polar domain, $\psi(x)$ the characteristic function of Ω^*, and $\psi_t(x) = t^{-n}\psi(x/t)$. Finally, let $H_t(x) = \int f(x+y)\psi_t(y)dy$, where it is assumed that $f(x) \in L_2(E_n)$, and that $f(x)$ is bounded. It is asserted: The Fourier transform of $f(x)$ vanishes outside Ω if and only if $H_t(x)$ is analytic and of exponential type $\leqq 1$ in t, for each x. It is an easy matter to verify that the case $n = 1$ implies the classical result of Paley and Wiener. (Received February 27, 1956.)

429t. E. M. Stein: *Inequalities of Bernstein and Kolmogoroff.*

A general method is given for passing from theorems involving the L_∞ norm to their analogues for the L_p norms. Among applications are the following. Let $f(x)$ be analytic and of exponential type σ. Suppose that $f(x) \in L_p(-\infty, +\infty)$, $1 \leqq p < \infty$. Then the following is well known: $\|f'(x)\|_p \leqq \sigma \|f(x)\|_p$. It is shown that $\|f'(x)\|_p = \sigma \|f(x)\|_p$ implies that $f(x) = 0$. The analogous statement for the sup norm is well known. Another application is the following: Let $f(x)$ be a function whose derivatives up to order n are continuous and lie in $L_p(-\infty, +\infty)$. It is shown that $\|f^{(k)}(x)\|_p^n \leqq C_{k,n} \|f(x)\|_p^{n-k} \cdot \|f^{(n)}(x)\|_p^k$. Here, $0 < k < n$, and $C_{k,n}$ are the same constants which are valid for the limiting case $p = \infty$. The case $p = \infty$ is due to Kolmogoroff (see Translation of the American Mathematical Society, no. 4). (Received February 27, 1956.)

430t. E. M. Stein: *Mean convergence below the critical index.*

Let $f(x) \in L_p(E_n)$ (E_n = Euclidean n-space), $n \geqq 2$, and let $S_R^\delta(f)$ denote the Bochner-Riesz spherical means of order δ of the "partial integrals" of the Fourier expansion of $f(x)$. The index $\delta_0 = (n-1)/2$ is known as the "critical" index. It is shown that if $1 < p < 2$, and $f \in L_p$, then $S_R^\delta(f)$ converges to f in L_p norm as $R \to \infty$, whenever $\delta > (2p^{-1} - 1)\delta_0$. A similar statement holds if $2 < p < \infty$. In particular, if $1 < p < \infty$, then there is convergence of $S_R^\delta(f)$ to f in L_p norm for some index δ below the critical index. An analogous theorem holds for multiple Fourier series. (Received February 27, 1956.)

431. F. M. Wright: *Sufficient conditions for certain C-fraction expansions.*

Let $\{k_j\}$ and $\{p_j\}$ ($j = 0, 1, 2, \cdots$), be given infinite sequences of real integers such that $k_0 = 1$, $p_0 = -1$, and $k_j \geqq 0$, $p_j \geqq 0$ for $j = 1, 2, 3, \cdots$. Let $\{\alpha_n\}$ ($n = 1, 2, 3, \cdots$) be the sequence of positive integers defined by the following rules: (i) $\alpha_1 = 1$; (ii) $\alpha_n = 1$ for $n = 2 \cdot \sum_{j=0}^{h} (k_j + p_j) + i$ ($i = 2, 3, \cdots, 2k_{h+1} + 1$; $h = 0, 1, 2, \cdots$); (iii) $\alpha_n = 2$ for $n = 2 \cdot \sum_{j=0}^{h} (k_j + p_j) + 2k_{h+1} + i$ ($i = 2, 3, \cdots, 2p_{h+1} + 1$, $h = 0, 1, 2, \cdots$). Certain assumptions made concerning the sequences $\{k_j\}$ and $\{p_j\}$ assure that the sequence $\{\alpha_n\}$ is infinite. The author considers here C-fractions of the form $\mathrm{K}_{(n)}[c_n w^{\alpha_n}/1]$ which are either nonterminating or terminating with $[2 \cdot \sum_{j=0}^{m+1} (k_j + p_j) + 1]$ partial quotients, where m is any given non-negative integer. Let $C(w)$ denote this fraction or this fraction with a partial quotient $cw/1$ attached according as this C-fraction is nonterminating or terminating. In a previous paper, the author developed and discussed certain results which hold in case a given formal power series is the corresponding power series of $C(w)$. This paper is essentially devoted to showing that certain assumptions concerning a given formal power series $\sum_{(n)} \mu_n w^{n+1}$ are sufficient for this power series to have $C(w)$ as its C-fraction expansion. (Received February 27, 1956.)

APPLIED MATHEMATICS

432t. R. W. Bass: *Equivalent linearization and the existence of periodic solutions of ordinary differential equations.*

A recent heuristic criterion, the "Nyquist diagram for nonlinear servomechanisms," is currently used by engineers as an indication of whether or not a given nonlinear differential equation has periodic solutions. The method, reminiscent of that of Kryloff-Bogoliuboff, has been developed in particular cases by involved "frequency response" arguments. Here the method is clarified and systematized by noticing that it results from time-averaging what the author calls the "instantaneous characteristic polynomial" to obtain the characteristic polynomial for a corresponding "linearized" equation. This formulation renders trivial the extension to difference-differential equations and many other cases not hitherto treated. Also a companion heuristic criterion for determining stability of the periodic solutions is presented. This criterion, based on certain new identities, extends Bulgakov's idea of time-averaging the variational equations. It is shown indirectly (by comparison with the explicit solutions) that both criteria are valid for a certain three parameter family of discontinuous difference-differential equations. By combining the method of undetermined Fourier coefficients with the Leray-Schauder theory, it is also proved directly that in a range of typical cases the first criterion is legitimate. (Received February 29, 1956.)

433. Bernard Friedman: *Virtual eigenvalues.* Preliminary report.

The resolvent $R(\lambda)$ of a class of self-adjoint ordinary differential operators over the interval $(0, \infty)$ may be extended to an analytic function of λ in a two-sheeted Riemann surface. The poles of the resolvent in the "physical" sheet are eigenvalues of the operator. For a class of operators the resolvent has poles in the second non-physical sheet. These are "virtual" eigenvalues and the corresponding solutions of the differential equations are "virtual" eigenfunctions. Even though these solutions are exponentially large for large x, they may be used to give an expansion of the solution of a partial differential equation. Applications to the Breit-Wigner nuclear level resonance formula, the Stark effect in Quantum Mechanics and the problem of α-particle decay are considered. (Received March 1, 1956.)

434. P. C. Hammer and W. H. Peirce (p): *Numerical integration over planar regions.* Preliminary report.

Hammer and Wymore have previously [Bull. Amer. Math. Soc. Abstract 61-4-318] reported on methods of obtaining numerical integration formulas exact for polynomials over certain symmetric regions. In this paper are presented explicit formulas for the circle, square, regular octagon, hexagon, and the arbitrary circular annulus. Using a confluence process, formulas for numerical integration over boundaries of some of these regions are established. From these, certain integration formulas for surfaces are established. Formulas for asymmetrical regions using mappings are established. (Received February 27, 1956.)

435. G. E. Hay: *On axially symmetric torsion.*

The problem of axially symmetric torsion reduces to the determination of a function F satisfying a certain second order partial differential equation, F being constant on the surface of the body and on the axis of revolution of the body. Cylindrical coordinates or spherical coordinates are usually employed. In the present paper, the equation satisfied by F is expressed in terms of general curvilinear coordinates. Certain nonorthogonal coordinates systems are then introduced; these lead to the solution of special problems. (Received February 29, 1956.)

436. Bayard Rankin: *An algebra for minimizing machine performance time within a class of algebraically equivalent programs.*

A program for a computing machine is a finite sequence, p, of real parameters together with a finite sequence, i, of instructions. Let PXI be a collection of pairs (p, i). The real functions $T(p, i)$ and $O(p, i)$ defined on PXI correspond respectively to the performance time and output for (p, i). The computing machine M is defined as the triplet (PXI, T, O). (p, i) and (p', i') are called algebraically equivalent (a.e.) if in the absence of round-off and overflow errors $O(p, i) = O(p', i')$. It is desired to find $(\bar{p}, \bar{i})$ within a class of a.e. programs such that $T(\bar{p}, \bar{i})$ is minimum. In this paper the author constructs an algebra of instructions for an exemplary three address machine. He uses the symbols A: add, S: subtract, M: multiply, D: divide, R: modify previous instruction, C: make a two-valued choice, $(\quad)^n$: perform bracketed sequence n times. By convention, an instruction on the left of another is performed first in time. We investigate laws with which to generate a.e. programs from known ones. Example, distributive law for addition: $(AR)^2M = (MRAR)^2$. By letting instructions depend on the output of other programs the nonlinear dynamic structure of programs is expressed as simultaneous equations. Partial sponsorship by the Office of Naval Research. (Received February 28, 1956.)

GEOMETRY

437. Eugenio Calabi: *Examples of 6-dimensional almost-complex and complex manifolds and their properties.*

It is known that any orientable, differentiable hypersurface imbedded in Euclidean 7-space R^7, considering the latter as the set of purely imaginary Cayley numbers, is *ipso facto* an almost-complex manifold. The integrability condition for the induced almost-complex structure to be derivable from a complex analytic one is expressed in terms of the second fundamental form of the hypersurface and of the invariants of the Cayley algebra, with the following results. (I) Every compact, orientable hypersurface in R^7 has a domain in which the only holomorphic functions in any subdomain are constants. (II) The only hypersurfaces for which the induced almost complex structure is integrable form a special class of minimal hypersurfaces, including the direct products of minimal surfaces in a distinguished $R^3 \subset R^7$ with the orthogonal R^4. Following this one constructs a compact, complex manifold, differentiably homeomorphic to the direct product of a closed surface of odd genus ≥ 3 with a 4-torus, which, contrary to the "trivial" model, admits no Kähler metric, and has 1st Chern class equal to zero. (Received January 25, 1956.)

438*t*. John DeCicco: *Velocity systems upon a surface.*

For a given field of force F on a surface S, there is defined a velocity family corresponding to a given constant speed $v_0 > 0$. If v_0 varies, the resulting set is a velocity system S_∞. The Lagrangian and Hamiltonian equations of an S_∞ are derived. A family of ∞^2 curves on the surface S is a velocity family if and only if the locus of centers of geodesic curvature constructed at each point P of S of the ∞^1 curves of the family that pass through P is a straight line, not passing through P. As applications, natural and isogonal families on the surface S are each characterized. Finally the transformation theory of these families is studied. (Received February 9, 1956.)

439. P. C. Hammer: *Overlapping areas of convex sets.* Preliminary report.

Let B and C be two planar convex sets of positive area. Define the point functional $f(x) = \text{Area } [B + x \cap C]$. Then f is quasi-concave, that is, $f(x_p) \geq \min [f(x_0), f(x_1)]$ where x_p is on the line segment x_0x_1. This result is generalized. Properties of quasi-concave functionals are developed. The infimum of a class of quasi-concave functionals is a quasi-concave functional. A monotonic function of a quasi-concave functional is a quasi-concave functional. A necessary and sufficient condition that a functional be quasi-concave is that its infimum on every compact set be the same as its infimum on the class of extreme points of the set. (Received February 27, 1956.)

440t. Shoshichi Kobayashi: *On holomorphic affine connections.*

The existence of a holomorphic affine connection on a compact complex manifold imposes a very strong restriction on the space. In fact, if there exists a holomorphic affine connection on a compact complex manifold M which admits a Kaehlerian metric, then all the Chern classes of M vanish. In particular, if M is compact homogeneous Kaehlerian, there exists a holomorphic affine connection on M if and only if M is a complex torus. It can be also proved that the homogeneous holonomy group of a holomorphic affine connection is a complex Lie subgroup of the complex general linear group. More generally, the holonomy group of a holomorphic connection in a holomorphic principal fibre bundle with complex Lie structure group G is a complex Lie subgroup of G It should be remarked (due to Serre) that there exists always a holomorphic connection in every holomorphic principal fibre bundle whose base space is a Stein manifold. (Received February 15, 1956.)

441t. Shoshichi Kobayashi: *On isometries of pseudo-Kaehlerian spaces.*

Every isometry of an irreducible pseudo-Kaehlerian space M of real dimension $2n$ preserves the almost complex structure or gives the conjugate structure except in the case where n is even, say $2m$, and the homogeneous holonomy group is contained in $Sp(m)$. This improves the previous result of the author (Bull. Amer. Math. Soc. Abstract 62-4-534) as well as that of Lichnerowicz (C. R. Acad. Sci. Paris vol. 239, p. 1344) and Schouten-Yano (Indagationes Math. vol. 17, p. 565). The proof is extremely simple. If $J: T(M) \to T(M)$ defines the almost complex structure and if f is an isometry of M, then $\delta f^{-1} \circ J \circ \delta f$ gives also an almost complex structure which commutes with every element of the holonomy group. The theorem follows immediately from the fact that the algebra of all linear transformations which commute with every element of an irreducible group of linear transformations of a real vector space is isomorphic to the field of real numbers, complex numbers or quaternions. (Received February 15, 1956.)

442. Albert Nijenhuis: *Derivations on differential forms.*

The graded ring $\mathfrak{R}$ of real-valued differential forms (scalar forms) on an n-dimensional manifold $\mathfrak{M}$ admits derivations D of every degree r, $-1 \leq r \leq n$, where D satisfies $D(\mathfrak{R}_p) \subset \mathfrak{R}_{p+r}$, $D(\phi + \psi) = D\phi + D\psi$, $D(\phi_p \wedge \psi_q) = D\phi_p \wedge \psi_q + (-1)^{pr} \phi_p \wedge D\psi_q$. If $D | \mathfrak{R}_0 = 0$ (D of the *first* kind), D is determined by a tangent vector-valued differential form (vector form) $L_{r+1}: D\phi_p = \phi_p \perp L_{r+1}$, with $\phi \perp L(u_1, \cdots, u_{p+r}) = (1/(p-1)!(r+1)!) \sum_{\alpha \in \mathfrak{S}_{p+r}} (-1)^{\alpha}(L(u_{\alpha_1}, \cdots, u_{\alpha_{r+1}}), u_{\alpha_{r+2}}, \cdots, u_{\alpha_{r+p}})$. If $Dd = (-1)^r Dd$ (D of the *second* kind) there is a vector r-form L such that $D\phi = [L, \phi] = d\phi \perp L - (-1)^r d(\phi \perp L)$. Every derivation is a sum of a derivation of the first, and one of the second kind. The derivations (of the first/second kind) form a graded module $\mathfrak{D}$ ($\mathfrak{D}^1/\mathfrak{D}^2$), where $\mathfrak{D}^1$ and $\mathfrak{D}^2$ are subalgebras. The vector form $[L, M]$ associated with the commutator of two derivations of the second kind determined by vector forms L and M is a differential concomitant whose existence was already known, but not its function. The identities between $[L, \phi]$ and $[L, M]$, $\phi \perp L$, etc., which express the Lie algebra nature of $\mathfrak{D}$ appear to be extremely useful in the theory of almost complex structures j, and also in almost-product structures. The operations defined there arise naturally in our scheme; e.g. $d''\phi = [I - ij, \phi]/2$, $T = [j, j]/2$, $d''d''\phi = -[T, \phi]$. (Received February 28, 1956.)

443. W. L. Stamey: *On generalised euclidean and non-euclidean spaces.*

A semimetric space is generalized {euclidean, r-hyperbolic}, i.e. has each of its n-dimensional subspaces congruent with {euclidean, r-hyperbolic} n-space, provided it is complete, convex, externally convex and has the weak {euclidean, r-hyperbolic} four-point property. (See L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford, at the Clarendon Press, 1953.) Similar definitions and results with respect to a diameterized class of spaces instead of externally convex spaces hold for generalized r-spherical and r-elliptic space. (See Blumenthal, loc. cit. and J. D. Hankins' University of Missouri doctoral dissertation, 1954.) Recently Blumenthal (*An extension of a theorem of Jordan and von Neumann*, Pacific Journal of Mathematics vol. 5 (1955)) has shown that for euclidean spaces a "feeble euclidean four-point property" can replace the weak euclidean four-point property. In the present note the generalized euclidean, r-hyperbolic, r-spherical, r-elliptic spaces are characterized among the complete, convex, semimetric spaces having segments which can be uniquely locally extended. It is shown that the appropriate feeble four-point property characterizes each of these generalized spaces among the above class of spaces. (Received February 28, 1956.)

Logic and Foundations

444t. M. L. Keedy: *On a relation which orders a set with a first element.* Preliminary report.

An element a in an abstract relation algebra, as developed in (Chin and Tarski, *Distributive and modular laws in the arithmetic of relation algebras*, University of California Publications in Mathematics, Vol. 1, No. 9, pp. 341–384), is defined to be an "order element" if it satisfies: $a; a = a$, $a \cdot \breve{a} = 1'$, and $a + \breve{a} = 1$. If, in addition, $a + 0 \neq 0$, then a is called a "first-point" order element. Such an element in a proper relation algebra over a set U is a reflexive linear order relation, and it orders the set U such that one member of U precedes all other members of U. In case the relation a orders a proper subset of U in a similar manner its arithmetic characterization is identical to the above, but is expressed in the subalgebra consisting of all relations included in $[(a+\breve{a}); 1] \cdot [1; (a+\breve{a})]$. (Received March 20, 1956.)

445t. A. R. Schweitzer: *An outline of potential theory.*

This paper consists of an introduction and two parts. The introduction has three sections A, B, C outlining a transition from geometry to mathematical physics. A. Geometry and dynamics. B. Dynamics and potential theory. C. Potential theory and mathematical physics. Part I has title: Chapters in potential theory. Chapter I. Potential theory and dynamics; Gravitation. Newton, Lagrange. Chapter II. Potential theory and analysis; harmonic functions. LaPlace. Chapter III. Potential theory and mathematical physics; Magnetism and electricity. Green. Part II has title: Potential theory and related subjects. For Part II the author refers to O. D. Kellogg, *Foundations of potential theory*, Berlin, 1929, pp. VIII–IX, pp. 94–97. On pp. 94–97 Kellogg comments on the Heine-Borel theorem with application to uniform convergence of series. In concluding his paper the author mentions lectures on potential theory delivered by David Hilbert in Göttingen, 1901–1902. (Received February 27, 1956.)

446t. A. R. Schweitzer: *On the foundations of mathematical analysis.*

The author holds that uniformity, including uniformity of convergence of series, is the most important concept in recent mathematical analysis. Reference is made to articles and treatises by W. F. Osgood including *Introduction to infinite series*, 3d ed., Cambridge, 1928; *A geometrical method for the treatment of uniform convergence and certain double limits*, Bull. Amer. Math. Soc., November, 1896, pp. 59–86. Reference is also made to E. J. Townsend, *Der Doppel Limes*, dissertation, Göttingen; G. D. Birkhoff, *Collected mathematical papers*, vol. III, p. 153. An extension of uniform convergence to "relatively uniform convergence" is due to E. H. Moore who has used his extension in the construction of a form of general analysis (New Haven, 1906). (Received February 27, 1956.)

STATISTICS AND PROBABILITY

447. G. E. Baxter (p) and M. D. Donsker: *On the distribution of the supremum functional for processes with stationary independent increments.*

Let $\{x(t), 0 \leq t < \infty\}$ be a separable stochastic process with stationary independent increments whose sample functions vanish at the origin. Such a process is characterized by $E\{\exp(i\xi x(T))\} = \exp(T\psi(\xi))$, $0 \leq T < \infty$, where $\exp(\psi(\xi))$ is the Lévy-Khintchine representation of the characteristic function of an infinitely divisible distribution. For $f(u, \lambda) = \int_0^\infty \int_0^\infty \exp(-uT - \lambda\alpha) d_\alpha \sigma(\alpha, T) dT$ with $\sigma(\alpha, T) = P\{\sup_{0 \leq t \leq T} x(t) < \alpha\}$, it is shown that for all positive u and λ; (1) if $\psi(\xi)$ is real-valued, $uf(u, \lambda) = \exp\{1/2\pi \int_u^\infty \int_{-\infty}^\infty \lambda\psi(\xi)/[(\lambda^2+\xi^2)s(s-\psi(\xi))]d\xi ds\}$; and (2) if $\psi(\xi)$ is complex and such that for some positive δ, $\int_{-\delta}^{\delta} |\psi(\xi)/\xi| d\xi < \infty$, $uf(u, \lambda) = \exp\{1/2\pi \int_u^\infty \int_{-\infty}^\infty \lambda\psi(\xi)/[\xi(\xi - i\lambda)s(s-\psi(\xi))]d\xi ds\}$. These results are used to calculate $\sigma(\alpha, T)$ in various examples. (Received February 23, 1956.)

448. S. C. Moy: *Conditional expectations of Banach space valued random variables and their properties.*

Let $(\Omega, \mathfrak{F}, \mu)$ be a probability space and $\mathfrak{X}$ be a Banach space. A random variable X is a function on Ω to $\mathfrak{X}$ which is strongly $\mathfrak{F}$-measurable. It is proved: If X is a random variable which is Bochner integrable and $\mathfrak{F}'$ is a σ-subalgebra of $\mathfrak{F}$, there is a unique random variable Y which is strongly $\mathfrak{F}'$-measurable and Bochner integrable and $\int_E Y d\mu = \int_E X d\mu$ for every $E \in \mathfrak{F}'$. Y is defined to be the conditional expectation of X relative to $\mathfrak{F}'$ and is designated by $E[X|\mathfrak{F}']$. The following properties of conditional expectations are established. 1. $\|E[X|\mathfrak{F}']\| \leq E[\|X\| |\mathfrak{F}']$. 2. Let $B_p(\mathfrak{X})$ be the Banach space of all functions X from Ω to $\mathfrak{X}$ which are strongly $\mathfrak{F}$-measurable and $\int \|X\|^p d\mu < \infty$. The mapping $T: TX = E[X|\mathfrak{F}']$ is a linear, norm decreasing, idempotent transformation of $B_p(\mathfrak{X})$ into itself. 3. By ϕ is a complex valued function on Ω which is $\mathfrak{F}'$-measurable and $\int |\phi|^q d\mu < \infty$ where $1/p + 1/q = 1$ then $E[X\phi|\mathfrak{F}'] = \phi E[X|\mathfrak{F}']$ for every $X \in B_p(\mathfrak{X})$. (Received February 27, 1956.)

449. J. M. Shapiro: *Some properties of infinitely divisible distributions*. Preliminary report.

The principal result is as follows: Let X be a random variable with an infinitely divisible distribution. Then either X has a unitary distribution or $P(|X| > A) > 0$ for any constant A. The proof uses some of the results of the basic limit theorems for sums of independent infinitesimal random variables. Note that this theorem gives a large class of distributions which are not infinitely divisible. More precise results as to when $P(X > A) > 0$ and $P(X < -A) > 0$ where $A > 0$ are also given. (Received February 27, 1956.)

TOPOLOGY

450. F. W. Anderson (p) and R. L. Blair: *Ideals in lattices of continuous functions*. I.

Let $L = L(X)$ be the lattice of all real-valued continuous functions on a completely regular space X, and let L_+ be the sublattice of all $f \geqq 0$. An ideal I of L_+ is an *R-ideal* if $f \in I$ implies $f \ll g$ for some $g \in I$ and $h \in I$ whenever $h \leqq f$ (cf. T. Shirota, Osaka J. Math. vol. 4 (1952)). An ideal of L_+ is a maximal R-ideal iff it consists of all $f \in L_+$ vanishing on an X-neighborhood of some unique $p \in \beta X$. Hence, $L(X)$ characterizes βX. Also, a class of "closed" R-ideals serves to characterize the lattice of regular closed subsets of X. If P is a prime ideal of L, it is "associated" with some unique point $p \in \beta X$ (cf. Kaplansky, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 617–623), and the set of all prime ideals of L associated with p is denoted by $\mathcal{P}(p)$. The space νX is then characterizable as the subspace of all $p \in \beta X$ for which $\mathcal{P}(p)$ has a countable cofinal subset. An investigation is made of a class of ideals important in the study of automorphisms of L, namely, those $\mathcal{P} \in P(p)$, called *nodes*, which are comparable with every $Q \in \mathcal{P}(p)$. (Received February 28, 1956.)

451t. F. W. Anderson and R. L. Blair: *Ideals in lattices of continuous functions*. II.

A point $p \in X$ is an *N-point* if the chain of nodes in $\mathcal{P}(p)$ (see previous abstract for terminology) is separable in the interval topology. If p is an N-point and $f, g \in L$, then $f(p) < g(p)$ iff $f \in P$, $g \notin P$ for some node in $\mathcal{P}(p)$. A topological characterization of N-points is given, and it is shown then that $p \in X$ is an N-point if (a) the first axiom of countability holds at p, (b) p has a connected neighborhood, or (c) p is a P-point (Gillman and Henriksen, Trans. Amer. Math. Soc. vol. 77 (1954)). In fact, p is a P-point iff $\mathcal{P}(p)$ is a chain separable in the interval topology. A point $p \in X$ is an M-point if it is either an N-point in X or a P-point in some regular closed subset of X. Again, it is possible to determine $f(p) < g(p)$ algebraically at M-points. Finally, the following statements are equivalent: (i) X consists entirely of M-points; (ii) every automorphism of L_+ sends strictly positive functions onto strictly positive functions; (iii) every automorphism of L is continuous in the m-topology. (Cf., also, Kaplansky, Amer. J. Math. vol. 70 (1948) pp. 626–634). (Received February 28, 1956.)

452. R. H. Bing: *A peculiar decomposition of E^3.*

An example is given of an upper semicontinuous decomposition G of E^3 such that G has only a countable number of nondegenerate elements, each of these elements has a complement topologically like the complement of a point, but the decomposition space associated with G is topologically different from E^3. Each element of G lies in a plane. The nondegenerate elements of G are topologically alike—in fact there is such a G where they are all pseudo-arcs. It is not known if there is such a G each of whose elements is a continuous curve. (Received February 28, 1956.)

453*t*. C. F. Briggs: *Locally normed commutative *-algebras.*

Call a linear algebra A, with identity, a locally normed algebra if it is a topological group under addition, and if the component of $O(B)$ is a Banach subalgebra, topologically open and closed in A. A is a *-algebra if there is a *-operation on A with the properties: (1) $(x^*)^*=x$, (2) $(ax)^*=\bar{a}x^*$, (3) $(xy)^*=y^*x^*$, (4) $(x+y)^*=x^*+y^*$, (5) B is a Banach *-subalgebra. A is B-related if $xx^*\,e$ has an inverse in B for every x in A, and A is regularly semi-simple if the intersection of the closed maximal left ideals is 0 and the union is the set of all singular elements. The set of all closed maximal ideals of A can be embedded topologically in the set of all maximal ideals of B if A is commutative, locally normed, *, and B-related, where the set of closed maximal ideals is topologized by embedding in the conjugate space of B as a Banach space. If M is the set of closed maximal ideals of A so topologized, then A is isomorphic and "isometric" to a subalgebra of $O(M)$ under the uniform topology. (Received February 27, 1956.)

454. E. H. Brown, Jr.: *Computability of homotopy groups.*

M. M. Postnikov associates with each arcwise connected space X a sequence of algebraically defined complexes $P_1(X)$, $P_2(X)$, $\cdots$ which he calls the natural system of X (Doklady Akad. Nauk SSSR. vol. 76, pp. 356–362.) The writer shows that if X is a finite simply connected simplicial complex, then a complex $P_{n,q}(X)$, $q>n>0$, homologically equivalent to $P_n(X)^q$ (q-skeleton) can be finitely constructed. $P_{n,q}(X)$ is constructed by induction on n and in the process of going from $P_{n-1,q}(X)$ to $P_{n,q}(X)\pi_n(x)$ is computed. This process gives an explicit finite procedure for calculating $\pi_n(X)$ but it must be emphasized that the techniques involved are much too complicated to be considered practical. (Received February 23, 1956.)

455. E. R. Fadell: *On extending the range of the covering homotopy theorem.*

The covering homotopy theorem (CHT) is said to hold for a space Y with respect to a triple $(X, B\pi)$, where $\pi: X\rightarrow B$ is a given map, if for any homotopy $H: Y\times I\rightarrow B$ and map $g: Y\times\{0\}\rightarrow X$ such that $\pi\circ g=H$ on $Y\times\{0\}$, there exists a homotopy $G: Y\times I\rightarrow X$ which extends g such that $\pi\circ G=H$ on $Y\times I$. An open cover $\{U_\alpha\}$ of a space Y is said to be normal if there exists, for each α, real-valued maps f_α on Y which are positive on U_α and 0 elsewhere. In this note the following theorem is proved. If $\{U_\alpha\}$ is a locally finite, normal, open cover of Y and if, furthermore, the CHT holds for each $\bar{U}_\alpha$ (closure of U_α) with respect to $(X, B\pi)$, then the CHT holds for Y with respect to (X, B, π). The following is an immediate corollary. The CHT for cells implies the CHT for metric spaces which are locally polyhedral, e.g., manifolds. If one desires to condition B, one can show using the above theorem that if B is a metric ANR, the CHT for cells implies the CHT for separable metric ANR's which are locally compact and finite dimensional. (Received February 27, 1956.)

456. Jesús Gil de Lamadrid: *Higher derivatives of mappings of topological vector spaces.* Preliminary report.

Let E and F be two such spaces. Denote F^E by $\mathfrak{F}_1(E, F)$ and define $\mathfrak{F}_n(E, F)$ by induction as $\mathfrak{F}_1[E, \mathfrak{F}_{n-1}(E, F)]$ for any n. Let Σ be a family of subsets of E. The Σ-topology of $\mathfrak{F}_n(E, F)$, which for $n=1$ reduces to the topology of uniform convergence over the sets of Σ, is defined by an induction process. If $(E)^n$ denotes the cartesian product of E taken n times, and Σ^n the family of products $A_1 \times A_2 \times \cdots \times A_n$, $A_i \in \Sigma$, it is shown that there exists a homeomorphic isomorphism M of $\mathfrak{F}_n(E, F)$ onto $\mathfrak{F}_1[(E)^n, F]$, the latter space with the Σ^n-topology. If τ is the Σ-topology of $\mathfrak{F}_1(E, F)$, the Σ-derivative $f' \in \mathfrak{F}_2(E, F)$ of $f \in \mathfrak{F}_1(E, F)$, defined by the author in an earlier paper (Bull. Amer. Math. Soc. vol. 61 (1955) p. 315), is called the Σ-derivative of f. By induction one can define the nth Σ-derivative $f^{(n)} \in \mathfrak{F}_{n+1}(E, F)$. In the literature the nth differential of f is ordinarily defined as an element of $\mathfrak{F}_1((E)^{n+1}, F)$. It is shown in this note that if Σ is the family of all (1) bounded sets, (2) finite sets, then $Mf^{(n)}$, when it satisfies the unusual linearity and continuity conditions, is respectively (1) the nth Fréchet differential when E is normed, (2) the nth Gâteaux differential. Thus this last theorem permits an extension of Fréchet differentiation to non-normed spaces. (Received February 27, 1956.)

457. J. G. Hocking: *A characterization of monotone maps on 2-manifolds.*

The following theorem is established: Let M be a compact 2-manifold (with or without boundary) and f be a map of M onto itself. Then f is monotone if and only if there exists an extension h of f mapping the cone $v^\circ M$ over M at a point v onto itself such that h is a homeomorphism on $v^\circ M - M$: This generalizes a result of Floyd and Fort, *A characterization theorem for monotone mappings*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 828–830. (Received February 28, 1956.)

458. G. P. Johnson: *Spaces of functions with values in a normed ring.* Preliminary report.

Let R be a commutative normed ring, G a locally compact Abelian group, and $B_1 = B_1(G, R)$ the space of Bochner integrable R-valued functions on G. It is shown that the space $\mathfrak{M}_{B_1}$ of regular maximal ideals of B_1 is homeomorphic to the Cartesian product of the maximal ideal spaces of R and $L_1(G)$ when each of the spaces is given its weak topology. An integral formula for the Fourier transform is obtained; and with some additional restrictions on R, a number of the theorems for $L_1(G)$ are extended to B_1. B_1 is semi-simple if and only if R is semi-simple, and in this case the linear space generated by the homomorphisms $\mathfrak{M}_{B_1}$ is weakly dense in the conjugate space of B_1. This leads to an investigation of the form of the general linear functional on B_1. (Received February 28, 1956.)

459. G. R. Livesay: *On the fixed points of a class of homotopic mappings of a triangulable manifold.*

Let $\{f_t\}$, $0 \leqq t \leqq 1$, be a class of homotopic mappings of a space M into itself. Define the map $F: M \times I \rightarrow M$ by $F(x, t) = f_t(x)$. Let $P: M \times I \rightarrow M$ be the projection. Then x is a fixed point of f_t if and only if $F(x, t) = P(x, t)$. Let $A = \{(x, t) \in M \times I \mid F(x, t) = P(x, t)\}$. D. G. Bourgin (*On some separation and mapping theorems*, Comment. Math. Helv. vol. 29, 3) has asked for conditions under which A contains a continuum containing a point of $M\times 0$ and a point of $M\times 1$. A partial answer is given by the theorem: If M is a compact triangulable manifold, and the Lefschetz number of $\{f_t\}$ is not zero, then A contains a continuum C which intersects $M\times 0$ and $M\times 1$. In fact if P is any map of $M\times I\rightarrow M$ such that the coincidence number of $F|M\times 0$ and $P|M\times 0$ is not zero, then A contains such a continuum. The proof consists of two parts: (a) If A contains no such continuum, then there exists a submanifold N of $M\times I$ such that the inclusion $i_*:H(N)\rightarrow H(M\times I)$ is onto, and $N\cap A=\emptyset$. (b) The coincidence number of $F|N$ and $P|N=$coincidence number of $F|M\times 0$ and $P|M\times 0$. Since (b) contradicts $N\cap A=\emptyset$, then A contains a continuum such as C. (Received March 1, 1956.)

460. M. J. Mansfield: *On countably paracompact normal spaces.*

The results of this paper are generalizations of well-known theorems of J. S. Griffen, J. L. Kelley, E. Michael, and A. H. Stone (cf. J. L. Kelley, *General topology*, New York, 1955, p. 156 and pp. 170–171), and of some as-yet-unpublished results of E. Michael. A collection $\mathfrak{A}$ of subsets of a topological space is *closure-preserving* if, for each subcollection $\mathfrak{B}\subset\mathfrak{A}$, the union of the closures is the closure of the union. THEOREM: If X is a normal space, then the following statements are equivalent: (1) X is countably paracompact; (b) each countable open covering of X has a countable, locally finite, closed refinement; (c) each countable open covering of X has a countable, closure-preserving, closed refinement; (d) each countable open covering of X has a σ-discrete closed refinement; (e) each countable open covering of X has a σ-locally finite closed refinement; (f) each countable open covering of X has a σ-closure-preserving closed refinement; (g) each countable open covering of X has a countable, open, star-refinement; (h) each countable open covering of X is even; (i) for each countable locally finite collection $\{A_i\}$ of subsets of X there exists a countable, locally finite, open collection $\{G_i\}$ such that, for each i, $A_i\subset G_i$. (Received February 27, 1956.)

461. J. M. Slye: *Collections whose sums are two manifolds.*

The purpose of this paper is to establish conditions that are both necessary and sufficient in order that an upper semicontinuous collection of arcs (or simple closed curves) filling up a nondegenerate locally connected metric space, and whose decomposition space is an arc, fill a 2-cell (or annulus, or Möbius strip, or Klein bottle). Two additional conditions are needed. First no point (or pair of points) separates the space. The second condition is contained in the following definition. If P is a point of a space S, S has property X at P if and only if there exists a connected open set D containing P such that if Q and Q' are two points of D and B is a compact closed set contained in D that does not contain $Q+Q'$ and is irreducible with respect to separating Q and Q' in S, and P' and P'' are points of B, then $B-(P'+P'')$ is not the sum of three mutually separated sets each of which has $P'+P''$ in its closure. (Received February 27, 1956.)

462t. Stephen Smale: *The effect of an open map on the fundamental group.*

Let X and Y be metric spaces where X is locally arcwise connected and Y is semilocally simply connected. Suppose that $f:(X, x_0)\rightarrow(Y, y_0)$ is an open, onto map and that the inverse images of compact sets are compact. Let g be a map of the unit interval I into Y with $g(0)=y_0$. Then it is proved that there exists a map $\bar{g}:(I, 0)$

$\rightarrow(X, x_0)$ such that $f\tilde{g}(1) = g(1)$ and $f\tilde{g}$ is homotopic to g with fixed end points. Using this result, by a method of M. L. Curtis (Ann. of Math. vol. 57 (1953) p. 239) it is proved that $\pi_1(Y, y_0)/f_\# \pi_1(X, x_0)$ is finite. Hence if $\pi_1(X, x_0)$ is finite, then so is $\pi_1(Y, y_0)$ or if $\pi_1(X, x_0)$ is finitely generated, then so is $\pi_1(Y, y_0)$. (Received February 23, 1956.)

463. R. F. Williams: *A note concerning dimension raising maps.*

An earlier result of the author is generalized to the following: if $f: X \rightarrow Y$ is a map, X compact, metric and n-dimensional, Y m-dimensional, then there is a subcontinuum $M \subset X$ such that dim $M \geqq n-1/2$ and dim $f(M) \leqq m/2$. As a corollary, if for some integer i, $2^i \leqq n$ and $m < 2^i + n$, then there is a nondegenerate continuum $M \subset X$ whose image has dimension at most 1. The proofs are simple. (Received February 27, 1956.)

J. W. T. YOUNGS,
Associate Secretary
WILLIAM PRAGER
Chairman, Program Committee for the Symposium

THE APRIL MEETING IN NEW YORK

The five hundred twenty-fourth meeting of the American Mathematical Society was held on Friday and Saturday, April 20–21, 1956 at Columbia University in New York City. The meeting was attended by about 330 persons, including 299 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor J. T. Schwartz of Yale University and New York University addressed the Society at 2:00 P.M. on Friday on *Riemann's method in the theory of special functions*, and Professor Fritz John of New York University addressed the Society at 2 P.M. on Saturday on *Continuation of solutions of partial differential equations*. These sessions were presided over by Professors Nelson Dunford and J. J. Stoker respectively.

Sessions for contributed papers were held at 3:15 P.M. on Friday, and at 10 A.M. and 3:15 P.M. on Saturday. Chairmen at these sessions were Professor G. K. Kalisch, Dr. Lawrence Markus, Professors G. Y. Rainich, Alex Rosenberg, Charles Fox, Emil Grosswald, E. G. Straus. The Employment Register was maintained by Professor W. M. Hirsch.

The Council of the Society met at 5:15 P.M. on Friday, reconvening after dinner.

The Secretary announced the election of the following sixty-five persons to ordinary membership in the Society:

Mr. Sheldon Buckingham Akers, Jr., A C F Electronics, Alexandria, Virginia;
Dr. Allen E. Andersen, University of Massachusetts;
Professor Tommie Marie Anderson, Tougaloo College;
Mr. Americo Aruffo, Philco Corporation, Philadelphia, Pennsylvania;
Mr. Stephen Joseph Barone, Polytechnic Institute of Brooklyn;
Mr. Stephen Barry Behman, State College, Pennsylvania;
Mr. Andrew Angelo Benvenuto, University of Illinois;
Mr. Albert T. Bharucha-Reid, University of California, Berkeley;
Mr. Rupert Dean Boswell, Jr., University of Georgia;
Miss Christine Viola Brannan, University of Houston;
Mr. Kenneth Lawrence Brinkman, Cloud Physics, Dugway Proving Ground, Utah;
Reverend Robert Ignatius Canavan, New York University;
Mr. Michael F. Capobianco, St. John's University;
Mr. Robert Carter, P. O. Box 903, Chicago 90, Illinois;
Mr. George Gilman Chapin, Remington Rand Univac, St. Paul, Minnesota;
Dr. Yun Shick Choi, Seoul National University, Seoul, Korea;
Mr. William Henry Colbert, Jr., University of Nevada;
Professor Samuel D. Conte, Wayne University;

Mr. James Russell Cretcher, Jr., Lockheed Aircraft Corporation, Van Nuys, California;
Mr. Henry Lawrence Crowson, University of Florida;
Professor Joseph U. Crum, Western Carolina College;
Dr. Bogomil Milko Cvetkov, Snowy Mountains Hydro-Electric Authority, Cooma, NSW, Australia;
Lt. Col. Halvor Thomas Darracott, Signal Corps Engineering Laboratories, Fort Monmouth, New Jersey;
Reverend Theodosius Laszlo Demen, St. Louis University;
Professor Augustus Isaac Dhar, Illinois Wesleyan University;
Mr. N. S. D'Andrea DuBois, Jr., Howard University;
Mr. William Preston Eames, Queen's University, Kingston, Ontario, Canada;
Mr. Edwin George Eigel, Jr., St. Louis University;
Dr. Kathryn Powell Ellis, Iowa City, Iowa;
Dr. Carl Clifton Faith, Michigan State University;
Miss Hilda Feist, Pennsylvania State University;
Mr. George David Findlay, McGill University, Montreal, Quebec, Canada;
Mr. Robert Sumner Fishman, University of Vermont;
Mr. Gerald Marvin Fleischner, Polytechnic Institute of Brooklyn;
Mr. Marion McGee Fulk, National Bureau of Standards (CEL), Boulder, Colorado;
Mr. Fernley L. Fuller, Grumman Aircraft Engineering Corporation, Bethpage, New York;
Mr. David Snellenberg Greenstein, University of Pennsylvania;
Mr. Edgar H. Grossman, Grossman and Miller, Vancouver, British Columbia, Canada;
Professor Norman Bray Haaser, University of Notre Dame;
Mr. John Leo Hammersmith, U. S. Naval Research Laboratory, Washington, D. C.;
Mr. Jean-Baptiste François Hocepied, 14 rue des Receveurs, Bruges, Belgium;
Mr. Bernard Ashworth Hodson, Auro Aircraft, Malton, Ontario, Canada;
Dr. Melvin Joseph Jacobson, Bell Telephone Laboratories, Whippany, New Jersey;
Mr. Morton Kupperman, Department of Defense, Washington, D.C.;
Dr. Andre Gilbert Laurent, Michigan State University;
Mr. Carroll Raymond Lindholm, Motorola, Inc.;
Professor Emmet Francis Low, Jr., University of Miami;
Mr. Robert Quinn Macleay, Christian College;
Professor Jean-Marie Maranda, University of Montreal;
Mr. James Walter Moeller, Holloman Air Force Base, New Mexico;
Dr. Jurgen Moser, Institute of Mathematical Sciences, New York University;
Dr. Pauline Mann Nachbar, Lockheed Aircraft Corporation, Van Nuys, California;
Mr. Walter Pressman, Signal Corps Engineering Laboratories, Fort Monmouth, New Jersey;
Dr. Elvira Strasser Rapaport, Stockbridge, Massachusetts;
Professor Robert Davis Richtmyer, Institute of Mathematical Sciences, New York University;
Dr. Shepley Littlefield Ross, University of New Hampshire;
Professor Zevi Walter Salsburg, The Rice Institute;
Mr. Roger Lee Shaw, University of California, Berkeley;
Professor Robert Wesley Sloan, University of New Hampshire;
Mr. Frank Marcellus Staley, Jr., The Fort Valley State College;
Dr. Arthur Steger, University of New Mexico;

Professor Frederick Max Stein, Colorado Agricultural and Mechanical College;
Mr. Ralph Surasky, University of South Carolina;
Mr. Charles Maynard Terry, National Security Agency, Washington, D. C.;
Reverend Clarence Joseph Wallen, St. Louis University.

It was reported that the following two-hundred thirty-six persons had been elected to membership on nomination of institutional members as indicated:

Alabama Polytechnic Institute: Professor Clayton V. Aucoin.

University of Alabama: Mr. Arnold L. Milner.

University of Arizona: Mr. LeRoy Austin Kenna, Mr. Bernard Marcus, and Dr. Arthur Henry Steinbrenner.

Brown University: Dr. Walter F. Freiberger, Professor Albert Edward Green, Dr. Heini Halberstam, Mr. Herbert Meyer Kamowitz, and Mr. Thomas Patrick Mulhern.

California Institute of Technology: Mr. Theodore Kerner Matthes and Mr. Charles Robsom Storey, Jr.

University of California, Berkeley: Dr. Frederick Ellis Alzofon, Mr. John James Harton, Mr. Lee Opert Heflinger, Mr. Jean-Pierre Imhof, Mrs. Carol Ruth Karp, Mr. Melvin Louis Katz, Jr., Mr. Thomas Charles Kipps, Mr. Otto Plaat, Mr. Robert Richard Read, Mr. Edward Scott Robbins, Mr. Lucien Roland Roy, Miss Mandakini J. Sané, and Mr. Aram John Thomasian.

University of California, Los Angeles: Mr. Alfred William Adler, Mr. Richard Lewis Dunn, Mr. Yoichiro Fukuda, Mr. Antranig Vaughn Gafarian, Mr. Tom Alan Magness, Mr. Robert Ralph Phelps, Mr. Kenneth Everett Ralston, and Miss Beatriz Rossello.

Case Institute of Technology: Mr. Donald Wilford Robinson.

University of Chicago: Mr. Hyman Bass, Mr. Robert James Blattner, Mr. Richard Earl Block, Mr. Paul Joseph Cohen, Mr. Platon Constantine Deliyannis, Mr. David James Foulis, Miss Cynthia Freedman, Mr. Leonard Gross, Mr. Laurence Raymond Harper, Jr., Mr. Ray A. Kunze, Mr. Eben Matlis, Mr. John J. McKibben, Mr. Benjamin Muckenhoupt, Mr. Donald Samuel Ornstein, Mr. Donald Lewis Reinken, Mr. Stephen Hoel Schanuel, Mr. Haruo Suzuki, and Miss Mary Catherine Weiss.

University of Colorado: Mr. Robert George Buschman, Mr. James Earl Householder, and Mr. Roy Ben Kriegh.

Columbia University: Mr. Richard Stanley Wasserman.

Duke University: Mr. Waleed A. Al-Salam, Mr. Charles Edward Brooks, Mr. Harry Herbert Corson, III, Mr. Arthur Louis Gropen, Mr. Julius Robert Johnson, Jr., Mr. Nosup Kwak, Mr. Stewart Marshall Robinson, Mr. William Rowley Shawver, Mr. Marvin Gene Sperry, and Mr. Robert Guy Van Meter.

University of Florida: Mr. Dmitri Elias Thoro.

Harvard University: Mr. Robert Daniel Mize Accola, Mr. Glen Eugene Bredon, Mr. Thomas Bulkley Knapp, Mr. Henry Jacob Landau, Mr. Robert Pearlman, Mr. Kenneth Franklin Simpson, Jr., Professor Saffet Süray, Professor Yuki Wooyenaka, and Mr. Mishael Zedek.

College of the Holy Cross: Reverend Raymond Joseph Swords.

University of Illinois: Mr. Walter Scott Bartky, Mr. Hans Peter Dembowski, Mr. Gene Howard Golub, Mr. Vytas B. Gylys, Mr. Frank John Hahn, Mr. James Tomei Joichi, Dr. Klaus Krickeberg, Mr. Indar Singh Luthar, Mr. Chih-Han Sah, Mr. William Mack Sanders, and Mr. Shoji Sato.

Indiana University: Mr. Terence Butler, Mr. Gabriel Margulies, and Mr. Dock Sang Rim.

Institute for Advanced Study: Dr. Michael Francis Atiyah, Dr. Hans Joachim Bremermann, Mr. Georg Kreisel, Professor Sigekatu Kuroda, Professor Shigeo Nakano, Dr. Georges Papy, Dr. Walter Otto Roelcke, Professor Jean-Pierre Serre, Dr. Tsuneo Tamagawa, and Dr. John Hunter Williamson.

The Johns Hopkins University: Mr. Cornelius Bright Baytop, Mr. William Milton Brown, Mr. Vernon E. Derr, Mr. Walter James Douglass, Jr., Professor Jun-ichi Igusa, Mr. Kenneth Frederick Ireland, Mr. Jay Andrew Horton Jacobs, Reverend C. Frederick Koeler, S. J., Mr. Donald Malick, Mr. Richard Ferris Muth, Mr. Frank Vernon Rigler, Dr. Joseph Andrew Silva, Mr. John Clay Stuelpnagel, and Mr. James Howard Young.

Kenyon College: Mr. Trevor Herbert Barker.

Lehigh University: Mr. Francis Clyde Oglesby.

University of Maryland: Mr. Norman William Bazley, Mr. Louis William Ehrlich, Mr. Irving Isadore Glick, Mr. Steven H. Schot, and Mr. Harry Shaw, Jr.

University of Miami: Mr. Ernst Paul Buchtenkirch, Miss Michelle Roberta Harrell, and Mr. Jack Segal.

Michigan State University: Mr. John David Barab, Jr., Mr. Robert Gene Brown, and Mr. Joseph Alphonse Meier.

University of Minnesota: Mr. G. Philip Johnson.

University of Nebraska: Mr. Mervin L. Keedy.

Northwestern University: Mr. S. Forrest Ebey, Mr. Alan John Heckenbach, and Mr. Daniel Saltz.

Ohio State University: Miss Virginia Sheldon Hanley.

Oklahoma Agricultural and Mechanical college: Miss Parsla Kleinhofs and Mr. William Edwin Pruitt.

Oregon State College: Mr. David Linus Clark.

University of Oregon: Mr. Abdur Rahman Ansari, Mr. Larry Clifton Hunter, Mr. Donald Palmer Peterson, and Dr. William Leray Roach, Jr.

The Pennsylvania State University: Miss Thelma Ruth Hobaugh and Professor Mary Lister.

University of Pennsylvania: Mr. Shaul Reuven Foguel and Mr. Thomas Harold MacGregor.

Princeton University: Mr. Franz Bingen, Mr. Peter Nicholas Burgoyne, Mr. Aubert Daigneault, Mr. Arthur Pentland Dempster, Mr. John Edwin Derwent, Mr. Giulio Fermi, Mr. Harry Furstenberg, Mr. Marvin Jay Greenberg, Mr. Frank Bardsley Knight, Dr. Simon Bernard Kochen, Mr. Constantine John Koutsopoulos, Mr. James George Larocque, Mr. James Carlyle Lillo, Mr. Raymond Charles Mjolsness, Dr. Josef Schmid, Mr. Carroll Max Shipplett, and Mr. Stephen Weingram.

Purdue University: Mr. Jack Edwin Forbes, Mr. Joseph Edmund Kist, Mr. Carl William Kohls, Mr. Maynard Joseph Mansfield, Mr. Robert Hull McDowell, and Mr. Theron D. Oxley, Jr.

The Rice Institute: Mr. Howard Benton Curtis, Jr., Mr. Joel Dyne Erdwinn, Mr. James Edward Scroggs, and Mr. Daniel Weiser.

University of Rochester: Mr. David MacGregor Burton, Mr. Robert Allan Hultquist, and Mr. Paul L. Kingston.

Rutgers University: Professor Morton J. Hellman, Professor Richard E. Henry, Professor Robert Edward Luce, Mr. Guy W. Ricker, Professor Francis-Aloysius Charles Sevier, and Mr. Aaron Siegel.

College of St. Thomas: Mr. Lawrence Lee Pinsonneault.

University of Southern California: Mr. John Donald Brooks, Mr. Frank Sidney Cater, Mr. Stanley J. Greif, Mr. Alemu Melaku, Dr. Karl Hans Roth, Mr. Bernard Whitney, and Mr. Abraham Zukerman.

Stanford University: Mr. Gerald Lee Alexanderson, Mr. James Edward Burke, Mr. H. Jay Davis, Mr. Augustus Jerome Fabens, Mr. Donald D. Fisher, Miss Laura Stith Ketchum, Professor Harold Levine, Dr. Johannes Carl Christian Nitsche, Dr. Vikramaditya Singh, Mr. Dale William Swann, Mr. Ju-kwei Wang, and Mr. John Edwin Wetzel.

Syracuse University: Mr. Arnold Grudin, Mr. Simon Hellerstein, Mr. William Jordan Jones, Mr. Norman Samuel Rosenfeld, and Mr. Kapbyung Yoon.

The University of Tennessee: Mr. Ralph David McWilliams.

The University of Texas: Mr. Ben Fitzpatrick, Jr., Professor James Prichard Jewett, Mr. Charles William Leininger, and Mr. James Howard Wells.

University of Toronto: Mr. Frans Handest, Mr. Donald Albert Lyon, Mr. Barry Miller Mitchell, Mr. Frank Arthur Sherk, Mr. Carl Frederick Templin, Mr. James Ray Vanstone, and Mr. Joshua Tsunekatsu Yamada.

Tulane University: Mr. Lee William Anderson and Mr. Arthur Bernard Simon.

University of Utah: Mr. Lawrence Fearnley.

Vanderbilt University: Mr. Harvey P. Carter.

University of Virginia: Mr. John Joseph Costello.

Washington University: Mr. Nai-chao Hsu.

University of Washington: Mr. Billy Lee Foster, Mr. Shoshichi Kobayashi, and Mr. Sam Cundiff Saunders.

Wayne University: Mr. Max Krolik, Dr. Fred Meyer, Mr. Harold Willis Milnes, Mr. David Dean Morrison, and Mr. Sadanand Verma.

Wellesley College: Miss Isabel Stewart Macquarrie.

Yale University: Mr. John Harold Ahlberg, Mr. Robert Allen Bonic, and Mr. Arshag Berge Hajian.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker-Vereinigung: Professor Herbert E. L. Bilharz, University of Würzburg, Germany, Dr. Friedrich Claus Huckemann, Harvard University, and Professor Emeritus Eduard Rembs, Technische Universität, Berlin, Germany; Indian Mathematical Society: Mr. Bhagwandas Dube, Government Engineering College, Jabalpur, India; Schweizerische Mathematische Gesellschaft: Dr. Michel Andre Kervaire, Berne University, Berne, Switzerland; Société Mathématique de France: Dr. Marcel Paul Schutzenberger, Centre National Recherche Scientifique, Paris, France; Swiss Mathematical Society: Dr. Urs Walter Hochstrasser, The American University and National Bureau of Standards, Washington, D. C.

The Secretary announced that by mail votes, Professors K. O. Friedrichs and Antoni Zygmund had been elected to the Executive Committee; Professor J. L. Doob had been elected a member of the

Editorial Committee for the Transactions and Memoirs as a replacement for Professor Kac for the period January 1–June 1, 1956; the U. S. Naval Postgraduate School, Worcester Junior College, Brandeis University, and the New Mexico College of Agriculture and Mechanic Arts had been elected to institutional membership.

The Council voted to approve a Summer Institute in 1957 on the topic Mathematical Logic.

The Council approved amendments to the by-laws to make it possible for a member, by paying an appropriate premium in addition to his dues, to receive either *Mathematical Reviews* or the *Transactions* as a privilege of membership in lieu of the *Proceedings*. The amount of the premiums will be set by the Trustees.

The Council voted to recommend the appointment of Professor J. L. Brenner as Executive Editor of *Mathematical Reviews*.

The Council voted to approve a Seminar on Mechanics to be held in the summer of 1957. The major objective of this Seminar will be instructional and it is expected that those who attend will have a background of information substantially equivalent to that of Ph.D.

The Secretary reported that Professor Marshall Stone had accepted an invitation to deliver the Gibbs Lecture in 1956; that Professor L. M. Graves had been appointed Editor of the Proceedings of the Applied Mathematics Symposium held at the University of Chicago, April 12–14, 1956; that the Advisory Board of the Applied Mechanics Reviews had been terminated as of December 2, 1955.

The Council voted to set meetings on April 5–6, 1957, at New York University; October 26, 1957, at the National Bureau of Standards, Washington, D. C.; and approved the dates November 30–December 1, 1956, for the fall meeting of the Southeast section at Lexington, Kentucky; and the dates August 26–30, 1957, for the Summer Meeting at Pennsylvania State University.

The Executive Director reported that the Headquarters of the Society will be moved to a new building, 190 Hope Street, Providence, Rhode Island.

The Council voted to authorize the sale of membership buttons.

The Council elected Professor W. S. Massey a member of the Editorial Committee for the Transactions and Memoirs as a replacement for Professor Samuel Eilenberg for the period June 1, 1956 to September 15, 1957.

The Council voted to request the Trustees to authorize two additional issues of the Proceedings during 1956.

The Secretary reported that Professor Reinhold Baer had submitted his resignation from the Editorial Board of the American Journal of Mathematics effective June 1, 1956.

The Council voted to approve a special session in Applied Mathematics at the Summer Meeting in Seattle, the session to consist of about five invited speakers.

The Council voted to omit the Colloquium Lectures in 1958.

The Council voted to discharge the Committee on the Role of the Society in Mathematical Publications.

Abstracts of the contributed papers follow. Those with "*t*" after the abstract number were presented by title. Where a paper has more than one author, the paper was presented by that author whose name is followed by "(p)". Mr. Akers was introduced by Professor Alonzo Church, Dr. Bareiss by Dr. R. M. Davis, Mr. Kobayashi by Professor C. B. Allendoerfer, Dr. Moser by Professor K. O. Friedrichs, and Professor Rector by Professor D. W. Hall.

ALGEBRA AND THEORY OF NUMBERS

464. S. S. Abhyankar: *Simultaneous resolution for algebraic surfaces.*

Let K be a two dimensional algebraic function field over an algebraically closed ground field of characteristic p and let K^* be a finite algebraic extension of K. Concerning the *simultaneous resolvability* of the pair (K, K^*) the following theorems are proved in this paper: *Theorem* 1. Let v^* be a zero dimensional valuation of K^*. Then there exists a *normal* model of K on whose K^*-normalization v^* has a simple center. *Theorem* 2. Assume that K^*/K is galois and $p=0$. Then there exists a *normal* model of K whose K^*-normalization is *nonsingular*. *Theorem* 3. Assume that K^* is a quadratic extension of K and $p\neq 2$. Then there exists a *nonsingular* model V of K whose K^*-normalization is also *nonsingular* and such that the branch locus on V is a nonsingular (in general reducible) curve. *Theorem* 4. Assume that K^* is a cubic cyclic extension of K and $p\neq 3$. Then we have the same conclusion as in Theorem 3. *Theorem* 5. Assume that K has a minimum model, i.e., that K is not the function field of a ruled surface. Let q be a prime number such that $q>3$ and $q\neq p$. Then there exist (lots of them) cyclic extensions K' of K of degree q such that there *does not exist* any *nonsingular* model of K having a *nonsingular* K'-normalization. (Received February 27, 1956.)

465. George Bachman: *Geometry in groups.*

Suppose that a finite group G is given which is considered as a group of motions of a point set. Let H be the subgroup of G leaving a given point P fixed. In order that G can be considered as a two-dimensional group of motions, it is demanded that the following axioms are satisfied: *Axiom* 1: H determines P uniquely and conversely. *Axiom* 2: Any motion has at most one fixed point unless it is the identity. *Axiom* 3: G is a transitive group of motions. It follows from these axioms that G must contain a subgroup H which is its own normalizer, and such that any two conjugates of H intersect only in the identity. A theorem of Frobenius gives us the existence of a normal subgroup which is interpreted as a subgroup of translations. A three-dimensional group of motions can be defined in a similar but more complicated manner. It is shown that Frobenius' theorem has an analogue in the three-dimensional case if certain additional assumptions are made. Examples can be given showing the necessity of all these conditions and showing that the theorems obtained are not empty. (Received February 28, 1956.)

466. W. E. Baxter: *Lie simplicity of a certain class of associative rings*. II.

Let A be a simple associative ring, characteristic not 2, either with its center $Z=(0)$, or of dimension greater than 16 over Z, and let A have an involution defined on it. Let K be the set of skew elements of A. In this paper, it is proven that if U is a proper Lie ideal of $[K, K]$ then U is contained in Z; that is, $[K, K]/Z\cap[K, K]$ is a simple Lie ring. In particular, if the involution is of the first kind then $[K, K]$ is a simple Lie ring. This generalizes well known results for matrices over fields. (Received February 3, 1956.)

467. David Gale: *The problem of rank in tensor products.*

Let $V_1, \cdots, V_n$ be vector spaces over a field F and let V_i have dimension d_i. Then any element $s\in\otimes_{i=1}^{n}(V_i)$ can be expressed in the form $s=\sum_{j=1}^{k}(V_i^j\otimes\cdots\otimes V_n^j)$. The minimum value of k over all such expressions is defined to be the *rank* of s. This agrees with the usual definition of rank for the case $n=2$. The problem of rank is then to determine the maximum of the ranks of all tensors s in $\otimes_{i=1}^{n}(V_i)$ as a function of the dimensions $d_1, \cdots, d_n$. This maximum is denoted by ρ. For $n=2$, the answer is classical, $\rho=\min(d_1, d_2)$. Some very special answers are obtained for the case $n=3$. (1) If $d_1d_2\leqq d_3$ then $\rho=d_1d_2$. (2) If $d_1=2$, $d_3=2d_2-1$ then $\rho=d_3$. (3) If $d_1=d_2=d_3=2$ then $\rho=3$. (4) The rank of a tensor may decrease if the base field F is extended. An example is given of a tensor having rank three over the real numbers and two over the complex numbers. (Received April 12, 1956).

468t. Karl Goldberg and Morris Newman: *Some free groups generated by matrices of order two.*

Let $A=(a_{ij})$ be a rational integral matrix of order two with determinant $d=\pm 1$, trace $t>0$, and $k=|a_{12}|-|a_{11}|-|a_{22}|-|a_{21}|\geqq 0$, such that if $d=1$ then $t\geqq 2$ and if $d=-1$ then $tk\geqq 2$. Let B be another matrix with these properties. Then A and B^T generate a free group. (Received February 13, 1956.)

469. Oscar Goldman: *Algebraic functions of several variables*. Part I. *Divisors*.

If K is a field of algebraic functions of n variables with a perfect constant field k, a *temporary constant field* of K/k will mean a subfield L of K, containing k, which is such that K/L is a field of algebraic functions of one variable. If $P(K/k)$ is the set of $(n-1)$-dimensional valuations of K/k, and $P(K/L)$ the subset of $P(K/k)$ of those which are trivial on L, then a *divisor* of K/k is defined as an integer-valued function on $P(K/k)$ whose restriction to $P(K/L)$ is a divisor of K/L for every temporary constant field L of K/k. The group of these divisors is shown to have certain properties upon which one may base an arithmetic theory of algebraic functions of several variables in a manner different from that of algebraic geometry. (Received February 28, 1956.)

470t. Edwin Hewitt: *Compact monothetic semigroups.*

A topological semigroup G is *monothetic* if it contains an element a such that $\{a, a^2, a^3, \cdots\}^-=G$. Let G be a compact monothetic (Hausdorff) semigroup. Then

G is of one of the following 3 types. I. $G=H$, where H is the (compact) character group of a subgroup of the reals modulo 1. II. $G=H\cup\{a, a^2, \cdots, a^s\}$, where H is as in I and s is a positive integer. Multiplication in G is: $a^ia^j=a^{i+j}$ for $i+j\leqq s$, $a^ia^j=b^{i+j}$ for $i+j>s$, where b is any element of H such that $\{b, b^2, b^3, \cdots\}^-=H$; $xa^i=a^ix=b^ix$ for $i\leqq s$ and $x\in H$; xy is as in H for $x, y\in H$. All points a^i are isolated and H has its original topology. III. $G=H\cup\{a, a^2, a^3, \cdots, a^n, \cdots\}$, where H is as in I. Multiplication in G is: $a^ia^j=a^{i+j}$ for all i, j; $a^ix=xa^i=b^ix$ for $x\in H$ and all i, where b is as in II; xy is as in H for $x, y\in H$. All points a^i are isolated and a generic neighborhood in G of $x\in H$ is $U(x)\cup\{a^i: b^i\in U(x)$ and $i\geqq n\}$, where $U(x)$ is an arbitrary neighborhood of x in H and n is an arbitrary positive integer Conversely, all topological semigroups of the form I, II, and III are compact and monothetic. (Received March 22, 1956.)

471t. Edwin Hewitt: *Compact semigroups with one-sided cancellation.*

Let S be a compact Hausdorff semigroup with the left cancellation law: $ax=ay\Rightarrow x=y$ for all $a, x, y\in S$. Then S is isomorphic and homeomorphic to a direct product $P\times E$, where P is a compact group and E is a compact semigroup in which $xy=y$ identically. E can be taken as the set of all idempotent elements of S. This set E is a compact subsemigroup of S in which $ef=f$ identically. P can be taken as Sa for an arbitrary $a\in E$. Sa is a compact subgroup of S. The mapping $(xa, e)\rightarrow xe$ is then an isomorphic homeomorphism of $(Sa)\times E$ onto S. (Received April 5, 1956.)

472t. J. H. Hodges: *Weighted partitions for general matrices over a finite field.*

For $\alpha\in GF(q)$, $q=p^n$, let $e(\alpha)=e^{2\pi i t(\alpha)/p}$ where $t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}}$. If $A=(\alpha_{ij})$ is a square matrix with elements in $GF(q)$, let $\sigma(A)=\sum\alpha_{ii}$. Consider the sum $S=S(B, W, R, A)=\sum e\{\sigma(UW+RV)\}$, where B, W, R, A, U and V are matrices having elements in $GF(q)$, A is nonsingular of order m, B is of order t, W and V are m, t, R and U are t, m, and the sum is over all U, V such that $UAV=B$. If $W=0$ and $R=0$, S is the number of solutions U, V of $UAV=B$, which the author has given in *Representations by bilinear forms in a finite field* (Duke Math. J. vol. 22 (1955) pp. 497–510). In the present paper it is shown that the sum S can be expressed in terms of certain Kloosterman sums defined for square matrices over $GF(q)$. A number of the properties of these Kloosterman sums are also given. (Received February 27, 1956.)

473. Arno Jaeger: *Power series representations of rings and modified differentiations.*

The modified theory of differentiations in the sense of Hasse and F. K. Schmidt (cf. J. Reine Angew. Math. vol. 190 (1952) pp. 1–21) is extended to certain types of rings, in particular simple and semi-simple rings, and generalized to include homomorphic mappings. A generalization of the concept of semi-derivations in the sense of Dieudonné is defined, and relations between them and the differentiations established. The totality of all differentiations is constructed by means of formal series, a generalization of the author's expansion theory, and the chain-rule-dependency. (Received February 29, 1956.)

474. G. F. Leger, Jr.: *On cohomology theory for Lie algebras.*

The first three cohomology groups $H^n(L, M)$ $(n=1, 2, 3)$ of a Lie algebra L with respect to an L-module M have well known interpretations. If L is an ideal of a Lie algebra G and M is also a G-module then G operates on $H^n(L, M)$. In this paper we devise operations of G on the interpretations of $H^n(L, M)$ which agree with the operations of G on $H^n(L, M)$. (Received February 29, 1956.)

475. A. B. Lehman: *Free modular lattices and linear graphs.*

Given a linear graph $P(a_1, \cdots, a_n)$ having branches $a_1, \cdots, a_n$, define the polynomials $p^{i,1}(a_1, \cdots, a_n)=\dot{a}_i(a_1, \cdots, a_n)$, $p^{i,j+1}(a_1, \cdots, a_n)=p^{i,j}(p^{1,1}, \cdots, p^{n,1})$ where $\dot{a}_i$ denotes the product of all sums of branches of P having the property that the elements of each sum constitute a series graph between the vertices of a_i when all other branches are deleted. Define $p_{i,j}$ analogously where a_{*i} is the sum of a_i and at most two product terms whose elements are all the branches other than a_i meeting at a vertex of a_i which is not a vertex of a_i. Theorem 1. $p^{i,j} \geqq p_{i,j}$ in a free modular lattice. If P is series-parallel it has a polynomial representation $p_i(a_1, \cdots, a_n)$ viewed from the vertices of a_i; with $+$ corresponding to series and $\cdot$ to parallel. Lemma. $p^{i,j} \geqq p_i = p^{i,k}$ for $k \geqq n$. Theorem 2. If $P(a_1, \cdots, a_n)$ and $Q(a_1, \cdots, a_n)$ are series-parallel graphs such that Q may be derived from P by a sequence of identifying vertices (shorting) and transpositions of terminals of two-terminal subgraphs (flips) then $p_i \geqq q_i$ in a free modular lattice. As $a(b+c(d+e)) \geqq ae(b+cd)$, in a lattice, is equivalent to the modular law, each application of the law may be performed by a single short in a linear graph. Theorem 3. If $r \geqq s$ in a modular lattice, then there exist P, Q and i such that $r=p_i$ and $s=q_i$ in a free lattice, and P and Q satisfy the conditions of Theorem 2. Hence linear graphs may be used to generate free modular lattice inequalities; all free modular lattice inequalities may be computed in terms of linear graphs. Also see Bull. Amer. Math. Soc. Abstract 61-2-311. (Received February 28, 1956.)

476. G. de B. Robinson: *The degree f_λ of an irreducible representation $[\lambda]$ of S_n.*

The original Frobenius-Young formula (FY) for f_λ was recently modified (Frame, Robinson and Thrall, Canadian Journal of Mathematics vol. 6 (1954) pp. 316–324) to yield $f_\lambda = n!/H$, where H is the product of the lengths of all the hooks in $[\lambda]$. Another version of (FY) has been given by Feit (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 740–744) who showed that $f_\lambda = n!|z_{ij}|$, where $z_{ij}=1/(\lambda_i-j+i)!$ This determinantal form can be deduced immediately from the operator equation (Robinson and Taulbee, P.N.A.S. vol. 40 (1954) pp. 723–726) $[\lambda]=(1-R_{ij})[\lambda_1]\cdot[\lambda_2]\cdot \cdots \cdot[\lambda_h]$, where each factor $(1-R_{ij})$ appears once only and R_{ij} raises a node from the disjoint constituent $[\lambda_j]$ to $[\lambda_i]$. Taking the degree on each side we have f_λ expressed as a linear sum $(\pm)$ of the degrees of the permutation representations on the right. We can distinguish "determinantal" terms, which yield the terms in the expansion of $n!|z_{ij}|$, and "nondeterminantal" terms, all of which cancel. This simple direct proof of (FY) emphasizes the importance of the permutation representations of S_n as a basis for the irreducible representations. A similar operator approach is applicable to skew diagrams $[\lambda]-[\mu]$, for which Feit's formula also holds. (Received February 20, 1956.)

477. Alex Rosenberg (p) and Daniel Zelinsky: *Global dimension of tensor products.*

Let A be an algebra over a field K with nilpotent radical N. Set $\bar{A}=A/N$ and suppose further that $[\bar{A}:K]<\infty$. Theorem: If $\bar{A}$ is separable, then for an arbitrary algebra B over K, l.gl.dim $A\otimes B=$ l.gl.dim $A+$l.gl.dim B. If $\bar{A}$ is inseparable then l.gl.dim $A\otimes B$ is infinite or is equal to l.gl.dim $A+$l.gl.dim B. In proving the second half of the theorem we show, under the same hypotheses on A and B: Let M be a left $\bar{A}\otimes B$ module and N a left $A\otimes B$ module. Suppose that for $n>$l.dim$_B M$ one has $\operatorname{Ext}^n_{A\otimes B}(M, N)\neq 0$, then also $\operatorname{Ext}^{n+1}_{A\otimes B}(M, N)\neq 0$. (Received February 29, 1956.)

478. G. B. Seligman: *Characteristic ideals and the structure of Lie algebras.*

Let L be a Lie algebra of finite dimension over an arbitrary field F. An ideal I in L is characteristic if $ID\subseteq I$ for every derivation D of L. L possesses a unique maximal solvable characteristic ideal R, and L/R contains no solvable characteristic ideal other than (O). We say that L/R is characteristic semi-simple (css.). Every css. Lie algebra M contains a nonzero unique maximal characteristic ideal S with the property that S is a direct sum of characteristic ideals in S which, in their own right, contain no proper characteristic ideal. The restriction to S of $ad(x)$, x in M, defines an isomorphism of M into the Lie algebra of all derivations of S. The "characteristic simple" summands of S have the property that every proper ideal in such an algebra is nilpotent. An example is known where R does not coincide with the ordinary radical of L. It is possible to develop an analogous theory for restricted Lie algebras. (Received February 10, 1956.)

479. Ruth Rebekka Struik: *A relation between subgroups of a free product.*

A relation between subgroups of a free product is established. Let F be the free product of the groups $A_1, A_2, \cdots, A_p$. Let (R, S) be the commutator subgroup generated by $r^{-1}s^{-1}rs$, $r\in R$, $s\in S$. Let $m_1, m_2, \cdots, m_k$ be non-negative integers. Let ${}_0A_i$ be the normal subgroup in F generated by A_i, and let ${}_{m+1}A_i=({}_mA_i, F)$. Let $c({}_{m_1}A_{i_1}, {}_{m_2}A_{i_2}, \cdots, {}_{m_k}A_{i_k})$ be an arbitrary complex commutator of weight k in the components ${}_{m_j}A_{i_j}$ [see P. Hall; *A contribution to theory of groups of prime power order*, Proc. London Math. Soc. vol. 36 (1933) pp. 29–95]. Let $c_k({}_{m_1}A_{i_1}, {}_{m_2}A_{i_2}, \cdots, {}_{m_k}A_{i_k})$ $=\prod c({}_{m_1}A_{i_1}, {}_{m_2}A_{i_2}, \cdots, {}_{m_k}A_{i_k})$ where the product is taken over all possible ordered k-tuples $({}_{m_1}A_{i_1}, {}_{m_2}A_{i_2}, \cdots, {}_{m_k}A_{i_k})$, the jth element of which is of the form ${}_{m_j}A_{i_j}$, $1\leq i_j\leq p$; however, k-tuples in which $A_{i_1}=A_{i_2}=A_{i_3}=\cdots=A_{i_k}$ are excluded. Let $(A_i)_F$ be the subgroup of F generated by the subgroups (A_i, A_j), $i\neq j$. It is proved that $c({}_{m_1}F, {}_{m_2}F, \cdots, {}_{m_k}F)\cap{}_0(A_i)_F=c_k({}_{m_1}A_{i_1}, {}_{m_2}A_{i_2}, \cdots, {}_{m_k}A_{i_k})$ unless $c({}_{m_1}F, {}_{m_2}F, \cdots, {}_{m_k}F)={}_{m_1}F$ in which case ${}_{m_1}F\cap{}_0(A_i)_F={}_{m_1-1}(A_i)_F$. This relation is useful in computing a large number of the verbal products of S. Moran. (Received February 27, 1956.)

480t. Ruth Rebekka Struik: *On the finiteness of certain products of groups.*

It is shown that certain products of finite groups are infinite. Using the notation of Bull. Amer. Math. Soc. Abstract 61-3-400, let $G=A*B$; let m, n be fixed non-negative integers; let $A(m, n)B = A * B/({}_mA_G, {}_nB_G)({}_nA_G, {}_mB_G)$. Let A, B be finite groups. It is proved that if $A/(A, A)$ and $B/(B, B)$ are both nontrivial groups, then (A, B) in $A(m, n)B$ is infinite for $m \geqq 1$ and $n \geqq 1$; hence $A(m, n)B$ is infinite for $m, n \geqq 1$. If $A = (A, A)$ and $B = (B, B)$, then $A(m, n)B$ is the direct product of A and B for all m, n. If $A = (A, A)$ or $B = (B, B)$, then $A(m, n)B$ is finite for all m, n; in this case (A, B) in $A(m, n)B$ is Abelian. The proof utilizes identities and inequalities of subgroups of free products. Since the nilpotent product of two finite groups is finite [O. N. Golovin, Mat. Sbornik vol. 27 (69) (1950) pp. 427–453], this shows that $A(m, n)B$ for $m, n \geqq 1$ is a different type of product from the nilpotent products. (Received February 27, 1956.)

481. A. W. Tucker: *Dual lattice face structure in polar polyhedral cones.*

Let C be the polyhedral convex cone of all non-negative linear combinations of the vectors of a finite set S in Euclidean vector n-space, and C^* the polar cone of all vectors nonacute-angled to each vector of S. For each subset s of S form the set $F^*(s)$ of all vectors obtuse-angled to each vector of s and orthogonal to each vector of $s' = S - s$. Each vector of C^* belongs to one and only one $F^*(s)$, called an (open) *face* of C^*. Under set-inclusion the system L of those s for which $F^*(s) \neq \emptyset$ constitutes a relatively complemented lattice satisfying the Jordan-Dedekind chain condition, but generally not even semi-modular. For each s' form also the set $F(s')$ of all vectors expressible as fully positive linear combinations of the vectors of s' but not so expressible via any larger subset of S. [$F(\emptyset) = \emptyset$ or 0 according as 0 belongs or not to some other $F(s')$.] Each vector of C belongs to one and only one $F(s')$, called a *face* of C. For each s and $s' = S - s$, $F^*(s)$ and $F(s')$ are both vacuous *or* are open convex sets ($\neq \emptyset$) in complementary orthogonal subspaces. So the system L' of those s' for which $F(s') \neq \emptyset$ is the *dual* lattice of L. (Received April 10, 1956.)

482*t*. P. S. Wolfe: *Games over polyhedra.*

Let A be an m by n matrix, E_m and E_n the sets of all m-component row vectors and n-component column vectors respectively. Let X, Y be (not necessarily bounded) intersections of finite families of half-spaces in E_m, E_n respectively. Let $v_1 = \sup_{x \in X} \inf_{y \in Y} xAy$, $v_2 = \inf_{y \in Y} \sup_{x \in X} xAy$ ($+\infty$, $-\infty$ admitted). (A, X, Y) has the *value* $v_1 = v_2$ iff the equality holds. THEOREM: If (A, X, Y) has a finite value then xAy has a saddle point. If it does not have a value then $v_1 = -\infty$ and $v_2 = +\infty$. There exist (A, X, Y) having: finite value; value $+\infty$; value $-\infty$; no value. The last case does not obtain unless $m, n \geqq 2$. COROLLARY: If either X or Y is bounded, then (A, X, Y) has a value. (Received January 30, 1956.)

ANALYSIS

483. M. G. Arsove: *Mass distributions for products of subharmonic functions.*

The class of all differences of locally bounded subharmonic functions on a region Ω forms an algebra $\mathfrak{A}$ under pointwise multiplication [Trans. Amer. Math. Soc. vol. 75 (1953) p. 347]. Hence, associated with the product $w = w_1w_2$ of two functions in $\mathfrak{A}$ there is a unique mass distribution m. The problem of determining m in terms of the mass distributions m_1 and m_2 for w_1 and w_2 is here studied by means of the Dirichlet integral and approximation techniques based on the work of Brelot, Cartan, Deny, and

Evans. If ω is a subregion bounded by a simple closed curve in Ω, then $m(\omega)=\int_\omega w_1 dm_2 +\int_\omega w_2 dm_1-(1/\pi)D_\omega(w_1, w_2)=\int_\omega h_1 dm_2+\int_\omega h_2 dm_1-(1/\pi)D_\omega(h_1, h_2)$, where D_ω denotes the Dirichlet integral over ω and h_1 and h_2 are the solutions of the Dirichlet problem on ω for boundary functions determined by w_1 and w_2. This and similar results lead to subalgebras of $\mathfrak{A}$. For example the class of all δ-subharmonic functions having mass distributions given by continuous density functions is a subalgebra of $\mathfrak{A}$, and the density functions are related by $\rho_{w_1w_2}=w_1\rho_{w_2}+w_2\rho_{w_1}-(1/\pi)[(\partial w_1/\partial x)(\partial w_2/\partial x) +(\partial w_1/\partial y)(\partial w_2/\partial y)]$. (Received February 27, 1956.)

484*t*. Stefan Bergman: *On a representation of stream functions of flows of a compressible fluid.*

The integral operator $\mathrm{P}_{\mathrm{II}}(f)$ of the second kind (see (A) Amer. J. Math. vol. 70 (1948) p. 856, and vol. 74 (1952) p. 444) transforms analytic functions of a complex variable f defined at $Z=a$, Re $a<0$ into solutions $\psi^{(1)}$ of the differential equation (1) $\psi^{(1)}_{ZZ^*}+F\psi^{(1)}=0$. Here $Z=\lambda+i\theta$, $Z^*=\lambda-i\theta$, λ is a function of the Mach number, θ is the angle which the velocity vector forms with the positive x-direction, $F=\sum_{n=1}^{\infty} C_n \exp(2n\lambda)$ is a function which is singular at $\lambda=0$. P_{II} is defined for $[|\lambda|<3^{1/2}|\theta|]$. Replace (1) by (2) $\psi^{(1)(\alpha)}_{ZZ^*}+F_\alpha\psi^{(1)(\alpha)}=0$, $F_\alpha=\sum C_n \exp(2n\lambda)/\Gamma(1+2n\alpha)$ and form the operator $\lim_{\alpha\to 0} \int_{-1}^{1} E_\alpha(Z, \overline{Z}, t)f(Z(1-t^2)/2)dt/(1-t^2)^{1/2}$ (see (A), p. 872); then one obtains the analytic continuation of the integral operator $\mathrm{P}_{\mathrm{II}}(f)$ to the parts of the subsonic region where $\psi^{(1)}$ is regular and to those segments of the sonic lines where $\psi^{(1)}(0, \theta)$ and $\{(-\lambda)^{1/3}[\partial\psi^{(1)}(\lambda, \theta)/\partial\lambda]\}_{\lambda=0}$ are continuous. The operator $\mathrm{P}_{\mathrm{II}}(f)$ with the generating functions $E^{(\kappa)}_{\mathrm{II}}=\sum_{n=1}^{\infty} q^{(n\kappa)} (-t^2Z)^{-n+1/3-(2/3)\kappa}$, $\kappa=1, 2$ (see (A), p. 878 and p. 888) yields the continuation of $\mathrm{P}_{\mathrm{II}}(f)$ to the supersonic region. In this way a representation of the stream function in the whole domain of its existence can be obtained. (Received February 20, 1956.)

485. Jerome Blackman: *Convolutions with rational kernels.*

The convolution equation $f(u)=\int k(u-x)d\alpha(x)$ is considered under the conditions: (i) $k(x)=p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials of degree p and q respectively. The distance from the set of zeros of $q(x)$ to the real axis $=d>0$. (ii) $q-p=n\geqq 1$. (iii) $\hat{k}(x)$, the Fourier transform of k, has no real zeros. (iv) $\alpha(x)$ is locally of bounded variation. (v) lim $\int_{-A}^{B} k(u_0-x)d\alpha(x)$ exists as A, $B\to\infty$ for some u_0, $|\mathrm{Im}\ (u_0)|>d$. Under these conditions a specific formula for $\alpha(x)$ is obtained, thereby also showing uniqueness. (Received February 16, 1956.)

486. F. E. Browder: *Regularity properties at the boundary of solutions of elliptic boundary value problem.*

Let K be a linear elliptic differential operator of order $2m$ on the domain G of E^n with boundary B of class C^{2m}. If the coefficients of K are of class C^0 on the closure $\overline{G}$ of G, then for $p>1$, there exists a constant $\psi_p>0$ such that for $f\in L^p(G)$, if u is a solution of $Ku=f$ under null Dirichlet boundary conditions, then $\|D^{2m}u\|_{L^p(G)} \leqq\psi_p\{\|f\|_{L^p(G)}+\|u\|_{L^1(G)}\}$, for any derivative $D^{2m}u$ of order $2m$. From a theorem of Sobolev, it follows that if $p>n/2m$, the derivatives of u of order less than $2m$ satisfy Hölder condition on G. More generally if B is of class C^{2m+r} $(r\geqq 0)$ while the coefficients of K are of class C^r on $\overline{G}$, there exist $\psi_{p,r}>0$ such that $\|D^{2m+r}u\|_{L^p(G)}\leqq\psi_{p,r} \cdot\{\|f\|_{H^{r,p}(G)}+\|u\|_{L^1(G)}\}$. The proofs combine theorems of Calderon-Zygmund on singular integral operators with a new integral representation of u near a boundary point. The results are extended to a general class of elliptic boundary value problems. (Received February 28, 1956.)

487. Lamberto Cesari and R. E. Fullerton (p): *A smoothing process for contours.*

Let S be a Fréchet surface defined by a continuous mapping $T:J\to E_3$ where J is a closed simple Jordan region in E_2 and let f be a real-valued continuous function defined in E_3. For any real number t let $C(t)=[w\in J\,|\,f(T(w))=t]$ and $\rho(t)$ the generalized length of the image of the contour $C(t)$ [L. Cesari, *Surface area*, Princeton, 1956]. Let $\tilde{\Gamma}$ be the topological space having as points maximal continua of constancy $g\in\Gamma$ of T in J under the usual hyperspace topology. Let $\tilde{G}\subset\tilde{\Gamma}$ be the set of all elements of $\tilde{\Gamma}$ such that $C(t)=\bigcup_{g\in G} g$ and let $g_1, g_2\subset C(t)$, $g_1\neq g_2$. Then there exists in $\tilde{\Gamma}$ at least one irreducible continuum $\tilde{\gamma}$ joining g_1 and g_2 such that $g\in\tilde{\gamma}$ implies $g\subset C(t)$. The continuum $\gamma=\bigcup_{g\in\tilde{\gamma}} g$ in J is called a smoothed contour relative to g_1 and g_2. It is shown that this method of constructing a smoothed contour is equivalent to a second method in which the contour is constructed by eliminating inessential parts of $C(t)$ and also that γ has local properties of uniqueness and γ is an arc in $\tilde{\Gamma}$. Extensions to surfaces defined as mappings from a 2-manifold are considered and other properties of smoothed contours are investigated. (Received February 29, 1956.)

488*t*. K. T. Chen: *Integration of paths. A unique representation of paths by noncommutative formal power series.*

Let α be an irreducible piecewise smooth continuous path in the n-dimensional real affine space E with coordinates $\langle x_1, \cdots, x_n\rangle$. There are differentials $d\pi_1, \cdots, d\pi_m$ such that $\int_\alpha d\pi_1 \cdots d\pi_m \neq 0$ where $d\pi_i=\sum f_{ij}(x)dx_j$ with each $f_{ij}(x)$ being a polynomial in $x_1, \cdots, x_n$. It follows that some $\int_\alpha dx_{i_1}\cdots dx_{i_p}\neq 0$. Let $\theta(\alpha)=1+\sum_{p=1}^{\infty}\sum\int_\alpha dx_{i_1}\cdots dx_{i_p}X_{i_1}\cdots X_{i_p}$ and $\theta^*(\alpha)=1+\sum_{p=1}^{\infty}\sum\int_\alpha dx_{i_p}\cdots dx_{i_1}X_{i_1}\cdots X_{i_p}$. Then $\theta(\alpha)\neq 1$ and $\theta^*(\alpha)\neq 1$. It is further proved that if, for two irreducible piecewise smooth continuous paths α and β, one has $\theta(\alpha)=\theta(\beta)$ or $\theta^*(\alpha)=\theta^*(\beta)$, then the path β, subject to change of parameter, can be obtained from α by a translation of E. (Received February 27, 1956.)

489*t*. K. T. Chen: *Integration of paths in a differentiable manifold.*

Let M be an n-dimensional manifold of class $r\geqq 1$. A path α represented by $\alpha(t)$, $a\leqq t\leqq b$, is required to be piecewise smooth with only finitely many jump discontinuities. The product $\alpha\cdot\beta$ of two paths α and β is the path α followed by β; the inverse α^{-1} is obtained from α by changing the orientation of α. For any differentials $d\omega_1, \cdots, d\omega_p$ in M, set $\int_\alpha d\omega_1=\int_{t=a}^{t=b} d\omega_1(\alpha(t))$ and, for $p\geqq 2$, $\int_\alpha d\omega_1\cdots d\omega_p=\int_{t=a}^{t=b}\left[\int_{\alpha[a,t]} d\omega_1\cdots d\omega_{p-1}\right]d\omega_p(\alpha(t))$ where $\alpha[a, t]$ denotes the portion of α with the parameter ranging from a to t. Given m differentials $d\omega_1, \cdots, d\omega_m$ in M, let $X_1, \cdots, X_m$ be noncommutative indeterminates and define $\theta(\alpha)=1+\sum_{p=1}^{\infty}\sum\int_\alpha d\omega_{i_1}\cdots d\omega_{i_p}X_{i_1}\cdots X_{i_p}$ and $\theta^*(\alpha)=1+\sum_{p=1}^{\infty}\sum\int_\alpha d\omega_{i_p}\cdots d\omega_{i_1}X_{i_1}\cdots X_{i_p}$, where $i_1, \cdots, i_p$ run over $1, \cdots, m$. By straightforward verification, $\theta(\alpha\cdot\beta)=\theta(\alpha)\theta(\beta)$ and $\theta^*(\alpha\cdot\beta)=\theta^*(\beta)\theta^*(\alpha)$. Furthermore, both $\log\theta(\alpha)$ and $\log\theta^*(\alpha)$ are Lie elements. A path α is irreducible if α is not in the form $\alpha=\alpha'\cdot\beta\cdot\beta^{-1}\cdot\gamma$, where α', β, γ are paths. It is shown that, given any irreducible path α, there exist differentials $d\omega_1, \cdots, d\omega_m$ in M for which $\theta(\alpha)\neq 0$ and $\theta^*(\alpha)\neq 0$. (Received February 6, 1956.)

490. W. F. Eberlein: *Approximation and functional analysis.* Preliminary report.

The prototype of the problems considered is that of best uniform approximation of a function f by an exponential sum $\sum_1^n A_m e^{-t_m s}$ $(0 \leq s < \infty)$ where *both* the coefficients (A_m) and the exponents (t_m) are indeterminate—in contrast to the classical theory in which the n-exponents are assigned in advance. Existence and uniqueness theorems are obtained. There are specializations to the theory of numerical integration and generalizations to abstract harmonic analysis on groups. (Received February 27, 1956.)

491t. Albert Edrei: *A density theorem for entire functions possessing deficient values.*

Let $f(z) = \sum c_n z^{\lambda_n}$ be an entire function of finite order ρ and let $M(r)$ and $n(r)$ denote, respectively, the maximum modulus and the number of zeros of $f(z)$ in the disk $|z| \leq r$. If (1) $n(r) = O(\log M(r))$, then (2) $\Delta(1) \geq 1/(2\rho)$, where $\Delta(1)$ is the maximum density of the sequence $\{\lambda_n\}$. Using a result of Pfluger [Comment. Math. Helv. vol. 19 (1947) pp. 91–104], the above theorem implies the following corollary. Let σ denote the sum of the deficiencies of the finite deficient values of $f(z)$. The inequality (2) remains true if the condition (1) is replaced by the assumption $\sigma = 1$. It would be interesting to know whether (2) is still true if it is only known that $f(z)$ is of finite order and possesses some finite deficient value. (Received March 1, 1956.)

492. Charles Fox: *A classification of kernels which possess integral transforms.*

If $K(s) = \int_0^\infty k(x) x^{s-1} dx$, $K(s)$ is the Mellin transform of $k(x)$. It is well known that if $K(s)K(1-s) = 1$ then $k(x)$ can be the kernel of a generalized Fourier transform In this paper kernels for which $K(s)K(1-s) \neq 1$ are considered. If $K(s) = H(s)/(m+ns)$ where $H(s)H(1-s) = 1$, m, n are constants it is proved that (1) $f(x) = \int_0^\infty k(ux) g(u) du$, (2) $g(x) = a\int_0^\infty k(ux) f(u) du + b(d/dx)\int_0^\infty xk(ux)(d/du)\{uf(u)\} du$, $a = m^2 + mn$, $b = n^2$. An example of such a kernel is $k(x) = x\int_x^\infty (\cos u/u^2) du$. It is then shown that kernels with Mellin transforms given by $K(s) = H(s)/(m+ns)P\{s(1-s)\}$ and $K(s) = H(s)/P\{s(1-s)\}$, $H(s)H(1-s) = 1$ and P denotes a polynomial form, also have transforms analogous to (1), (2) but more complicated. In the second case the transforms are symmetrical and a Parseval theorem exists of the form $\int_0^\infty f(x)^2 dx = \int_0^\infty g(x)^2 dx$ It is then possible to form a classification in which the type of transform for which $k(x)$ is a kernel is related to the properties of its Mellin transform $K(s)$. Thus Fourier kernels form class 1, kernels for (1) (2) form class 2 etc. One can frequently go from one class to the next by convolution. (Received February 17, 1956.)

493. Alvin Hausner: *Group algebras of vector-valued functions.*

Let $B = B(G, X)$ denote the set of all strongly measurable functions defined over a locally compact abelian group, G, with values in a complex commutative Banach algebra with or without an identity, X, which are such that $|f| = \int_G |f(a)| da < \infty$. B is a complex commutative Banach algebra under natural addition, scalar-multiplication, and multiplication defined by $f * g(a) = \int_G f(b) g(a-b) db$; the integral is a Bochner integral with respect to Haar measure. The following is proved: (1) The radical of B consists of functions with values in the radical of X a.e. so that B is semi-simple if and only if X is semi-simple; (2) The space of regular maximal ideals in B, topologized in the Gelfand sense, is homeomorphic with the topological product of the dual group of G and $\mathfrak{M}(X)$, the space of regular maximal ideals in X in the Gelfand topology; (3) B is regular if and only if X is regular; (4) Every proper closed ideal in B is contained in a regular maximal ideal if X is semi-simple, regular, and if the set of $x \in X$

with $\phi_M(x)$ having compact support in $\mathfrak{M}(X)$ is dense in X(ϕ_M denotes the canonical homomorphism from X onto the complex numbers associated with $M \in \mathfrak{M}(X)$). (4) generalizes Wiener's tauberian theorem from $L(G)$ to $B(G, X)$. The paper also considers kernels and hulls, representations of B, and certain homomorphisms from B to $L(G)$. (Received February 20, 1956.)

494*t*. Edwin Hewitt: *Invariant means for compact semigroups.*

Let S be a compact Hausdorff semigroup, $\mathfrak{C}(S)$ the linear space of all continuous real functions on S. For $\phi \in \mathfrak{C}(S)$ and $a \in S$, ${}_a\phi$ and ϕ_a are the functions such that ${}_a\phi(x) = \phi(ax)$ and $\phi_a(x) = \phi(xa)$, respectively, for $x \in S$. A non-negative linear functional M on $\mathfrak{C}(S)$ of norm 1 is a left [right] invariant mean if $M({}_a\phi) = M(\phi)$ $[M(\phi_a) = M(\phi)]$ for all $\phi \in \mathfrak{C}(S)$ and $a \in S$. Let K be the least [compact] ideal in S, and E the [nonvoid and compact] set of idempotents in K. Theorem 1. $\mathfrak{C}(S)$ admits an invariant mean M iff K is a group. In this case M is unique, and has the form $M(\phi) = \int_K \phi(x) d\lambda(x)$, where λ is normalized Haar measure on K. Theorem 2. If M is a right invariant mean and N is a left invariant mean, then $M = N$, and Theorem 1 applies. Theorem 3. If K is not a group, $\mathfrak{C}(S)$ may still admit a left invariant mean. This happens iff $eK = K$ for all $e \in E$. In this case Ke is a group for all $e \in E$ and E is a compact semigroup in which $xy = y$ identically. K is isomorphic to $(Ke) \times E$. The left invariant means on $\mathfrak{C}(S)$ are just the functionals M of the form $M(\phi) = \int_E \{ \int_{Ke} \phi(x) d\lambda_e(x) \} d\mu(e)$, where λ_e is normalized Haar measure on the group Ke and μ is any non-negative Borel measure on E such that $\mu(E) = 1$. (Received April 5, 1956.)

495. F. C. Huckemann: *On the strength of indirectly critical points of Riemann surfaces.*

The structure of the critical points of Riemann surfaces is related to the distribution of values of the generating function. It is an old problem [which is yet far from a solution] to determine this relation. Results so far available in this direction concern mostly directly critical points. It seems, therefore, useful to consider the simplest types of indirectly critical points and to investigate their contribution to the order as well as to the exceptional values of the generating function. After the introduction of a certain measurement for the strength of directly and indirectly critical points of a certain class, it is seen that indirectly critical points of positive strength on parabolic Riemann surfaces behave in some sense similarly to (the directly critical) logarithmic winding points. On surfaces of bounded characteristic, on the other hand [where directly critical points outside the closure of the boundary values of the generating function are always exceptional (Lehto)], this is not so; there are, however, also (strong) indirectly critical points on such surfaces which make the corresponding value exceptional. (Received February 28, 1956.)

496. Robert C. James: *A characterization of reflexivity.*

It is known that a reflexive Banach space has the property that each linear functional attains its sup on the unit sphere. It is shown that the converse of this is true for separable Banach spaces. Also, that a Banach space is reflexive if each linear functional attains its sup on the unit sphere of each separable subspace which has a basis. These results are related by the theorem that any nonreflexive Banach space has a nonreflexive subspace with a basis. It follows that a separable Banach space is reflexive if and only if for any bounded closed convex set U which does not intersect the unit sphere S there is a hyperplane which separates S and U and is disjoint from each of S and U. (Received March 1, 1956.)

497. W. T. Kyner: *Stability of invariant manifolds.*

J. McCarthy has generalized the periodic surface problem that has been studied by S. P. Diliberto and his students. Let $T:(W, V)\rightarrow(W, V)$ be a differentiable homeomorphism where V is a compact differentiable submanifold of the manifold W. V is called an invariant manifold of T and is said to be stable if for any differentiable family $T_\gamma: W\rightarrow W$ such that $T_0=T$, there is a continuous family $V_\gamma(|\gamma|<c)$ of submanifolds such that $T_\gamma: V_\gamma\rightarrow V_\gamma$ and $V_0=V$. McCarthy [Bull. Amer. Math. Soc. Abstract 61-2-264] has given sufficient conditions for stability. His result includes the (nondegenerate) periodic surface results of M. D. Marcus, but not those of G. Hufford and the author. The assumption that a certain pair of linear transformations associated with the homeomorphism T have uniformly bounded inverses implies stability. This extension of McCarthy's theorem includes the above-mentioned periodic surface results. (Received February 28, 1956.)

498. R. E. Lane: *Linear operators on quasi-continuous functions.*

Of the linear transformations from the set of all quasi-continuous functions to the set of all functions, the author considers those which satisfy a special boundedness condition, and shows that the transform of a quasi-continuous function is quasi-continuous and is the sum of two integrals; he gives a condition on the transformation which is necessary and sufficient for the transform of each quasi-continuous function to be an integral. He gives conditions sufficient for the transform of each quasi-continuous function to have various properties, such as continuity, bounded variation, continuous first derivative, etc., and gives examples of transformations for smoothing data and obtaining derivatives of the smoothed data. (Received December 22, 1955.)

499. P. D. Lax: *An abstract stability theorem.*

The following stability theorem for abstract differential equations is proved: Let H be a Hilbert space, D a nonpositive self-adjoint operator, $u(t)$ a solution of $du/dt=(D+K(t))u$, where $K(t)$ is bounded and $\int_0^\infty\|K(t)\|dt<\infty$. Conclusion: $u(t)$ unless identically zero tends to zero no faster than some finite exponential. Take H to be the space of harmonic functions in a solid sphere under L_2 scalar product, D to be $-\sum x_i\partial/\partial x_i$ and $u(t)$ the harmonic function whose boundary values are $w(rx)$, $r=e^{-t}$, and where $w(x)$ is a solution of $\Delta w=Nw$, N of first order. One can show that $\|du/dt-Du\|\leq O(r)\|u\|$; hence, application of the abstract stability theorem gives a new proof of Claus Muller's theorem on the finiteness of zeros of solutions of such elliptic equations (see also E. Heinz, Hartman-Wintner, and Aronszajn for general second order equations). (Received March 29, 1956.)

500. W. S. Loud: *Periodic solutions of the equation $x''+cx'+g(x)=\epsilon e(t)$.*

Let $x_0(t)\neq 0$ be a solution of (1) $x''+g(x)=0$ of period L. Using a technique given by Coddington and Levinson (*Theory of ordinary differential equations*, 1955, Chapter 14) it is shown that under appropriate conditions (2) $x''+g(x)=\epsilon e(t)$, with $e(t)$ of period L and small, has a unique solution of period L in some neighborhood of $x_0(t)$. $e(t)$ must be orthogonal over a period to $x_0'(t)$ and not orthogonal to $x_0''(t)$. The variation equation (3) $y''+g'(x_0(t))y=0$ must have no solution of period L linearly independent of $x_0'(t)$. The same results hold for the damped equation (4) $x''+cx'+g(x)$

$=\epsilon c(t)$ where c is small compared to ϵ. The stability properties of the solutions of (3) and (4) can be deduced from the algebraic sign of $\int_0^L c(t)x_0''(t)dt$ and the hardening or softening character of $g(x)$. ($g(x)$ is hardening if the periods of solutions of (1) decrease with amplitude, softening if they increase.) Existence and stability properties of numerous subharmonic solutions of (2) and (4) can be deduced. Results are obtained in cases in which some of the above conditions are not satisfied. The research for this paper was supported in part by the Office of Ordnance Research. (Received February 28, 1956.)

501*t*. F. I. Mautner: *Geodesic flows on symmetric Riemann spaces.*

Let M be a locally symmetric Riemann space (in the sense of E. Cartan) of negative curvature and finite volume. Denote by B the manifold of all tangent vectors of unit length to all points of M. The ergodic parts of B under the geodesic flow are determined explicitly, in terms of the group G of isometries of the universal covering space $\tilde{M}$ of M. The geodesic flow is ergodic in B if and only if the universal covering space $\tilde{M}$ is of rank one in the sense of E. Cartan. It follows that there exist compact complex analytic locally symmetric Kaehler manifolds of finite volume and negative curvature for which the geodesic flow is not ergodic in B. The spectrum of the geodesic flow is studied and it is proved to be absolutely continuous under certain conditions. The method used is the reduction of the problem to that of unitary representations of semi-simple Lie groups and properties of invariant measures on homogeneous spaces. (Received February 23, 1956.)

502*t*. B. C. Meyer and H D Sprinkle: *Two nonseparable complete metric spaces defined on I.*

Two non-separable, complete metric spaces are defined on certain equivalence classes of the set of all subsets of $[0, 1]$. For each of the spaces, the subspace consisting of the equivalence classes of measurable sets has cardinality $\mathfrak{c}$, is perfect, and is nowhere dense in the space; its complement has cardinality $\mathfrak{f}$, is open, and is dense in the space. (Received February 7, 1956.)

503. Josephine Mitchell: *Orthogonal systems in matric spaces.* Preliminary report.

Let the domain $D=E\{z|I-zz^*>0\}$, where z is an n by n matrix of complex numbers, z^* its conjugate transpose, I the identity and 0 the zero matrix. A proper part of the boundary of D is the set $B=E\{z|zz^*=I\}$. For $n=2$ a complete orthonormal system with respect to the inner product $(f, g)=\int_D f\bar{g}\,dV_D$ (dV_D being Euclidean volume element on D) for analytic functions on D with $(f, \bar{f})<\infty$ is (1) $\{\phi^{(\nu)}_{pqr}(z)=Nz^p_{ab}z^q_{cd}z^r_{ef}F(-\nu, -\nu-p; q+r+4-2\nu; Z)\}$ ($\nu=0, \cdots, m=\min(q, r)$; $p, q, r=0, 1, 2, \cdots$), where $Z=z_{cf}z_{ed}/z_{cd}z_{ef}$, the z's are distinct elements of the matrix z with $c\neq e$ and $d\neq f$ and F is the hyper geometric function. The normalizing factor N equals $\{(4(\nu+1)/\pi^4)C_{r-\nu+1,1}C_{q-\nu+2,2}C_{p+q+r+3-\nu,p+\nu}C_{q+r+3-\nu,\nu+1}\}^{1/2}$. The system (1) with a different normalizing factor is also a complete orthonormal system for functions f, analytic on D and defined on B, with respect to the inner product $\int_B f\bar{g}\,dV_B$. For $n>2$ and the inner product (f, g) the system (1) with appropriate modifications is orthonormal but not necessarily complete. Let $s(z)=\sum_{p,q,r=0}^{\infty}\sum_{\nu=0}^{m}a^{(\nu)}_{pqr}\phi^{(\nu)}_{pqr}(z)$ be an orthogonal series formed from (1). If $s(I)$ exists, then $s(rI)$ is uniformly convergent for $0 \leq r \leq 1$ and $\lim_{r \to 1-} s(rI) = s(I)$. This is a special case of a more general theorem which assumes that $s(I)$ is Cesàro or Abel summable. (Received February 28, 1956.)

504. Jürgen Moser: *The analytic invariants of an area-preserving unstable mapping.*

The statement presented is concerned with area-preserving mappings M: $x_1 = f(x, y) = \lambda x + \cdots$, $y_1 = g(x, y) = \lambda^{-1} y + \cdots$, $f_x g_y - f_y g_x = 1$; $\lambda^2 > 1$, in the neighborhood of an unstable fixed point which we assume to be $x = y = 0$. The functions $f(x, y)$, $g(x, y)$ are assumed to be real analytic functions in some neighborhood of the origin. Then there exists an area-preserving transformation $x = \phi(\xi, \eta)$, $x_1 = \phi(\xi_1, \eta_1)$, $y = \psi(\xi, \eta)$, $y_1 = \psi(\xi_1, \eta_1)$, which transforms M into the "normal form" $\xi_1 = \xi U(\xi\eta)$, $\eta_1 = \eta U^{-1}(\xi\eta)$, where $U(\xi\eta) = \lambda + U_1 \xi\eta + \cdots$ is a function of the product $\xi\eta$. Furthermore ϕ, ψ, U are real analytic functions for sufficiently small ξ, η. In a weaker form this theorem was known to G. D. Birkhoff (Acta Math. vol. 43 (1920) pp. 1–119) namely for *formal* power series ϕ, ψ, U of ξ, η. The present theorem guarantees the *convergence* of these series. The proof uses Cauchy's majorant method. The coefficients of $U(\xi\eta)$ turn out to be the full system of invariants of M under the group of real analytic area-preserving transformations. (Received March 8, 1956.)

505. Dr. G. O. Peters: *Boole polynomials and numbers of the second kind.*

Charles Jordan in his book *Calculus of finite differences* (Budapest, 1939) defines Boole polynomials of order one. Nörlund in his book *Vorlesungen über Differenzenrechnung* (Berlin, 1924) defines Bernoulli and Euler polynomials of higher order. The author (Bull. Amer. Math. Soc. Abstract 62-1-24) defined the Boole polynomials of higher and negative order. Interchanging the roles of Δ and D and replacing factorials of X by powers of X, the author calls a new set of polynomials, Boole polynomials of the second kind. The author (Bull. Amer. Math. Soc. Abstract 62-2-213) defined the operator $\daleth$ (Dalet) where $\daleth = 1 + 1/2\, D$. The author defines the Boole polynomials of the second kind of degree ν and order n, as the polynomial $Z_\nu^n(X)$ satisfying the equation $\daleth^n Z_\nu^n(X) = X^\nu$. The Boole polynomials of the second kind of degree ν and order $-n$, is defined as the polynomial $Z_\nu^{-n}(X)$ satisfying the equation $Z_\nu^{-n}(X) = \daleth^n X^\nu$. The Boole numbers of the second kind are defined by the equation $Z_\nu^n(X) = \sum_{i=0}^{\nu} \binom{\nu}{i} (Z_i^n/2^i) X^{\nu - i} = (Z^n/2 + X)^\nu$ where n can be either positive or negative. Some properties of these Appell polynomials are found and the values of the Boole numbers are determined. The symmetry of the polynomials is also found. An interesting polynomial relation is found relating Euler, Boole and Bernoulli polynomials of the second kind. (Received February 29, 1956.)

506. Alexander Peyerimhoff: *On a class of matrix transformations.*

In a recent paper (J. London Math. Soc. vol. 29 (1954) pp. 459–476) H. C. Chow proved that a triangular matrix transformation $y_n = \sum_{\nu=0}^{n} a_{n\nu} x_\nu$ which transforms every sequence $x_n = O(1)$ into $\sum |y_n| < \infty$ has the property $\sum |a_{nn}| < \infty$. This is a special case of the following theorem: A matrix transformation $y_n = \sum_{\nu=0}^{\infty} a_{n\nu} x_\nu$ transforms every sequence $x_n = o(1)$ into $\sum |y_n| < \infty$ if and only if there exists a constant K with $\sum_{n=0}^{\infty} \left| \sum_{\nu \in \mathfrak{M}_n} a_{n\nu} \right| \leq K$ for every sequence $\{\mathfrak{M}_n\}$ of sets of non-negative integers such that $\mathfrak{M}_n \cap \mathfrak{M}_m = 0$ if $\mathfrak{M}_n \neq \mathfrak{M}_m$ (this formulation does not exclude the case $\mathfrak{M}_{n_1} = \mathfrak{M}_{n_2} = \mathfrak{M}_{n_3} = \cdots$), and K independent of $\{\mathfrak{M}_n\}$. This condition cannot be replaced by $\sum_{n,\nu} |a_{n\nu}| < \infty$. (Received February 29, 1956.)

507. V. C. Poor: *Residue of a complex function, a contour integral.*

This paper contains a mathematical deduction of a logarithmic residue formula from the fundamental results obtained in previous papers. It reduces the two formulae for residues over an area to a single contour integral. Its application to some special residues is included. (Received February 27, 1956.)

508*t*. V. C. Poor: *The logarithmic residue.*

This paper contains a brief development of the formula for the residues of a subclass of the class of complex functions. The formula is a contour integral of the logarithmic type, which completes the theory of residues of complex functions. Applications to complex functions are briefly but adequately discussed. (Received February 13, 1956.)

509. J. J. Price: *Certain classes of orthonormal step functions.*

Let $n_1, n_2, n_3, \cdots$ be a sequence of positive integers, $n_i \geqq 2$, and let G_i be a cyclic group of order n_i. There exists on the unit interval an orthonormal system of step functions $S(n_1, n_2, n_3, \cdots)$ which is essentially the set of characters of the group $G_1 \times G_2 \times G_3 \times \cdots$. $S(2, 2, 2, \cdots)$ is the Walsh system. $S(\alpha, \alpha, \alpha, \cdots)$ has been studied recently by Chrestenson. When $n_1, n_2, n_3, \cdots$ is unbounded, the behavior of the Fourier series of an integrable function with respect to $S(n_1, n_2, n_3, \cdots)$ is considerably different from the behavior in the above cases. The following theorems are proved for $n_1, n_2, n_3, \cdots$ unbounded. (1) There exist functions in Lip 1 whose Fourier coefficients are not $O(1/n)$. (2) If the n_i do not increase too rapidly, the Fourier coefficients of any nonconstant absolutely continuous function are not $O(1/n)$. (3) There exists a continuous function whose Fourier series is not $(C, 1)$ summable at a given point. (February 29, 1956.)

510*t*. R. T. Prosser: *Homogeneous W* algebras.*

A W* algebra is homogeneous if it is isomorphic with each of its direct summands. If A is a homogeneous W* algebra with center Z, then A contains a W* subalgebra F such that F is a factor algebra of the same type as A, and A is the W* tensor product of F and Z. Moreover, F is unique, up to equivalence. If it is true that every W* algebra is the W* direct sum of homogeneous algebras, then these results effectively reduce the classification problem of general W* algebras to that of simple factor algebras. (Received February 29, 1956.)

511. R. T. Prosser: *Representations of C* algebras.*

Let A be a C* algebra. We recall that the second adjoint A'' of A forms a W* algebra under the operations inherited from A. We show here that every C* representation of A may be lifted to a W* representation of A'' essentially by taking weak closures. This process generates a natural one-one correspondence between the C* representations of A and the W* representations of A''. It follows that the C* representations of A may be characterized (up to equivalence) by suitable multiplicity functions defined on the set of central projections of A''; or, equivalently, by multiplicity functions defined on the set of norm-closed central supports of the state space Ω of A. These results include most of the standard theorems on representations of C* algebras, and settle a number of open questions. In particular, the multiplicity theory recently developed by R. V. Kadison is a consequence. (Received February 29, 1956.)

512. R. A. Raimi: *Mean values and Banach limits.*

Let E be a Banach space of essentially bounded measurable real-valued functions on the real line, $\|f\| = \text{ess sup} |f(x)|$. Let E contain all the uniformly continuous bounded functions, and all translates of functions in E. Let $T_\alpha : f(x) \rightarrow (1/2\,\alpha)\int_{-\alpha}^{\alpha} f(x+t)dt$, and let $V = \{f \in E \mid T_\alpha f$ converges in $E\}$. Then $\lim_\alpha T_\alpha f$ is a constant, $m(f)$, defining a continuous linear functional on V. The class M of all norm-preserving extensions of $m(f)$ to all of E is studied. ϕ is in M if and only if ϕ is linear and $\lim_\alpha \inf_x T_\alpha f(x) \leq \phi(f) \leq \lim_\alpha \sup_x T_\alpha f(x)$ for all f in E (the limits exist). Alternate expressions, involving the geometry of E and its conjugate space, are given. All the elements of M are *Banach Limits* (translation-invariant positive functionals of norm one); if E contains only uniformly continuous functions, the converse is true. (Received February 16, 1956.)

513. B. L. Reinhart: *Harmonic integrals on almost product manifolds.*

An almost product manifold V is a compact, connected C^∞ manifold provided with a C^∞ direct sum decomposition of the tangent bundle $T(V)$ into integrable subspaces $PT(V)$ and $QT(V)$; local coordinates $(x^1, \cdots, x^p, y^{p+1}, \cdots, y^n)$ exist consistent with the structure. A natural bigrading is induced on the differential forms. This decomposition induces a splitting of the exterior derivative $d = d' + d''$ and its adjoint $\delta = \delta'\delta''$; and special Laplacians $\Delta' = d'\delta' + \delta' d'$, $\Delta'' = d''\delta'' + \delta'' d''$, and $\tilde{\Delta} = \Delta' + \Delta''$. The operators d', δ', and Δ' differentiate with respect to x only. Examples show that the Green's operator for Δ' is not in general everywhere defined on $\mathfrak{L}_2(V)$, and that the analogue of Hodge's theorem is false. It is here proved that the Green's operator is defined for every C^∞ section of the sheaf $E = D^r_x \wedge D^s_y$, where D^r_x is the sheaf of germs of r-forms depending on x alone, provided V has a torsionless metric. The proof employs a method of Morrey and Eells (Ann. of Math. vol. 63 (1956) pp. 91–128) applied to the Hilbert space generated by the C^∞ sections of a sheaf coherent in the sense that locally there exists a finite number of forms $\gamma(y)$ generating each stalk as a D^r_x-module. The necessary parametrix is constructed by a method of Kodaira (Ann. of Math. vol. 50 (1949) pp. 587–665). The operator $\tilde{\Delta}$ has the property that its solutions are precisely the harmonic forms of pure type; in the case of a torsionless metric, $\Delta = \Delta$. (Received February 29, 1956.)

514t. Walter Rudin: *Boundary values of continuous analytic functions.*

Let K and C denote the closure and boundary, respectively, of the open unit disc U. The following extension theorem is proved: If E is a closed subset of C, if the Lebesgue measure of E is 0, if g is a complex-valued continuous function on C, and if T is a closed Jordan region such that $g(E) \subset T$, then there exists a function f which is continuous on K and analytic in U, such that $f(K) \subset T$ and $f(z) = g(z)$ for all $z \in E$. (Received February 23, 1956.)

515. Robert Schatten: *On the "trace-class" of operators.*

Consider the Banach space of completely continuous operators on a Hilbert space $\mathfrak{H}$. Its conjugate space may be interpreted as the "trace-class," i.e., a Banach space of some operators A on $\mathfrak{H}$ where the norm of A is given by the trace of $(A^*A)^{1/2}$. It is natural therefore to consider the space $\mathfrak{C}$ of completely continuous operators on a given Banach space $\mathfrak{B}$ and identify its conjugate space $\mathfrak{C}^*$ with the corresponding "trace-class." Under some reasonable assumptions a nontrivial evaluation of the norm of an operator in $\mathfrak{C}^*$ is given. (Received February 17, 1956.)

516t. V. L. Shapiro: *On Green's theorem.*

Let D be a bounded domain in the (x, y)-plane, let C be its boundary, and let E be a closed set of logarithmic capacity zero contained in $D+C$. Then by means of the Riemannian theory of multiple trigonometric series, it is shown in this paper that Green's theorem $\int_C A dy - B dx = \int_D (A_x + B_y) dx dy$ holds under the following four assumptions: 1. The boundary C consists of one or several simple closed rectifiable curves. 2. The functions $A(x, y)$, $B(x, y)$ are defined and continuous in the closure $\bar{D} = D+C$. 3. The functions A and B have total differential in $D-E$. 4. The sum $\Phi(x, y) = A_x + B_y$ is Lebesgue integrable over D. Furthermore, it is shown in this paper that this theorem cannot be extended to the case where E is of measure zero and positive logarithmic capacity. This theorem extends a result recently obtained by Bochner (Math. Zeit. 1955). Analogous results also hold in n-dimensional Euclidean space ($n \geq 2$). (Received February 23, 1956.)

517. J. R. Shoenfield: *Infinite dimensional manifolds.* Preliminary report.

Let B be a Banach space. A B-manifold of class C^k is a space S with a class of mappings $[\phi_\alpha | \alpha \in A]$ of B into S such that each ϕ_α is one-one; the ranges of the ϕ_α cover S; and each $\phi_\alpha^{-1}\phi_\beta$ is differentiable of class C^k in the sense of Frechét. Many elementary properties of differentiable manifolds hold for such manifolds. There is a natural definition of the tangent space at a point x of S; it is a Banach space isomorphic to B. Affine connections and Riemann tensors may be defined for such manifolds. Sample theorem: In a Riemannian B-manifold, the geodesics issuing from a point simply cover a neighborhood of that point. The definition of Lie groups may be extended to such manifolds. The basic theorems of Lie groups hold, except that it is necessary in some cases to add the following hypothesis: Every closed subspace of B has a complementary closed subspace. (Received February 29, 1956.)

518. Seth Warner: *Equivalence of a problem in topological algebra with Ulam's measure problem.*

An *Ulam ultrafilter* $\mathfrak{F}$ on a set A is an ultrafilter such that the intersection of a countable family of members of $\mathfrak{F}$ is again a member of $\mathfrak{F}$. Call the following assertion Axiom U: For any set A and any Ulam ultrafilter $\mathfrak{F}$ on A, there exists $a \in A$ such that $\mathfrak{F}$ is the class of all subsets of A containing a. (Equivalently, for any set A and any nonzero measure μ defined on all subsets of A taking on only the values 0 and 1, there exists $a \in A$ such that $\mu(X) = 1$ if and only if $a \in X$.) A Hausdorff locally m-convex algebra E is *i-bornological* if any homomorphism from E into any other locally m-convex algebra carrying bound, idempotent subsets of E into bound sets is continuous. Axiom U is equivalent to the following: If $\{E_\alpha\}$ is any family of i-bornological algebras, each with identity, $\prod_\alpha E_\alpha$ is i-bornological. Recent advances in logic show that if the usual axioms of set theory are consistent, Axiom U may be added without destroying consistency. Recent advances in topology show that Axiom U is equivalent with any of the following: (1) Every Hausdorff topological space admitting a compatible, complete uniform structure is a Q-space in the sense of Hewitt; (2–3) Every paracompact [respectively, metrizable] space is a Q-space. (Received February 27, 1956.)

519. Albert Wilansky: *Banach algebra and summability.*

The set C of conservative matrices is a semi-simple Banach algebra, $\|A\| = \sup_n \sum_k |a_{nk}|$; a closed subalgebra of the space of endomorphisms of m (bounded sequences). Trivial fact: a maximal linear subspace of an algebra is an ideal if it is a subalgebra. Corollaries (also provable directly). 1. *The set of conull matrices is an ideal in* C. Proof. The product of conull matrices is conull. 2. $\chi(A) = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk}$ *is a multiplicative function of* A. For a sequence x, let (x) be the class of matrices in C which sum x. Theorem. (x) *is a left ideal iff* x *is bounded*, (x) *is a right ideal iff* x *is convergent*, (x) *is closed iff* (x) *is bounded*, $I \in \overline{(x)}$ *if* x *is unbounded*. It is known that $(x) \subset (y)$, y divergent, imply $(x) = (y)$. *Each* a_{nk} *is a continuous function of* A, *so also is each* $\lim_n a_{nk}$. Corollary: *The subset of multiplicative matrices is closed*. Let F be Banach algebra of matrices of finite norm. Theorem. C *has inverses in* F (i.e. $A \in C$, $A^{-1} \in F \Rightarrow A^{-1} \in C$). A conull matrix may have a right inverse in F (compare Corollary 1). Mercer's theorem implies that the $(C, 1)$ matrix lies in closure of maximal group. (Received February 27, 1956.)

520. Y. K. Wong: *On unbounded infinite matrices and their inverses.*

Let the columns of A consist of a sequence of Hilbert vectors such that every finite subset is (right) linear independent. The elements of A lie in a locally compact, connected topological field satisfying the second axiom of countability. By the orthogonalization process, $B = AT$ where $T(i, j) = 0$ for $i > j$ and $T(i, i) > 0$. Let B^* be the conjugate-transpose of B. Then $B^*B = I$ and $BB^* = E$ which is idempotent, hermitian, nonnegative-definite, and dependent only on A. There exists a unique matrix L_0 such that $L_0E = L_0$ and $L_0A = I$ if and only if all columns of T^* are Hilbert vectors. Then $L_0 = TB^*$ which is independent of T and B. L_0 is bounded if and only if T is bounded. Similarly if the columns of A^* are Hilbert vectors which are finitely (right) linearly independent, $C = SA$, $CC^* = I$ and $C^*C = F$ which is idempotent, hermitian and nonnegative-definite. There exists uniquely R_0 such that $FR_0 = R_0$ and $AR_0 = I$ if and only if all columns of S are Hilbert vectors. Then $R_0 = C^*S$. R_0 is bounded if and only if S is bounded. If the bounded matrices L_0, R_0 exist such that $L_0A = I = AR_0$ and $AL_0 = E$, $R_0A = F$, then $A^{-1} = L_0 + R_0(I - E) + Z = R_0 + (I - F)L_0 + Z$, where $ZA = AZ = 0$ and Z is bounded. (Received February 21, 1956.)

APPLIED MATHEMATICS

521. E. H. Bareiss: *On the numerical solution of the Boltzmann transport equation.* I.

An iterative method for the numerical solution of the Boltzmann transport equation for solving neutron transport problems is described. Several "Flexible Transport Theory Routines" have been coded for the UNIVAC and NORC and are currently used by the Westinghouse Atomic Power Division in their nuclear reactor design program. The problems are identified by the following common characteristics: One-dimensional, slab geometry; arbitrarily many regions per reactor cell; single energy with isotropic and anisotropic scattering and sources. The reactor cells may have reflecting and free boundaries. In the first code, acceleration of convergence may be obtained by Aitken's δ^2-process applied on the entire field of the solution. In the other codes, there is no provision for the extrapolation of the approximate solutions. (February 27, 1956.)

522. G. F. Carrier and R. C. DiPrima (p): *On the unsteady motion of a viscous fluid past a semi-infinite flat plate.*

The flow past a semi-infinite flat plate is considered when the flow at infinity is of the form $U_0+U_1 \exp (i\omega t)$, where U_0 and U_1 are constants. The equations of motion governing a viscous incompressible fluid are linearized by a modification of the Oseen linearization. Two fourth order partial differential equations are obtained governing the steady flow and the time dependent flow. These equations are solved by the use of Fourier transforms and the Wiener-Hopf technique. Results valid near the leading edge and far down the plate are obtained. (Received February 21, 1956.)

523*t*. Nathaniel Coburn: *A class of compressible flows.*

The purpose of this paper is to express the characteristic relations of an irrotational, isentropic, supersonic fluid in intrinsic form for the case when one family of of characteristic surfaces is ∞^1 parallel planes and to study the resulting flows of the fluid. Two types of flow are investigated: (1) the bicharacteristics are concentric circles; the stream lines are helices; (2) again, the bicharacteristics are concentric circles; the stream lines lie on cones and are a type of generalized helix. In each case, the Mach number varies with the radius of the bicharacteristic. (Received April 9, 1956.)

524. Ruth M. Davis (p) and Elizabeth Cuthill: *Improving convergence of the successive over-relaxation method.*

In solving numerically either eigenvalue or boundary value problems involving a system of n elliptic partial differential equations of the form (1) $\nabla\cdot D_i\nabla\phi_i-A_i\phi_i+B_i\phi_{i-1}=0$, $\nabla\cdot D_1\nabla\phi_1-A_1\phi_1+\eta B_n\phi_n=0$, $i=2, 3, \cdots, n$, one is led to consider systems of linear equations which can be written in matrix notation as (2) $A\phi_i=B\phi_{i-1}$, $C\phi_1=\eta D\phi_n$, $i=2, 3, \cdots, n$. The linear iterative method of successive overrelaxation developed by D. Young has been adapted for use in solving the system (2). An optimum over-relaxation factor ω_k is computed, using a Rayleigh Quotient, at each iteration which can be incorporated into succeeding calculations to aid convergence. To further hasten the convergence of slowly convergent sequences of solution approximations for each individual equation of (2) use is made of Aitken's δ^2 process. Additional extrapolatory techniques are applied to the successive approximations to the entire solution of the system (2) with favorable results. (Received February 27, 1956.)

525. I. J. Epstein: *A study of integrals and integral equations arising in diffraction theory.*

The problem of the diffraction of a normally incident plane wave by a circular aperture in a plane screen is treated here. The development is according to Bouwkamp and Magnus. The field in the aperture is determined by an infinite system of linear equations. The coefficients of these equations are shown to have a simple asymptotic behavior as the wavelength λ of the incident wave goes to zero. The system of linear equations is shown to be equivalent to an integral equation which is solved explicitly in the limit case $\lambda=0$. A method for determining correction terms for small values of $\lambda\neq 0$ is given and the order of magnitude of the correction term is investigated. Some of the incidental results are listed separately since they are of some interest as cosine transforms. (Received February 21, 1956.)

526. B. A. Fleishman: *The random convection model of turbulent diffusion.*

The fundamental problem of turbulent diffusion is to find the mean mass density distribution, $\langle s(x, y, z, t)\rangle$, of matter inserted in a turbulent velocity field, in terms of the statistical characteristics of the field. By combining a purely kinematical approach with the velocity-ensemble approach of the mathematical theory of turbulence, it is possible to formulate such problems as initial value problems for partial differential equations with random coefficients. If some dispersible matter (assumed not to be subject to molecular diffusion) is convected by a vector velocity field $\bar{v}(x, y, z, t)$, its mass density, $s(x, y, z, t)$, must satisfy the convection (or continuity) equation $s_t = -div(s\bar{v})$. Upon the choice of appropriate initial and/or boundary conditions an initial value problem is formulated, for a particular convection field $\bar{v}(x, y, z, t)$. Now let $\bar{v}(x, y, z, t)$ belong to an ensemble of functions (which is identified with the real turbulent velocity field). Then the expected mass density distribution, $\langle s(x, y, z, t)\rangle$, is found by solving the initial value problem with an arbitrary member $\bar{v}(x, y, z, t)$ of the velocity ensemble, then taking an ensemble average of the result. In general, the procedure is to solve by iteration an integral equation equivalent to the initial value problem, and then average. (Received February 27, 1956.)

527. H. M. Gurk: *Finite solutions of simple games.*

An n-person ***simple game*** is defined (using the (0, 1) normalization) by a characteristic function v taking on only the values 0 and 1. The desirable coalitions of such a game are the ***minimal winning coalitions***, i.e., the members of the collection $W^m = \{S/v(S) = 1$, but $v(T) = 0$ if $T \subset S$, $T \neq S\}$. von Neumann and Morgenstern have defined a natural, finite solution for such games, the ***main solution*** (*Theory of games and economic behavior*, Princeton, 1947, Chap. 10). Such solutions are based on the desire of players to form minimal winning coalitions and represent success for player i ($i = 1, \cdots, n$) by $x_i \geq 0$, failure by zero. The question has been asked whether these are the only finite solutions (i.e., only a finite number of imputations) for simple games. A ***natural solution*** of a simple game is defined as a finite solution V such that $\alpha \in V \Rightarrow$ there exists $T \in W^m$ such that $\mathfrak{I}_\alpha = \{i/\alpha_i > 0\} \subset T$. Natural solutions are exhibited for some simple games which have no main solutions, and some lemmas are given which are useful in finding such solutions. Other finite solutions are also exhibited. All known finite solutions are shown to belong to a class of ***naturally-derived solutions***. The conjecture is offered that all finite solutions of simple games are naturally-derived. A simple game is given which has no such solution. (Received February 29, 1956.)

528. J. R. Isbell and F. J. Wagner (p): *Military evaluation and statistical decision.*

This is an essay toward a military decision theory analogous to thermodynamics rather than to statistical mechanics. Such a theory would presumably turn on state variables. This paper gives a clear definition of one state variable already shadowily recognized and achieves a shadowy recognition of another variable or complex of variables which are so new that they may be meaningless. The former variable is termed *confidence*, occupies the same place in the formal theory as a priori bounds on the strategies of nature, but has a clear operational definition. The latter variable has no name nor formal status but seems to be recognizable in military contexts as that dimension which stretches between pure statistical uncertainty and pure game-theoretic competition. These two ideas are employed in axiomatic and heuristic comparisons of several proposed decision rules. (Received March 12, 1956.)

529. C. C. Lin: *On uniformly valid asymptotic solutions of the equation of hydrodynamic stability and other differential equations involving a turning point.* Preliminary report.

The asymptotic solution of a class of fourth order differential equations containing a large parameter, including the equation of hydrodynamic stability for parallel flows, is treated in a manner different from those of Wasow (Ann. of Math. vol. 58 (1953) pp. 222–253) and Langer (private communication). The equation is first transformed into the standard form (2) $L(\psi)=\lambda^2 M(\psi)$ where $L(\psi)=\psi^{iv}+\lambda^2(z\psi''+\alpha\psi'+\beta\psi)$ (α, β being constants), and $M(\psi)=Z\{a(z)\psi'+b(z)\psi\}+\lambda^{-1}\{c_0(z,\lambda)\psi''+c_1(z,\lambda)\psi'+c_2(z,\lambda)\psi\}$. In $M(\psi)$, $c_i(z,\lambda)=\sum_{j=0}^{\infty}c_{ij}(z)\lambda^{-j}$ $(i=0, 1, 2)$, and the functions $a(z)$, $b(z)$, and $c_{ij}(z)$ are regular in the neighborhood of $z=0$. The asymptotic solutions of (1) are then expressed in terms of the solutions of $L(u)=0$, which can be explicitly solved by the method of Laplace transformation. It is shown that if ψ is written in the form (2) $\psi=C_0u+C_1u'+C_2u''+C_3u'''$, then $C_i(z,\lambda)$ can be formally obtained in the form $C_i(z,\lambda)=\sum_{j=0}^{\infty}C_{ij}(z)\lambda^{-j}$, where $C_{ij}(z)$ are regular in the neighborhood of $z=0$. Proof of the existence of actual solutions of (1) in the form (2) are being carried out. (Received February 21, 1956.)

530t. Cathleen S. Morawetz: *On the nonexistence of continuous transonic flows past profiles.* II.

The following result is proved: There cannot exist two steady two-dimensional irrotational continuous transonic symmetric flows, with the same Mach number at infinity, past two symmetric convex profiles which differ from each other (by a finite amount) only on an arc cut out by two intersecting characteristics of one of the flows and containing the point of maximum velocity for that flow, if it is assumed that the flows are close to each other. The latter condition amounts to imposing a certain natural bound on the differences of the velocities and accelerations For an earlier partial result see C. S. Morawetz, *On the non-existence of continuous transonic flows past profiles* I, Communications on Pure and Applied Mathematics vol. 9 (1956) no. 1. (Received April 4, 1956.)

531. G. Y. Rainich: *Conditional covariance and theories of light.*

In the *general* cases the electromagnetic field at a point is given by a six-vector and a particle by a four-vector. Here we consider *singular* cases when both invariants of the six-vector and the invariant of the four-vector are zero. A singular four-vector is shown to be a conditional covariant of a singular six-vector; in fact, in this case the square roots of the diagonal elements of a matrix derived from the six-vector are the components of the four-vector. The four-vector in turn determines the six-vector but only up to an arbitrary angle. When we consider the situation in a neighborhood of a point and take into account Maxwell's equations a *constant* field of four-vectors determines a *periodic* electromagnetic field. This seems to establish a mathematical connection between the corpuscular and the wave theory of light. (Received February 28, 1956.)

532. Domina E. Spencer: *The foundations of universal time.*

In accordance with Einstein's theory of relativity, it is generally believed today that the concept of simultaneity is untenable for moving observers. This paper reexamines the concept of simultaneity and the synchronization of clocks. The possibility of establishing universal time is approached from a postulational point of view. Two types of synchronization are defined: quasi-synchronization and complete synchronization. The conditions under which each type of synchronization is possible are derived It is shown that quasi-synchronization is always possible but that complete synchronization is permissible only if certain limitations are imposed on the law of transformation of the velocity of light and on the relative motion of the observers. The results are compared with the work of Einstein, Milne, and Grünbaum. Practical discrepancies betwen quasi-synchronization and complete synchronization are also evaluated. (Received February 28, 1956.)

GEOMETRY

533. Theodore Frankel: *The homology class of an isometric flow.*

Let v be a vector field on a compact, orientable Riemannian manifold M. If v generates isometries of the Riemannian metric (i.e. if v is a Killing vector) and if there are any closed integral curves of v then we have the following. Theorem: (i) all closed integral curves lie in dependent homology classes, (ii) the homology class of a closed integral curve is determined (up to a nonzero multiple) by de Rham's theorem, (iii) if v vanishes at some point of M then all closed integral curves bound. Corollary: if $M=G/g$, G a compact Lie group, g a closed subgroup, then all closed orbits of a one parameter subgroup of G are in dependent homology classes. (Received March 26, 1956.)

534t. Shoshichi Kobayashi: *Nondegenerate pseudo-Kaehlerian spaces.*

A pseudo-Kaehlerian space M is said to be nondegenerate if the restricted homogeneous holonomy group contains the operator J of the almost complex structure. The following results are obtained. (1) An irreducible pseudo-Kaehlerian space with nonvanishing Ricci curvature is nondegenerate. (2) A simply connected and complete nondegenerate pseudo-Kaehlerian space is the direct product of irreducible nondegenerate pseudo-Kaehlerian spaces. (3) Every affine transformation of an irreducible nondegenerate pseudo-Kaehlerian space preserves the almost complex structure or gives the conjugate structure. (4) Every element of the connected component of the unit of the affine transformation group of a complete nondegenerate pseudo-Kaehlerian space is an isometry and preserves the almost complex structure. (Received January 20, 1956.)

LOGIC AND FOUNDATIONS

535t. S. B. Akers, Jr.: *A full disjunctive normal form for the monadic functional calculus of first order.*

In using machine techniques for the processing and analysis of logical expressions, it is often desirable to be able to assign binary numbers to the logical expressions involved so that a complete isomorphism exists between these binary designation numbers and the classes of logically equivalent expressions. Such an assignment is immediately possible with expressions in the propositional calculus because of the existence of a full disjunctive normal form for such expressions. For the monadic functional calculus of first order, a corresponding full disjunctive normal form is obtained by adapting the normal form of Herbrand (*Recherches sur la théorie de la démonstration*, published in Warsaw in 1930 in the Series, Travaux de la Société des Sciences et des Lettres de Varsovie, Class III, No. 33) and of Quine (*O Sentido da Nova Lógica*, São Paulo, Brazil, 1944) to the case that propositional variables and free individual variables are present; and it is shown that this normal form leads to an assignment of binary designation numbers in a way that is useful for machine techniques. The rules for expanding expressions into this normal form are given, and the satisfiability of the resulting disjuncts is shown. This work was sponsored in part by the USAF, Contract AF19(604)-1582. (Received February 6, 1956.)

536*t*. M. O. Rabin: *Construction of test groups.*

When studying recursive solvability questions for groups one needs the following algebraic construction: Given a group G by a presentation $\Pi=(x_1, \cdots, x_n : r(x))$ and a word $w(x) \in G$, to construct a presentation Π_w defining a ***test group*** G_w such that (1) if $w(x)=1$ in G then G_w reduces to the trivial group, (2) if $w(x) \neq 1$ then G is imbedded in G_w. G_w may thus serve to test whether $w(x)=1$. Let Π_0 be $(x_1, \cdots, x_{n+1} : r(x))$ (i.e. add one free generator to π), then the following statements hold. Lemma 1: Let H_0 be presented by Π_0 and let $u(x)$ have infinite order in H_0; then H_0 is isomorphically imbedded in the group H_1 defined by $\Pi_1=(x_1, \cdots, x_{n+1}, t, a, s, b : r(x), u(x)t=t^2u(x), ta=a^2t, u(x)s=s^2u(x), sb=b^2s)$, moreover the subgroup of H_1 generated by $a, b, x_1, \cdots, x_{n+1}$ is isomorphic to $(a, b) * H_0$ ($*$ denotes free product). Lemma 2: In (a, b) the elements $b^i a b^{-i}$, $i=0, \cdots, n+1$, $b^{n+2}aba^{-1}b^{-n-2}$ satisfy no nontrivial relation. Lemma 3: H_0 is imbedded in H_2 obtained from $(c, d) * H_1$ by adding the relations: $a=c$, $x_i b^i a b^{-i}=d^i c d^{-i}$, $i=1, \cdots, n+1$, $b^{n+2}aba^{-1}b^{-n-2}=d^{n+2}cdc^{-1}d^{-n-2}$. Note that the presentation Π_2 of H_2 can be obtained explicitly from Π_0 and $u(x)$. *Theorem: The presentation* Π_w *of the test group* G_w *is obtained from* Π_2 *by specialising* $u(x)$ *to be* $x_{n+1}w(x)x_{n+1}^{-1}w(x)$. *I.e. the group* G_w *thereby defined has the properties* (1) *and* (2) *required of a test group*. (Received April 12, 1956.)

537*t*. M. O. Rabin: *Recursive unsolvability of group theoretic problems.*

Combining Novikov's theorem on the recursive unsolvability of the word problem for groups (*Doklady Akad. Nauk SSSR. vol.* 85 (1952) *pp.* 709–712) with the construction of test groups given in the previous abstract, one can prove the following: *Theorem: Let P be a group theoretic property such that there exists at least one finitely presented group which has P and at least one finitely presented group which does not possess P and which cannot be imbedded in a group having P, then there does not exist any generally effective procedure for deciding for every presentation* Π *whether the group* G_Π *defined by it has the property P*. As corollaries one obtains the following results: *There does not exist a generally effective procedure for recognizing from a presentation* Π *whether the group* G_Π *defined by it is a* (this statement should be prefixed to each of the following) (1) *trivial group*, (2) *cyclic group*, (3) *finite group*, (4) *locally infinite group*, (5) *free group*, (6) *commutative group*, (7) *solvable group*, (8) *group expressible as nontrivial free product*, (9) *group expressible as nontrivial direct product*, (10) *group for which the word problem is solvable*, (11) *group which can be presented using only one relation*. (Received April 12, 1956.)

STATISTICS AND PROBABILITY

538t. Charles Fox: *Some applications of Mellin transforms to the theory of bivariate statistical distributions.*

If $F(r, s) = \int_0^\infty\int_0^\infty f(x, y)x^{r-1}y^{s-1}dxdy$ then $F(r, s)$ is the double Mellin transform of $f(x, y)$. This can be inverted to (1) $f(x, y) = (1/(2\pi i)^2)\int_{k-i\infty}^{k+i\infty}\int_{k-i\infty}^{k+i\infty} F(r, s)x^{-r}y^{-s}drds$. The following results are proved: let (ξ, η) be a pair of random bivariate variables with frequency function $f(x, y)$, assumed symmetrical about the x and y axes. Then $E(|\xi|^{r-1}|\eta|^{s-1}) = 4\ F(r, s)$, where E is the expectation function. Hence, if the expectation function is known one can, by (1), compute the frequency function. If (ξ_1, η_1) and (ξ_2, η_2) are two pairs of bivariate random variables (with symmetrical frequency functions) then the frequency function of the pair $(\xi_1\xi_2, \eta_1\eta_2)$ has Mellin transform $4F_1(r, s)F_2(r, s)$ where $F_1(r, s)$ is the Mellin transform of the frequency function of (ξ_1, η_1) and $F_2(r, s)$ is that of (ξ_2, η_2). Thus, on inverting $4F_1(r, s)F_2(r, s)$ by (1) we can compute the frequency function of the distribution of $(\xi_1\xi_2, \eta_1\eta_2)$. Similarly the frequency function of $(1/\xi_1, 1/\eta_1)$ has Mellin transform $4F_1(2-r, 2-s)$ and of $(\xi_1/\xi_2, \eta_1/\eta_2)$ has Mellin transform $4F_1(r, s)F_2(2-r, 2-s)$ and so the frequency functions of these distributions can be computed by (1). Results of this nature can be obtained for various products and ratios of the random variables (ξ_1, η_1) and (ξ_2, η_2). The case of unsymmetrical frequency functions is also considered. (Received February 17, 1956.)

539. Eugene Lukacs: *A question raised by D. Dugué.*

The rectangular distribution has the characteristic function $(\sin t)/t$; it is not infinitely divisible. For every positive integer n it has the factor $[\sin (t/n)]/(t/n)$. Therefore it provides an example for a distribution which is not infinitely divisible but has an enumerable infinity of different factors which depend on n D. Dugué [Annales de l'Institut Henri Poincaré vol. 12 (1951) pp. 159–169] raised the question whether there exists a distribution which is not infinitely divisible but a non-enumerable set of factors. A characteristic function is constructed which has this property so that Dugué's question can be answered in the affirmative. (Received February 27, 1956.)

TOPOLOGY

540. R. D. Anderson: *A characterization of the universal curve.*

The universal curve is a one-dimensional continuous curve which is an analog of the Cantor middle third set and is obtainable by punching rectangular "holes" out of a cube in a particular regular fashion. The author gives a characterization of the universal curve in terms of a sequence of finite coverings of a one-dimensional continuum M by continua. The essential conditions may be considered to be strong forms of the statements that M have no local cut points and that M be locally nonplanar, in fact, in a sense that M have a local anti-Jordan curve property. The characterization is used in another paper (abstract for February 1956 meetings) to establish continuous collections of universal curves. The technique of the argument is also used to show the n-point homogeneity of the universal curve as previously announced by the author (Bull. Amer. Math. Soc. Abstract 59-2-249). (Received February 28, 1956.)

541t. J. D. Baum: *P-recurrence in topological dynamics.*

The notion of recurrence [cf. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Providence, 1955, Chap. 7] in a "direction," the analog of positive or negative recurrence is generalized to transformation groups whose phase groups have a structure more general than the reals or integers. The notion of regional recurrence is similarly treated. Inheritance theorems are proved; and the relation between recurrence and asymptoticity and incompressibility is explored. Applications of these notions to other topics in topological dynamics are given. (Received February 27, 1956.)

542. E. E. Floyd: *Closed coverings in Čech homology theory.*

If α is a finite closed covering of a compact space X, there is a projection homomorphism π_α of the Čech group $H_n(X)$ into the group $H_n(\alpha)$ of the nerve. We say that α, β determine $H_n(X)$, α and β finite closed coverings with $\alpha<\beta$, if and only if π_α maps $H_n(X)$ isomorphically onto the image of the projection $\pi_{\beta\alpha}:H_n(\beta)\rightarrow H_n(\alpha)$. Sufficient conditions are given that α, β determine $H_n(X)$. These conditions are in terms of the groups of the intersections of elements of finite closed coverings. Applications are made to homology local connectedness, regular convergence, and the Vietoris mapping theorems. In the proof of the fundamental theorem, the Kelley-Pitcher theory of coverings of complexes [Ann. of Math. vol. 48 (1947) pp. 682–709] is generalized to pairs (X, α) where X is a compact and α is a finite closed covering of X. (Received December 12, 1956.)

543. O. H. Hamilton: *Fixed points for certain noncontinuous transformations on N-cells, a counterexample.*

An example is given of a noncontinuous transformation T of a closed N-cell I into itself such that the graph mapping, $g(x)=(x, T(x))$ transforms locally connected subsets of I onto connected subsets of $I\times I$, but such that T leaves no point of I fixed. There is given also a necessary and sufficient condition on the transformation T in order that g shall transform connected subsets of I onto connected subsets of $I\times I$. (Received February 27, 1956.)

544. Mary-Elizabeth Hamstrom and Eldon Dyer (p): *Certain completely regular mappings.*

An open mapping f of a compact metric space K onto a space M is *regular* iff for each positive number ϵ, there is a positive number δ such that if p and q are points in the inverse under f of a point of M at a distance apart of less than δ, there is an arc in that inverse containing p and q and having diameter less than ϵ. The mapping f is *completely regular* iff for each positive number ϵ, there is a positive number δ such that if p and q are points of M at a distance apart of less than δ, there is a homeomorphism of $f^{-1}(p)$ onto $f^{-1}(q)$ moving no point as much as ϵ. In this paper it is shown that if f is a completely regular mapping of a compact metric space K onto a closed interval I such that (1) for each point x of I, $f^{-1}(x)$ is an n-cell, and (2) $\bigcup_{x\in I} Bdf^{-1}(x)$ is the product of an $(n-1)$-sphere with I such that f is the projection map of this product onto I, then K is an $(n+1)$-cell. If $n\leq 4$, the conclusion holds even when condition (2) is omitted. And if f is a regular mapping of K onto an i-cell, $i=1$ or 2, such that each $f^{-1}(x)$ is a 2-cell, then K is an $(i+2)$-cell. An application of the preceding theorem is that if f is a mapping of Euclidean 3-space, E^3, onto a metric space

K such that for each point x of K, $f^{-1}(x)$ is a compact continuum lying in a horizontal plane and not separating that plane, then K is E^3. (Received February 28, 1956.)

545. L. J. Heider: *Generalized G_δ spaces.*

Let X denote a completely regular space, while νX denotes the Hewitt Q-space extension of X, and $C^*(X)$ denotes the class of all real-valued, bounded functions, defined and continuous on X. The space X is called a generalized G_δ space if, for each point p in X which is not a G_δ point, some element of $C^*(X-p)$ fails to be a restriction to $X-p$ of any element of $C^*(X)$. It may be shown that if X and Y are generalized G_δ spaces such that νX and νY are homeomorphic, then X and Y are homeomorphic. Moreover, if X is a generalized G_δ space and Y is a completely regular space with νY homeomorphic to νX, then $X \subset Y \subset \nu X$. (Received February 27, 1956.)

546. J. F. Nash: *Point set topology based on connectivity.*

Instead of basing point set topology on open sets one can use connected sets. Open sets can be defined from connected sets in a canonical manner, but this canonical open set topology derivable from a connectivity topology is usually not the only one which yields the same set of connected sets. The approach via connectivity might give a more natural insight into some questions. (Received February 23, 1956.)

547t. C. D. Papakyriakopoulos: *On loops on the boundary of 3-manifolds.*

The following theorem is proved: *Let M be a 3-manifold, finite or not, with boundary N, not necessarily connected or finite. Let N' be a closed orientable component of N, and let q be a point of N' such that there exists a loop on N' based at q which is $\simeq 0$ in M, and not homotopic to* 0 *on N' (the symbol $\simeq$ means homotopic to). Then there exists a system G_i, $i=1, \cdots, n(<\infty)$ of simple ($=$without multiple points) loops on N' based at q, $\simeq 0$ in M, not homotopic* to 0 *on N' and such that: Any loop on N' based at q, and $\simeq 0$ in M, is homotopic on N' to a product of transforms of the G_i^ϵ ($\epsilon=\pm 1$) by loops on N' based at q.* The proof of this theorem is based on the theorem contained in the author's abstract *On a lemma of Kneser* [Bull. Amer. Math. Soc. Abstract 62-2-279]. Applications of this theorem will be given in a subsequent abstract. (Received February 10, 1956.)

548. C. D. Papakyriakopoulos: *On solid toruses of higher genus.* Preliminary report.

The following theorem is proved: *Let M be a finite 3-manifold with nonvacuous orientable boundary N such that $\pi_1(M, N)=1$. Then M is aspherical, acyclic in dimension >1, and $\pi_1(M)$ is a free group on g generators, where g (≥ 0) is the genus of N.* The proof of this theorem is based on the theorem contained in the author's abstract *On loops on the boundary of 3-manifolds* [Bull. Amer. Math. Soc. Abstract 62-4-548], and on Hopf's paper *Fundamentalgruppe und zweite Bettische Gruppe* [Comment. Math. Helv. vol. 14 (1941–1942)]. (Received February 14, 1956.)

549t. R. W. Rector: *Fundamental linear relations for the seven ring.*

Birkhoff and Lewis in their comprehensive paper, *Chromatic polynomials*, (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 355–451) conclude their attack on the Four Color Problem with a partial analysis of the n-ring with special attention to the

6-ring and 7-ring. The present paper extends the list of linear relations for the 7-ring to 1505 equations and further demonstrates the presence of the predicted 126 linearly independent relations for the 7-ring. (Received March 12, 1956.)

550. Mary E. Rudin: *A property of the countable ordinals.*

Let W be the set of all countable ordinals with the usual topology. It is proved that there is a set E in W such that both E and $W-E$ intersect every uncountable closed subset of W. More precisely, for each limit ordinal w, let $f_n(w)$, for each positive integer n, be a countable ordinal less than w such that $\lim_{n\to\infty} f_n(w) = w$. Let $S(n, s)$ be the set of all w such that $f_n(w) = s$ (n is a positive integer and s is a term of W). Then at least one of the sets $S(n, s)$ has the desired property. (Received February 27, 1956.)

551. Walter Rudin: *Homogeneity problems in the theory of Čech compactifications.*

A topological space X is homogeneous if to every pair of points p, q in X there is a homeomorphism of X which carries p to q. If X is completely regular, the Čech compactification βX of X is a compact space in which X is dense, such that every continuous bounded real function on X can be extended to a continuous function on βX. The following question arises: Does the homogeneity of X imply the homogeneity of $\beta X - X$? Under the assumption of the continuum hypothesis, it is shown in this paper that the answer is negative even if X is a countable discrete space. This in turn leads to the result that $\beta X - X$ fails to be homogeneous whenever X is a locally compact Hausdorff space which is not sequentially compact. (Received February 23, 1956.)

R. D. Schafer,
Associate Secretary

THE APRIL MEETING IN MONTEREY

The five hundred twenty-fifth meeting of the American Mathematical Society was held at the U. S. Naval Postgraduate School in Monterey, California, on Saturday, April 28, 1956. Attendance was approximately 140, including about 110 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor H. C. Wang delivered an address on *Some aspects of transformation groups and homogeneous spaces.* He was introduced by Professor Z. W. Birnbaum. Presiding at the sessions for contributed papers were Professors Paul Garabedian, Ivan Niven, and Raphael Robinson.

Following are the abstracts of papers presented at the meeting, those whose numbers are followed by "*t*" having been given by title. Where a paper has more than one author, the paper was presented by the author whose name is followed by "(p)". Mrs. Butler was introduced by Professor Alfred Tarski, Mr. Hanf by Professor Bjarni Jonsson, and Professor Craig by Dr. R. L. Vaught.

ALGEBRA AND THEORY OF NUMBERS

552. Jean W. Butler: *On operations in finite algebras.*

Consider a finite set A with $n \geq 2$ elements. Let F_A^m be the set of all functions (m-ary operations) on $A \times \cdots \times A$ (m times) to A; F_A be the union $F_A^1 \cup F_A^2 \cup \cdots \cup F_A^m \cup \cdots$. For $X \subseteq F_A$, $\overline{X}$ denotes the smallest $Y \subseteq F_A$ with the properties: $X \subseteq Y$; if $f \in Y$ and h is obtained from f by identifying or transposing two arguments, or by substituting a function $g \in Y$ for an argument, then $h \in Y$. X is *closed* iff $\overline{X} = X \subseteq F_A$; Y is a *basis of* X iff $\overline{Y} = X$ and $\overline{Z} \neq X$ whenever $Z \subset Y$. F_A has a finite basis (Post, 1921). Theorems: (I) *For every closed $X \subseteq F_A^1$ there is a largest closed $Y \subset F_A$ such that $Y \cap F_A^1 = X$.* (II) *There exist p closed sets $M_1, \cdots, M_p \subset F_A$ (p finite, depending on n) such that every closed $X \subset F_A$ is included in some M_i.* (III) *There is an integer p (depending on n) such that every basis of F_A has $\leq p$ elements.* (II) follows from (I), and (III) from (II). (I)–(III) generalize results of Post (Annals of Mathematics Studies, No. 5) for $n = 2$. Tarski has noticed that a modification in the proofs of (II) and (III) leads to more general results (II′) and (III′) differing from (II) and (III) only in that F_A is replaced by any closed set Z with a finite basis. (Received April 25, 1956.)

553*t*. Anne C. Davis: *On simply ordered relations with nontrivial automorphism groups.*

Let S be a simply ordered relation, let $G(S)$ be the group of automorphisms of S and let α be the order type of S. Theorem 1. $G(S)$ consists of more than one element if and only if α is representable in the form (1) $\alpha = \beta + \gamma \cdot (\omega^* + \omega) + \delta$, where $\gamma \neq 0$. Lemma 2. Suppose that $f, g \in G(S)$ and x belongs to the field of S. Let β be the type of the subrelation of S whose field consists of all elements $h(x)$, where h belongs to the subgroup of $G(S)$ generated by f and g Then either (i) $\beta=(\omega^*+\omega)^m\cdot\gamma^n$, with m a non-negative integer and $n\in\{0, 1\}$, or (ii) there exists a sequence $\delta_0\cdots\delta_n\cdots$ such that, for each n, $\beta=(\omega^*+\omega)^n\cdot\delta_n$. (Lemma 2 is due in part to a suggestion of Tarski.) Theorem 3. If S' is a scattered relation of type α', then $G(S')$ is non-Abelian if and only if α' is representable in the form $\alpha'=\beta'+\alpha\cdot(\omega^*+\omega)+\delta'$, with α satisfying (1). Theorem 1 answers Problem (a) of Bull. Amer. Math. Soc. Research Problem 60-3-10, proposed by Goffman. Theorem 3 gives a partial answer to Problem (b) mentioned therein, but in the general case that problem remains open. (Received March 6 1956.)

554. William Hanf: *Representations of lattices by subalgebras*. Preliminary report.

Following Birkhoff and Frink [Trans. Amer. Math. Soc. vol. 64 (1948) pp. 29–316] we call an element a of a complete lattice L inaccessible if $a\in K$ whenever K is a directed set such that $\sum_{y\in K} y=a$. *Given an algebra A with countably many operations of finite rank, the lattice L of all subalgebras of A has the following properties*: (i) *L is complete*, (ii) *every element of L is a sum of inaccessible elements*, (iii) $x\sum_{y\in K} y=\sum_{y\in K} xy$ *for any directed set K and* (iv) *the set of all inaccessible elements less than a given inaccessible element is countable. Conversely, every lattice L which satisfies* (i)–(iv) *is isomorphic to the lattice of all subalgebras of an algebra A with countably many operations of finite rank—in fact, A can be taken to be a commutative loop*. This supplements the result of Birkhoff and Frink [ibid. Theorem 2] that conditions (i)–(iii) are necessary and sufficient for a lattice to be isomorphic to the lattice of all subalgebras of an algebra with arbitrarily many operations of finite rank. The problem was suggested by B. Jónsson. (Received March 5, 1956.)

555. Arno Jaeger: *Multidifferentiations of finite dimension in commutative rings of nonzero characteristic*. Preliminary report.

The newly developed theory of differentiations of commutative rings of squarefree nonzero characteristic, having an identity and inverses to all nonzerodivisors, is generalized to multidifferentiations of finite dimension (cf. J. Reine Angew. Math. vol. 190 (1952) pp. 1-21). The concept of regularity plays the central role. Regular multidifferentiations can be approximated by systems of partial differentiations. The classes of chain-rule-dependent multidifferentiations are investigated. An addition is defined for commuting multidifferentiations of equal dimension, and all multidifferentiations of a class are exhibited. If there do not exist regular multidifferentiations of arbitrarily high dimension in a ring then each differentiation is shown to be dependent on any regular multidifferentiation of maximal dimension. (Received March 5, 1956.)

556t. S. A. Jennings and Rimhak Ree. *Derivations of the group algebras of a class of p-groups*.

Let G be a p-group of class 2 with $A^p=1$ for all $A\in G$, let Z be the center and C be the derived group of G. If $\Gamma=\Gamma(G)$ is the group algebra of G over a field Φ of characteristic p, we may consider the Lie algebra L of all derivations of Γ over Φ. We prove that L contains a solvable ideal S such that $L/S\cong J$, where J is the simple Lie Algebra of derivations of $\Gamma(Z/C)$. In particular, if $Z=C$, then L is solvable. (Received January 30, 1956.)

557. Bjarni Jónsson. *Universal relational systems.*

Consider systems $\mathfrak{A} = \langle A, R_0, \cdots, R_n\rangle$ with $R_i \subseteq A^{p(i)}$, n and $p(i)$ being finite. The set-operations are applied termwise to such systems, $\overline{\mathfrak{A}}$ denotes the cardinal of A, and $\mathfrak{B} < \mathfrak{A}$ means that $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$, i.e., that $\mathfrak{B} = \langle B, S_0, \cdots, S_n\rangle$ where $B \subseteq A$ and $S_i = R_i \cap B^{p(i)}$. Given a class K of systems, a system $\mathfrak{A} \in K$ with $\overline{\mathfrak{A}} = \aleph_\alpha$ is said to be $(\aleph_\alpha, K)$ universal if every $\mathfrak{B} \in K$ with $\overline{\mathfrak{B}} = \aleph_\alpha$ is isomorphic to a subsystem of $\mathfrak{A}$. Consider the following conditions on K: I. There exist $\mathfrak{A}$, $\mathfrak{B} \in K$ which are not isomorphic. II. If $\mathfrak{A} \cong \mathfrak{B} \in K$ then $\mathfrak{A} \in K$. III. For any $\mathfrak{A}$, $\mathfrak{B} \in K$ there exists $\mathfrak{C} \in K$ which contains subsystems isomorphic to $\mathfrak{A}$ and $\mathfrak{B}$. IV. If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A} \cap \mathfrak{B} \in K$, $\mathfrak{A} \cap \mathfrak{B} < \mathfrak{A}$ and $\mathfrak{A} \cap \mathfrak{B} < \mathfrak{B}$, then there exist $\mathfrak{C} \in K$ with $\mathfrak{A} < \mathfrak{C}$ and $\mathfrak{B} < \mathfrak{C}$ V. The union of any simply ordered subset of K is in K. VI$_\gamma$. If $\mathfrak{A} < \mathfrak{B} \in K$ and $\overline{\mathfrak{A}} < \aleph_\gamma$, then there exists $\mathfrak{C} \in K$ with $\mathfrak{A} < \mathfrak{C} < \mathfrak{B}$ and $\overline{\mathfrak{C}} < \aleph_\gamma$. *Results*: If I–V and VI$_0$ hold, then there exists an $(\aleph_0, K)$ universal system. Under the assumption of the Generalized Continuum Hypothesis, if I–V and VI$_1$ hold, then there exists an $(\aleph_\alpha, K)$ universal system for every $\alpha > 0$. I–V and VI$_1$ hold for the classes of all groups, groupoids, partially ordered systems, and lattices, but IV fails for semigroups and for distributive lattices. (Received March 5, 1956.)

558*t*. J. A. Kalman: *Absorption laws relating two binary associative operations.*

Let $(i \cdots j)$ be the family of all algebraic systems with binary associative operations $\wedge$ and $\vee$ related by the absorption laws $(i), \cdots, (j)$, where (1) $x \wedge (x \vee y) = x$, (2) $x \vee (y \wedge x) = x$, (3) $(y \vee x) \wedge x = x$, (4) $(x \wedge y) \vee x = x$, (5) $x \vee (x \wedge y) = x$, (6) $x \wedge (y \vee x) = x$, (7) $(y \wedge x) \vee x = x$, (8) $(x \vee y) \wedge x = x$. Call two such families *similar* if the systems of one may be obtained from those of the other by interchanging the operations and/or reversing one or both pairs of operands. *Results*. Every such family is similar to exactly one of (1), (12), (13), (15), (16), (17), (18), (123), (125), (126), (127), (128), (135), (136), (1234) = (12356), (1235), (1236) = (1267), (1237), (1256), (1258), (1268), (1357), (1368), (12357). In each case the laws indicated, together with the associative laws, are independent postulates for the corresponding family. The family of lattices is (1234) = (12356) (cf. Bull. Amer. Math. Soc. Abstract 61-4-537, I). Jordan's postulates (Arch. Math. vol. 2 (1949) pp. 56–59) for the family (3456) are not independent, since (3456) = (345) = (356). (Research supported in part by the University of New Zealand Research Fund.) (Received March 5, 1956.)

559*t*. Marvin Marcus: *Singular values of a product of matrices.*

Let (a) be a k-tuple of numbers and $E_r(a)$ be the rth elementary symmetric function of the (a). Let A, B, AB be n-square complex matrices with singular values $\alpha_i \geq \alpha_{i+1}$, $\omega_i \geq \omega_{i+1}$, $\lambda_i \geq \lambda_{i+1}$ respectively. Results: (i) *For* $1 \leq r \leq k \leq n$, $E_r(\lambda_1^2, \cdots, \lambda_k^2) \leq \prod_{j=1}^{r} \alpha_j^2 E_r(\omega_1^2, \cdots, \omega_k^2)$; $E_r(\lambda_n^2, \cdots, \lambda_{n-k+1}^2) \geq \prod_{j=1}^{r} \alpha_{n-j+1}^2 E_r(\omega_n^2, \cdots, \omega_{n-k+1}^2)$. (ii) *Let* $X = AB$, $\sigma \geq 0$, $\delta \geq 0$, $\sigma + \delta = 1$. *Let* λ_i^2, $\lambda_i^2 \geq \lambda_{i+1}^2$ *be the eigenvalues of the convex sum* $\sigma X^*X + \delta XX^*$. *Then for* $1 \leq r \leq k \leq n$, $E_r(\lambda_n^2, \cdots, \lambda_{n-k+1}^2) \geq (\prod_{j=1}^{r} \alpha_{n-j+1}^{\sigma} \omega_{n-j+1}^{\delta}) \cdot E_r^{\sigma}(\omega_n^2, \cdots, \omega_{n-k+1}^2) E_r^{\delta}(\alpha_n^2, \cdots, \alpha_{n-k+1}^2)$. These results extend a recent theorem of A. Horn (*On the singular values of a product of completely continuous operators*, Proc. Nat. Acad Sci. U.S.A. vol. 36 (1950) p. 374). (Received February 20, 1956.)

560. Marvin Marcus: *An extension of the results of K. Fan and H. Weyl concerning singular values.*

We use the notation of the above abstract (singular values of a product of matrices). Theorem. *Let A be an arbitrary complex n-square matrix with eigenvalues λ_i, $|\lambda_i| \geqq |\lambda_{i+1}|$. Let σ, $\delta \geqq 0$, $\sigma+\delta=1$ and denote the eigenvalues of the convex sum $\sigma A^*A+\delta AA^*$* by α_i^2, $\alpha_i^2 \geqq \alpha_{i+1}^2$. *If $1 \leqq r \leqq k \leqq n$ and $s \geqq 1$ then $E_r(\alpha_1^{2s}, \cdots, \alpha_k^{2s}) \geqq E_r(|\lambda_1|^{2s}, \cdots, |\lambda_k|^{2s})$*. H. Weyl obtained this result for $\delta=0$, $k=n$, $s>0$. (*Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) p. 408). K. Fan obtained this result for $r=k$. (*On a theorem of Weyl concerning the eigenvalues of linear transformations* II, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) p. 31). (Received February 20, 1956.)

561. G. C. Preston: *Rings of continuous p-adic valued functions.* Preliminary Report.

Let Q_p represent the field of p-adic numbers, X a compact Hausdorff space, and $C(X, Q_p)$ the ring of continuous functions from X to Q_p. If $f \in C(X, Q_p)$, $|f| = \max_{x \in X} |f(x)|$ is a norm on $C(X, Q_p)$ which takes on the values p^n only. Under this definition, $C(X, Q_p)$ is complete and is a topological Q_p-module (the mapping $q \times f$ onto qf is a bilinear, continuous mapping of $Q_p \times C(X, Q_p)$ onto $C(X, Q_p)$). If there are sufficiently many continuous mappings of X into Q_p, ($x \neq y$ implies there exists f such that $f(x) \neq f(y)$) then all maximal ideals of $C(X, Q_p)$ are "fixed" that is, a maximal ideal is of the form $\{f : f(x)=0\}$ for some $x \in X$. If X is completely regular with respect to Q_p, $X \cong \mathfrak{M}$ where $\mathfrak{M}$ is the space of all maximal ideals of $C(X, Q_p)$ with the standard topology. (See Gelfand and Silov, Mat. Sbornik, vol. 9 (1941) pp. 25–39). For the most part, proofs carry over from the theory of normed rings. Two questions which arise in this connection are the following: (1) Which Q_p-modules are representable as such function rings? (2) Which compact Hausdorff spaces are completely regular with respect to Q_p? (Received February 27, 1956.)

562. Herman Rubin and R. L. San Soucie (p): *On the center of certain rings.*

Jacobson has proved (Trans. Amer. Math. Soc. vol. 57 (1945) pp. 228–245), using Schur's lemma and the concept of transformation center of a ring, that the center of a simple (not necessarily associative) ring R ($R^2 \neq 0$) is either zero or a field. We offer a simple, short, direct proof of this result, using only elementary properties of a ring. Call a (not necessarily associative) ring R *primitive* in case R contains a maximal regular right ideal M containing no two sided ideals of R except the zero ideal. If R is associative, it is known that the center of R is a commutative integral domain. We give a very short proof of this fact in the general case, and believe the result to be new. Indeed, we prove more: a nonzero center element of R is not a zero divisor in R. Finally, we give a method for constructing an associative primitive ring whose center is an arbitrarily chosen commutative integral domain. (Received March 7, 1956.)

563. J. H. Walter: *A complete set of invariants for representation modules of algebras.*

Let $\mathfrak{A}$ be an algebra with an identity, which is of finite dimension over a field K. We assume that K is algebraically closed. Let V be a representation module for $\mathfrak{A}$ and let $V=V_1 \supseteq V_2 \supseteq \cdots \supseteq V_k=0$ be its upper Loewy series. Associated with the factors $V_r^* = V_r/V_{r+1}$ are the vector spaces $\mathfrak{A}$-Hom (F_i, V_r^*) pf $\mathfrak{A}$-homomorphisms of F_i into V_r where the F_i are the distinct irreducible representation modules of $\mathfrak{A}$. Let $\mathfrak{A}$-Hom$^*(F_i, V_r)$ denote the space dual to $\mathfrak{A}$-Hom (F_i, V_r). Then it is shown that equivalence classes of bilinear functions from $\mathfrak{A}$-Hom $(F_i, V_r)\times\mathfrak{A}$-Hom (F_j, V_s) to the first chain groups $C^1(\mathfrak{A}, \text{Hom}\,(F_j, F_i))$, introduced by Hochschild, provide a complete set of invariants for V. (Received April 25, 1956.)

564t. Morgan Ward: *Quadratic residues of Lucas functions.*

Let (U): $0, 1, P, \cdots$ and (V): $2, P, P^2-2Q, \cdots$ be the Lucas functions associated with the polynomial x^2-Px+Q, P and Q integers. Let p be a prime with positive rank of apparition r in (U). The distribution of quadratic residues of (U) and (V) modulo p is studied, and in particular, conditions for the appearance of at least one nonresidue in (U) or (V) are found. The problem is trivial for r large (i.e. $p/r<9$). For r small (i.e. $r<9$) definitive results are obtained when P^2-4Q is a residue of p. The criteria for appearance of a nonresidue depend on the quadratic, cubic or biquadratic character modulo p of simple arithmetic functions of r. (Received March 15, 1956.)

ANALYSIS

565. E. J. Akutowicz: *The spectral resolution of Watson transforms.*

If ϕ belongs to $L^2(0, \infty)$, the (complex) Watson transform $W\phi$ of ϕ is defined by $\int_0^t W\phi(u)du=\int_0^\infty (k(tu)/u)\phi(u)du$, where $k(u)/u$ belongs to $L^2(0, \infty)$, and $\int_0^\infty (k(t'u)\bar{k}(t''u)/u^2)du=\min\,(t', t'')$. The operator W is then unitary in $L^2(0, \infty)$. G. Doetsch (*Die Eigenwerte und Eigenfunktionen von Integraltransformationen*, Math. Ann. vol. 117 (1939)) has obtained several results pertaining to the point spectrum of W. In this note the full spectral resolution of W is obtained in explicit form. This is done by taking account of the behavior of the Poisson integral (with respect to the unit circle) arising from the operator $Q_\lambda=W(W-\lambda I)^{-1}$, $|\lambda|\neq 1$. (Received March 5, 1956.)

566. F. H. Brownell: *Asymptotic distribution of eigenvalues for the lower part of the Schrodinger operator spectrum.*

From Weyl's law for the distribution of eigenvalues of the n-dimensional membrane problem it is to be expected that $N(\lambda)=\sum_{\lambda_j\leq\lambda}1$ satisfies $N(\lambda)=[(2\pi^{1/2})^n\Gamma(2^{-1}n+1)]^{-1}\cdot\int_{V(x)\leq\lambda}[\lambda-V(x)]^{n/2}d\mu_n(x)$ asymptotically as $N(\lambda)\to+\infty$ with increasing λ for the Schrodinger eigenvalue equation $-\nabla^2u_j(x)+V(x)u_j(x)=\lambda_ju_j(x)$ over $x\in R_n$. euclidean n-space with n-dimensional Lebesgue measure μ_n, with $u_j\in L_2(R_n)$. Recently this has been proved only for $n=1$ by Titchmarsh subject to rather stringent smoothness conditions on $V(x)$, and in addition the condition that $V(x)$ be monotone increasing in $|x|$ with limit $+\infty$ as $|x|\to+\infty$. We use Titchmarsh's results to obtain the stated asymptotic relation when $n\geq 3$ for radial $V(x)=q(|x|)$ under weaker conditions on $q(r)$ over $0<r<+\infty$. It should be possible to prove this relation for general $V(x)$ without this radial restriction by using Courant's domain comparison methods and known perturbation results. (Received March 1, 1956.)

567t. E. A. Coddington: *On self-adjoint ordinary differential operators.*

Let $L=p_0D^n+p_1D^{n-1}+\cdots+p_n$, $D=d/dx$, where the p_k are complex-valued functions on an open real interval $a<x<b$ of class C^{n-k}, $p_0(x)\neq 0$. Assume that L coincides with its Lagrange adjoint. Let S be the operator in $L^2(a, b)$ with domain C_0^∞, the set of functions of class C^∞ vanishing outside some closed bounded subinterval of (a, b), and put $Su=Lu$, $u\in C_0^\infty$. Assuming that S has a self-adjoint extension H,

it is shown how to define self-adjoint boundary value problems on closed bounded subintervals δ of (a, b) in such a way as to produce, in the limit $\delta \to (a, b)$, the spectral matrix associated with the expansion theorem and Parseval equality for H. This matrix is related to Green's function for $H-l$, $Il \neq 0$, which is shown to be a limit of Green's functions for the problems defined on the subintervals of (a, b). Finally it is shown how the spectral family of projections $E(\lambda)$ associated with H can be represented in terms of the spectral matrix and solutions of $Lu = \lambda u$. This representation implies, among other things, the expansion theorem and Parseval equality. (Received February 27, 1956.)

568*t*. E. A. Coddington: *On maximal symmetric ordinary differential operators.*

The results of the preceding paper are extended to all maximal symmetric extensions H of S, which need not be self-adjoint. The generalized resolvent and generalized resolution of the identity (in the sense of M. A. Naimark) replace the resolvent and spectral family of projections, respectively. The domains of H and its adjoint H^* are characterized by certain boundary conditions. (Received February 27, 1956.)

569*t*. E. A. Coddington: *Some Banach algebras.*

Let $\{\phi_k\}$ be a complete orthonormal set in an L^2 space of complex-valued functions, and suppose, in addition, that $\phi_k \in L^2 \cap L^1$ for all k. If the complex constants ν_k satisfy certain conditions, and multiplication is defined by $f * g = \sum (f, \phi_k)(g, \phi_k)\nu_k\phi_k$ for $f, g \in L^1$, then it can be shown that L^1 with this multiplication is a commutative Banach algebra A which is regular. These algebras contain simple examples of regular Banach algebras which are not self-adjoint. Under additional assumptions on the ϕ_k the algebra A is semi-simple, and satisfies the hypothesis for the generalized Wiener theorem. (Received February 27, 1956.)

570*t*. I. I. Hirschman, Jr.: *A characterization of minimal projections which commute with left translations.* Preliminary Report.

Let G be a compact topological group with elements x, u, etc. A bounded linear transformation P of $L^1(G)$ into itself such that $P^2 = P$, $P^* \subset P$ is called a projection. It is said to be minimal if $\|P\| = 1$. Let H be a closed subgroup of G and $dm(u)$ Haar measure on H normalized by the condition $m(H) = 1$. Let $g(u)$ be a rank 1 character on H. Then the formula $Pf(x) = \int_H f(xu^{-1})g(u)dm(u)$ defines a minimal projection commuting with left translations. Conversely every minimal projection commuting with left translations is of this form. (Received March 5, 1956.)

571*t*. I. I. Hirschman, Jr.: *On the Young-Hausdorff-Riesz theorem for compact groups.* Preliminary Report.

Let G be a compact group and let, for each $\alpha \in A$, $[g_{ij}^{\alpha}(x)]_{i,j=1,2,\ldots,r(\alpha)}$ be an irreducible unitary representation of G of rank $r(\alpha)$, A being a complete set of inequivalent representations. Let $c_{ij}^{\alpha}(f) = \int_G f(x)\overline{g_{ij}^{\alpha}(x)}dx$ where dx is Haar measure on G normalized by the condition that G has measure 1. The analogue of the Young-Hausdorff-Riesz theorem asserts that $[\sum_{\alpha \in A} r(\alpha)^{1-q/2}(\sum_{i,j=1}^{r(\alpha)} |C_{ij}^{\alpha}(f)|^2)^{q/2}]^{1/q} \leq \|f\|_p$, $(1 \leq p \leq 2,\ p^{-1}+q^{-1}=1)$. Let p, $1<p<2$, be fixed. A function $f(x)$ such that equality obtains in the above equality is said to be a maximal function in $L^p(G)$. Let H be an open and closed normal subgroup of G, $g(x)$ a rank one character on H, u an element of G, and c a complex constant. Then $f(x)=c\chi_{uH}(x)g(u^{-1}x)$, where $\chi_{uH}(x)$ is the characteristic function of uH, is maximal in $L^p(G)$. Conversely every maximal function is of this form. The corresponding result for locally compact Abelian groups was obtained by E. Hewitt and the author in the Amer. J. Math. vol. 76 (1954) pp. 839–851. (Received March 5, 1956.)

572. W. Jurkat and A. Peyerimhoff (p): *On Fourier constants of functions in Lipschitz classes.*

A real-valued function $f(t)$ with period 2π is said to belong to Lip (k, p) $(0<k\leqq 1, p\geqq 1)$, if $f\in L_p$ and if $\int_0^{2\pi}|f(t+h)-f(t)|^p dt=0(h^{kp})$ for $h\to 0$. The notation $f(t)\sim\sum_{-\infty}^{+\infty} c_n e^{int}$ will be used and it will be assumed that $0<k<1$, $1<p\leqq 2$. The following theorem is proved: If $f\in$ Lip (k, p), then $(*)\sum_1^\infty n\alpha_n|n^k c_n|^q<\infty$ for all sequences $0<\alpha_n\searrow$, $\sum\alpha_n<\infty$ $(1/p+1/q=1)$, and all possible relations $\sum_1^\infty n^\delta|\sum_{\nu\leqq n} A_{n-\nu}^{\beta-1}\nu c_\nu|^\gamma<\infty\,(\beta-1, \gamma>0)$ are consequences of $(*)$. For $p=2$ $(*)$ is equivalent with $f\in$ Lip $(k, 2)$. This theorem contains results by Bernstein, H. C. Chow, Hardy-Littlewood, Sunouchi, Szász, Zygmund, concerning the convergence of $\sum|c_n|^\beta$, $\sum n^\delta|c_n|$ or the absolute Cesàro summability of $\sum c_n$. If $f\in$ Lip (k, p), and if c_n^* denotes the sequence $|c_0|, |c_1|, |c_{-1}|, \cdots$ arranged in decreasing order, then $c_n^*=0(n^{-k-1/q})$. If, on the other hand, $a_n\searrow$ and $a_n=0(n^{-k-1/q})$, then $\sum a_n\cos n\phi$ and $\sum a_n\sin n\phi$ are Fourier series of functions of Lip (k, p). This leads to the following theorem: A necessary and sufficient condition that a sequence $\{c_n\}$ $(\bar{c}_n=c_{-n})$ represents for some variation of the arguments and arrangement of the c_n's a sequence of Fourier constants of a function $f\in$ Lip (k, p) is $c_n^*=0(n^{-k-1/q})$. There are similar theorems in the case $p>2$. (Received March 7, 1956.)

573. V. A. Kramer: *Asymptotic perturbation series.*

Let $H(t)$ be an operator defined on a Hilbert space by the series $H(t)=H_0+tH_1+t^2H_2+\cdots$. Suppose that $\mathfrak{D}$, the domain of $H(t)$, is dense, that H_0 is self-adjoint and bounded below by 1, that H_i for $i>0$ is symmetric and bounded below by 0, and that $t\geqq 0$. Then $H(t)$ has a Friedrich's extension, $\bar{\bar{H}}(t)$. Suppose that H_0 is the Friedrich's extension of its own contraction to $\mathfrak{D}$. If λ_0 is an isolated eigenvalue of H_0 and is less than the smallest accumulation point of the spectrum of H_0, then $\bar{\bar{H}}(t)$ has an eigenvalue $\lambda(t)$ which converges to λ_0. The method of identification of coefficients leads to formal series (possibly finite) for this eigenvalue and the associated eigenvectors; $\lambda(t)=\lambda_0+t\lambda_1+t^2\lambda_2+\cdots$, $\phi(t)=\phi_0+t\phi_1+t^2\phi_2+\cdots$. A set of conditions sufficient for these expansions to be asymptotically valid is derived by appeal to an earlier study of the series expansion of $\bar{\bar{H}}(t)^{-1}\phi$. [See Bull. Amer. Math. Soc. Abstract 62-1-54] The main result is that if ϕ_i is defined for $0\leqq i\leqq M$ and ϕ_i satisfies inverse conditions for order $M-i$, then the eigenvector expansion is valid to order M. An auxiliary theorem then allows us to expand the eigenvalue to order $2M$. Corresponding results are obtained for the degenerate case. Kato [J. Fac. Sci. Imp. Univ. Tokyo, Sect. I, vol. 5 (1951) pp. 145–225] has solved the problem for operators with only two terms using a different method. (Received March 7, 1956.)

574. L. B. Rall: *The extension of the quadratic formula to Banach spaces.*

Given a Banach space X, a symmetric bilinear operator B in X, a linear operator L in X, and a point v in X, the expression $Qu=Buu+Lu+v=0$ is called a quadratic equation in X. If $2Bs+L=0$, or for $(2Bs+L)^{-1}$ existing for some s in X, the quadratic equation is transformed into $Bxx+y=0$ or $B'xx+Ix+y=0$, respectively, where $u=x+z$. The set $F(B)$ is composed of all elements w of X such that $(Bw)^2-B(Bww)=0$. The equation $Bxx+y=0$ has a solution x in $F(B)$ if and only if $(-By)^{1/2}$ exists, $Bx=(-By)^{1/2}$, and $(-By)^{1/2}x+y=0$. The equation $B'xx+Ix+y=0$ has a solution x in $F(B')$ if and only if $(I-4B'y)^{1/2}$ exists, $B'x=-I/2+(I-4B'y)^{1/2}/2$, and $[I+(I-4B'y)^{1/2}]2x+y=0$. (Research sponsored by the U. S. Army Office of Ordnance Research, under the terms of Contract DA 04-200-ORD-177.) (Received March 7, 1956.)

575. R. M. Redheffer: *On pairs of harmonic functions.*

A characteristic value k for a region R is a constant such that the problem $\Delta u+ku=0$, $u=0$ at the boundary, has a nontrivial solution. (a) Let u and v be harmonic functions which map a region R into a region R^* contained in a circle of radius r. Then $\inf\,(u_x^2+u_y^2+v_x^2+v_y^2)\leq kr^2/2$ where k is the smallest characteristic value for R. (b) Let u and v be harmonic in a bounded region R, and let M and m be constant. Suppose $u^2+v^2\geq M$ at the boundary and $u^2+v^2=m$ at an interior point, P. If d is the maximum distance from P to the boundary of R, then $\sup\,(u_x^2+u_y^2+v_x^2+v_y^2)\geq 2(M-m)/d^2$, $(x, y)\in R$. That the latter result is sometimes optimum is shown by (c): Let u and v be twice-differentiable in a circle R of radius r. Suppose $u^2+v^2\geq M$ at the boundary and $u^2+v^2=0$ at the center. Then the value of $\inf_{(u,v)}\sup_{(x,y)}(u_x^2+u_y^2+v_x^2+v_y^2)$ is M/r^2 when u and v are unrestricted; it is $2M/r^2$ when u and v are harmonic; and it is $2M/r^2$ when u and v are conjugate harmonic. (d) Let $u(x, y)$ and $v(x, y)$ be harmonic in a region R, and let their "conjugateness" be measured by $\alpha+v_x^2=v_y^2-u_x^2-u_y^2$, $\beta=2(u_xv_x+u_yv_y)$ (so that $\alpha=\beta=0$ when $u+iv$ is analytic). Let M and m be constants with $M\geq m>0$. Suppose $u^2+v^2\geq M$ on the boundary, $u^2+v^2=m$, a minimum, at an interior point, P. Then $\sup\,(\alpha^2+\beta^2)^{1/2}\geq(2m/D^2)\log\,(M/m)$ where D is the maximum diameter of R. The sharper inequality $\sup\,[(\alpha^2+\beta^2)^{1/2}/(u^2+v^2)]\geq(2/d^2)\log M/m$ is also true, where d is the maximum distance from P to the boundary. (Received January 27, 1956.)

576t. R. M. Redheffer: *On entire solutions of nonlinear equations.*

An entire solution $u(x, y)$ of a partial differential equation (*) is a function which is twice-differentiable and satisfies * at every point of the (x, y)-plane. (a) Let $h(x)$ be continuous for $-\infty<x<\infty$ and define $H(t)=\int_0^t h(x)dx$. Then the partial differential equation $\Delta u=(u_x^2+u_y^2)h(u)$ has a nonconstant entire solution $u(x, y)$ if and only if the integrals $\int_0^\infty e^{-H(t)}dt$, $\int_0^\infty e^{-H(-t)}dt$ are both divergent. (b) Let $r(t)$ be a continuous positive function for $t\geq t_0$ and define $R(t)=\int_{t_0}^t r(\xi)d\xi$. If $\int_{t_0}[R(t)]^{-1/2}dt<\infty$ then every entire solution of $\Delta u\geq r(u)$ is constant. (c) Let $u(x, y)$ be an entire solution of $\Delta u\geq r(u)+s(u)(u_x^2+u_y^2)$ such that $\sup u=m<\infty$. If there exists an $\epsilon>0$ such that $r(u)+as(u)>0$ for $0<a<\epsilon$, $m-u<\epsilon$, then u is constant. (d) Suppose there is a sequence of compact sets R_n such that $(x^2+y^2\leq n^2)\subset R_n$ and such that $\sup u/\log\,(x^2+y^2)\to 0$ as $n\to\infty$, the sup being taken on the boundary of R_n. If Δu is an entire solution of $\Delta u\geq 0$, then u is constant. (Under the additional hypothesis that $r(t)$ is twice-differentiable, increasing, and positively convex, the result (b) is due to Wittich. See Math. Zeit., 1944). (Received January 27, 1956.)

577t. R. M. Redheffer: *On the growth of solutions of nonlinear equations.*

Write $\rho^2=x^2+y^2$, $\rho>0$, and define $M(\rho)=\sup u(x, y)$ on $x^2+y^2=\rho^2$. Then (a)

For $\rho \geqq \rho_0$ and $u > u_0$ let u be twice differentiable and satisfy $\Delta u \geqq r(u) + s(u)(u_x^2 + u_y^2)$, $\limsup M(\rho)/\rho \leqq 0$, $\rho \to \infty$. Suppose there is a fixed positive sequence $a_i \to 0$ such that $\liminf u_i[r(u_i)/a_i + a_i s(u_i)] > 0$ whenever $u_i \to \infty$. Then u is bounded from above for $\rho \geqq \rho_0$. (b) For $\rho \geqq \rho_0$ and $u > u_0$ let u be twice differentiable and satisfy $\Delta u \geqq r(u) + s(u)(u_x^2 + u_y^2)$, $\limsup M(\rho)/\rho^2 \leqq 0$, $\rho \to \infty$. Suppose there is a constant A such that, for $u > u_1$, we have $2(r(u)s(u))^{1/2} \geqq 1 + 1/\log (Au)$ when $u^2 s(u) \geqq r(u)$, $r(u)/u + us(u) \geqq 1 + 1/\log (Au)$ when $u^2 s(u) \leqq r(u)$, $0 \leqq r(u)$. Then u is bounded from above, for $\rho \geqq \rho_0$. (c) Let v be a twice differentiable function satisfying $\Delta v \leqq 0$ exterior to a bounded region R, and let P be a point exterior to R and exterior to the circle $\rho = 1$. Then there is a simple polygonal arc, extending from P to infinity, upon which $v/\log \rho$ is bounded from above. (d) For $u > u_0$, $v > v_0$ and $\rho > \rho_0$ let $u(x, y)$ and $v(x, y) = v(\rho)$ be twice differentiable in an unbounded region R and satisfy $v\Delta u \geqq u\Delta v$, $\sup u \leqq A$ at the boundary, $\lim v(\rho) = \infty$ as $\rho \to \infty$. If $\limsup M(\rho)/v(\rho) = m > 0$ as $p \to \infty$, then $\liminf M(\rho)/v(\rho) = m$. Here $M(\rho) = \sup u$, $(x^2 + y^2 = \rho^2$, $(x, y) \in R)$. The hypothesis may be replaced by $v\Delta u - u\Delta v \geqq f(x, y, u, u_x, u_y, v_x, v_y)$ with mild restrictions on f. (Received January 27, 1956.)

578. W. M. Stone: *On a class of Bessel-like functions*. Preliminary Report.

Studies of the behavior of radar receiver systems have led to the 4th order differential equation $(D^2 + m^2)(D^2 + m^{*2})y = y \exp(-2x)$, where m and m^* are complex conjugates. Taking $H_m(\lambda_n x) = y_m(\lambda_n x) y_{m^*}(\lambda_n) - y_{m^*}(\lambda_n x) y_m(\lambda_n)$, $H_m'(\lambda_n) = 0$, it follows that the set $\{x^{1/2} H_m(\lambda_n x)\}$ is complete $L^p(0, 1)$, $1 \leqq p < \infty$, Re $(m) > -1/2$. Many formulas and theorems, particularly the orthogonality property, are closely analogous to those of the Bessel functions. In particular, infinite sums and infinite products involving the λ_n are readily evaluated in closed form. (Received March 6, 1956.)

APPLIED MATHEMATICS

579. W. Karush (p) and A. Vazsonyi: *A minimization problem involving a finite sequence and its first differences.*

The problem is to minimize $F(x)$ where $F(x) = \sum_{i=0}^{n} f_i(x_i) + \sum_{i=1}^{n} [\alpha \max \{x_i - x_{i-1}, 0\} - \beta \min \{x_i - x_{i-1}, 0\}]$, $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta > 0$ are constants, and $x = (x_0, x_1, \cdots, x_n)$ is subject to $a_i \leqq x_i \leqq b_i$, $i = 0, 1, \cdots, n$. It is assumed that $f_i(u)$ is continuous on the interval $I_i : a_i \leqq u \leqq b_i$ and *ditonic* on I_i, i.e., for some $m_i \in I_i$, f_i is nonincreasing on $a_i \leqq u \leqq m_i$ and is nondecreasing on $m_i \leqq u \leqq b_i$. An algorithm is given for obtaining a minimizing vector x. The problem is first reduced to finding the minimum of a function $G(t) = \sum_{i=0}^{r} g_i(t_i)$, $g_i(u)$ continuous, $\{t_i\}$ an arbitrary alternating sequence, i.e., $t_0 \leqq t_1$, $t_1 \geqq t_2$, etc. (or $t_0 \geqq t_1$, $t_1 \leqq t_2$, etc.). The reduced index r equals the number of montonic sections of $\{m_i\}$. The latter problem is then solved by iteration. For the cases when the f_i are (1) piecewise analytic, (2) piecewise linear, (3) convex, the method of solution variously simplifies. (Received March 12, 1956.)

580. W. R. Wasow: *On the accuracy of finite difference approximations to Dirichlet problems involving piecewise analytic boundary values.*

The known appraisals of the truncation error $v = U - u$ involved in replacing the solution u of a boundary value problem for a partial differential equation by the solution U of a difference equation are either restricted to very special—usually rectangular—domains, or they assume the boundedness of three or more partial derivatives of u in the whole domain R under consideration. This is unsatisfactory, since most computational problems deal with boundaries or boundary values that have corners. As a first step to overcome this difficulty it is proved that for the simplest finite difference approximation to Dirichlet's problem for Laplace's equation the order of the error is not affected by jumps in the first derivative of the boundary function. More precisely, it is assumed that the boundary C is a simple closed analytic curve, and that the boundary function $f(s)$ is continuous and piecewise analytic. The approximating function U satisfies $U(x+h, y)+U(x-h, y)+U(x, y+h)+U(x, y-h) = 4U(x, y)$, if all five grid points are in $R+C$. At grid points near the boundary U is assigned the prescribed boundary value at a nearby point of C. Then the truncation error is $O(h)$ except possibly near the boundary. The main tool is an asymptotic analysis of Green's function for the discrete problem. (Received January 3, 1956.)

GEOMETRY

581. Iacopo Barsotti: *Abelian varieties over fields of positive characteristic.*

Let A be an n-dimensional nonsingular abelian variety over the algebraically closed field k of characteristic $p\neq 0$; let $\mathfrak{A}$, $\mathfrak{G}$ be the groups of the $(n-1)$-dimensional cycles on A which are, respectively, arithmetically equivalent $(\equiv)$ to 0, or algebraically equivalent $(\approx)$ to 0; let p^f be the number of the distinct $P\in A$ such that P^p is the identity of A; let Γ be the k-module of the factor sets of A into a 1-dimensional vector variety over k, and let Γ_0 be the k-module of the elements of Γ which are associated to the identity of Γ. The following results are proved: (1) A has torsion 1, that is, $\mathfrak{A}=\mathfrak{G}$; (2) Γ/Γ_0 is a k-module of order n; (3) the k-module of the differentials of the first kind on A has a k-independent basis of the form $\{x_1^{-1}dx_1, \cdots, x_f^{-1}dx_f, dx_{f+1}, \cdots, dx_{f+e}, \omega_{f+e+1}, \cdots, \omega_n\}$, where $x_i\in k(A)$, $e>0$ if $f<n$, and each ω_i is not a linear combination, with coefficients in k, of differentials of the first kind of the types dx or $x^{-1}dx$. Results (1) and (2) are generalizations of similar results, previously found by the author, valid in the case of characteristic zero. By means of examples, it is proved that either relation, $e+f<n$, or $e+f=n$, may actually occur. (Received March 23, 1956.)

582. V. L. Klee, Jr.: *The structure of semispaces.*

When L is a real linear space and p a point of L, a "*semispace*" at p in L is a maximal convex subset of $L\sim\{p\}$. This notion was recently introduced by Hammer, who showed that the class of all semispaces in L is the smallest intersection-base for the class of all convex proper subsets of L. The present note studies the structure of semispaces, and of sets which are the intersection of countably many semispaces. It is proved that each semispace in L is generated in a simple way by an ordered set of linear functionals, and L is of countable dimension if and only if each semispace can be generated by the set of "coördinate" functionals associated with a basis. For a convex subset C of a space of countable dimension, the following assertions are equivalent: (i) C is the intersection of a countable family of semispaces; (ii) whenever a family of convex sets has intersection C, so has some countable subfamily. (Received March 2, 1956.)

583. C. O. Oakley (p) and R. J. Wisner: *Flexagons.*

A flexagon (the paper model of which is a network of equilateral triangles folded into a flexible hexagon, discovered by R. Feynman, A. H. Stone, B. Tuckerman, and J. W. Tukey) is defined abstractly. Let m be a positive integer. For $m=1$, the single permutation of the integer 1 is called a pat of degree 1. For $m=r+s>1$, the permutation $A_rA_{r-1}\cdots A_2A_1b_sb_{s-1}\cdots b_2b_1$, where $A_i=a_i+s$, of the integers 1 to m is called a pat of degree m if the permutation $a_1a_2\cdots a_r$ of the integers 1 to r and the permutation $b_1b_2\cdots b_s$ of the integers 1 to s are pats of degree r and s, respectively. A flexagon is an ordered pair of pats of degrees p and q, and the order N of the flexagon is $N=p+q$ (this being the number of faces obtainable under a pinching operation). A basis of pats arises from which all flexagons are generated and models can be made from a single straight strip of equilateral triangles by folding and gluing. Equivalence of flexagons is defined and the number of equivalence classes of flexagons of order N is found to be $V_{N-1}/N+\{V_K/2\}+\{2V_L/3\}$, where $V_M=\binom{2M-1}{M}/(2M-1)$ and $N=2K=3L$ with inclusion of a braced term when and only when K or L is integrable. (Received March 6, 1956.)

Logic and Foundations

584. C. C. Chang and Anne C. Davis (p): *Arithmetical classes closed under direct product.*

For terminology see Tarski, Indag. Math. vol. 16 (1954) pp. 572–578. As is known, (i) If a class $K\in AC_\delta$ and if K is characterized by conditional sentences, then K is closed under direct product (Horn, J. Symbolic Logic vol. 16 (1951) p. 17, Th. 4). Tarski asked whether the converse of (i) holds (cf. Bing, Proc. Amer. Math. Soc. vol. 6 (1955) p. 836). By I and II below, it is evident that the converse fails. Given a class of relational systems, $\mathfrak{A}_i=\langle A_i, R_i\rangle$, where $i\in I$ and each R_i is a binary relation, a new system $P'_{i\in I}\mathfrak{A}_i=\langle A, R\rangle$ is defined: elements of A are equivalence classes of the functions f on I such that $f(i)\in A_i$ for each $i\in I$, f and g belonging to the same class whenever $f(i)=g(i)$ for all but finitely many indices i; two equivalence classes with representatives f and g are in the relation R iff $\langle f(i), g(i)\rangle\in R_i$ for almost all $i\in I$. Clearly, the operation P' is definable on elements of an arbitrary similarity class. I. If $K\in AC_\delta$ and K is characterized by conditional sentences, then $P'_{i\in I}\mathfrak{A}_i\in K$ whenever I is infinite and $\mathfrak{A}_i\in K$ for each $i\in I$. II. The class of atomistic Boolean algebras, although closed under direct product, is never closed under P'. (Received March 5, 1956.)

585t. William Craig: *Further remarks on finite axiomatizability using additional predicates.*

For notation see the two abstracts of Vaught's papers below. Theorem 1: If T is f.a.$^+$, then T is strongly f.a.$^+$. The proof uses Theorem (I) and Lemma (II) of Vaught's abstracts. Corollary: If $S\in \mathrm{PC}$ then also $\phi(S)\in \mathrm{PC}$ where $\phi(S)=\bigcap(X \mid S\subseteq X\in \mathrm{AC})$. Theorem 2: If T is axiomatizable and if some $(ER_1)\cdots(ER_k)F$, F being a first-order sentence, has the same finite models as T and no infinite models other than those of T, then T is f.a.$^+$. The proof depends on (K) only. Corollary: If T is not f.a.$^+$, then the set T_f of those first-order sentences which are true of every finite model of T is not f.a.$^+$. Theorem 3: If T has only a finite number of nonisomorphic finite models and if T is axiomatizable, then T is f.a.$^+$. This is an easy extension of (K). (Received April 23, 1956.)

586. Solomon Feferman: *Degrees of unsolvability correlated to theories with standard formalization.*

For terminology and notation see Tarski, Mostowski, Robinson, *Undecidable Theories*, and Post, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316. For any $\alpha \subseteq \omega$, the following conditions define a consistent theory $\mathrm{T}[\alpha]$ with standard formalization: Symbols of $\mathrm{T}[\alpha]$ are the unary predicate symbol A and the distinct individual constant symbols $c_0, \cdots, c_n, \cdots$. Axioms of $\mathrm{T}[\alpha]$ are the sentences: (i) $(c_n \neq c_m)$ for $n \neq m$, (ii) $\bigwedge x_0 \cdots \bigwedge x_{n-1} \bigvee x_n \bigvee x_{n+1} [A(x_n) \wedge \sim A(x_{n+1}) \wedge \prod_{i<n} (x_i \neq x_n \wedge x_i \neq x_{n+1})]$ for any n, and (iii) $A(c_n)$ for any $n \in \alpha$. Let α^* be the set of Gödel numbers of those sentences ϕ provable from the axioms of $\mathrm{T}[\alpha]$ in first order predicate calculus with identity. If α is recursively enumerable, so is α^*. The following theorems hold for arbitrary $\alpha \subseteq \omega$. (I) *α is 1-1 reducible to α^**. (II) *α^* is reducible to α by unbounded truth tables*. (III) *α and α^* have the same degree of unsolvability*. (IV) *$\mathrm{T}[\omega]$ is consistent and decidable; hence $\mathrm{T}[\alpha]$ is not essentially undecidable*. Theorem (II) is obtained by an application of the method of eliminating quantifiers (see, e.g., Tarski, *A Decision Method for Algebra and Geometry*, 2d ed. (1951) p. 50, Note 11). Using the announced solution of Post's problem by Friedberg (Bull. Amer. Math. Soc. Abstract 62-3-362) Theorem (III) provides a negative answer to a conjecture of Myhill, Zeitschr. f. math. Logik u. Grundlagen d. Math. vol. 1 (1955) p. 98. (Received February 28, 1956.)

587. L. A. Henkin: *A nondenumerable formal system.* Preliminary report.

Let F be a first-order functional calculus, Γ a set of its individual constants. F is Γ-*complete* if, for every formula $B(x)$ containing x as sole free variable, $\vdash B(\alpha)$ for all $\alpha \in \Gamma$ implies $\vdash (Vx)B(x)$. M is a Γ-*model* of F if all formal theorems of F are true of M and each element of M is denoted by a constant of Γ. F is Γ-*saturated* if every sentence of F which is true of all Γ-models is provable in F. We have previously shown that for *denumerable* systems F, Γ-completeness and Γ-saturation are equivalent. We now have an example of a *nondenumerable* system F (and a set Γ having the same cardinality as the set of all formulas of F) which is Γ-complete, has many Γ-models, but is not Γ-saturated. A formal notion (*strong* Γ-completeness) is developed, which is equivalent to the semantical concept of Γ-saturation for all consistent systems. (Received March 5, 1956.)

588. R. L. Vaught: *Finite axiomatizability using additional predicates.* I.

Consider theories T, formalized in classical first order logic with identity, having finitely many predicates. T is called *finitely axiomatizable using additional predicates* (*f a.*$^+$) if T has an extension T' such that (i) T' has finitely many additional predicates, (ii) T' is finitely axiomatizable, and (iii) a sentence of T is valid in T' iff it is valid in T. If, moreover, all models of T are obtainable from models of T' by dropping the additional relations, T is called *strongly f.a.*$^+$. (Thus in Tarski's terminology (Indag. Math. vol. 16 (1954) p. 584), T is strongly f.a.$^+$ iff the AC_δ of models of T is a PC.) Kleene's work (Memoirs of the Amer. Math. Soc., no. 10, 1952) established: (K) *If all models of T are infinite, then T is axiomatizable* (*= recursively axiomatizable*) *iff T is f.a.*$^+$. By the device of using "finite set theory with individuals" where Kleene used "number theory"—a possibility noted by Tarski—we obtain the stronger: Theorem (I). *If all models of T are infinite, then T is axiomatizable iff T is strongly f.a.*$^+$. A corollary is an affirmative answer to question (iii) of Łoś (Coll. Math. vol. 3 (1954) p. 62) for the case of axiomatizable theories, having only infinite models. (Received March 5, 1956.)

589*t*. R. L. Vaught: *Finite axiomatizability using additional predicates*. II.

For notation see preceding abstract. Lemma (II). *If T' is any extension of T satisfying* (i), (ii), *and* (iii), *then all finite models of T are obtainable from models of T' by dropping the additional relations*. Theorem (III). *If T is f.a.$^+$, and if T^*, the extension of T obtained by adding axioms excluding all finite models, is decidable* (*in particular, if T is $\aleph_\alpha$-categorical*) *then T is decidable*. (III) slightly generalizes a result of Henkin (Indag. Math. vol. 17 (1955) p. 326). From a recent result of Tarski (J. Symbolic Logic vol. 19 (1954) p. 158) follows: Theorem (IV). *In* (I) *and in Kleene's result on logic without identity, one additional binary predicate suffices*. Using (II) and an idea of Ehrenfeucht, one sees that the problem of characterizing those theories which are f.a.$^+$, when restricted to theories having only the identity predicate, virtually coincides with an (unsolved) problem proposed by Scholz (J. Symbolic Logic vol. 17 (1952) p. 160). Some partial results (unpublished) of Mostowski on the Scholz problem may, therefore, be translated, yielding, in particular, the fact that *the hypothesis of* (*K*) *cannot be omitted*. (Received March 5, 1956.)

STATISTICS AND PROBABILITY

590. E. G. Kimme: *On the convergence of sequences of stochastic processes*.

Let $\{x_n(t), 0 \leq t \leq 1\}$, $n = 1, 2, \cdots$ be a sequence of separable stochastic processes with independent increments such that $p\{x_n(0) = 0\} = 1$. Let $\{x(t), 0 \leq t \leq 1\}$ be a separable stochastic process with independent increments, no fixed points of discontinuity, and such that $p\{x(t) = 0\} = 1$. Two modes of convergence of $\{x_n(t), 0 \leq t \leq 1\}$ to $\{x(t), 0 \leq t \leq 1\}$ are defined. The stronger of these, called uniform convergence in distribution, suffices for the convergence in distribution of $F[x_n(\cdot)]$ to $F[x(\cdot)]$, for certain classes of functionals, among which is $F[f] = \sup_{0 \leq t \leq 1} f(t)$. It is shown that the usual ordinary convergence in distribution of the sequence of stochastic processes under discussion implies uniform convergence in distribution thereof when the processes all have stationary increments as well. These results are used to obtain an extension of Gnedenko's necessary and sufficient condition for the weak convergence of sums of independent random variables to the case where $x_n(t) = \sum_{j=1}^{k_n t} x_{nj}$, $\{x_{nj}, 1 \leq j \leq k_n\}$, $n = 1, 2, \cdots$, being a sequence of finite sequences of infinitesimal independent random variables ("independent within rows"). This generalizes some results of Donsker, Kac and Erdös, Udagawa, and others. (Received March 22, 1956.)

591. R. B. Leipnik: *Statistics of irreversible processes*.

A stochastic model is set up for irreversible processes of transport and collision of particles in the presence of radiation and external parameters. Time-independent distributions are found under a variety of conditions, generalizing results of Moyal. Time-dependent distributions are found under more restrictive conditions, and questions of ergodicity, convergence to equilibrium, and recurrence times are answered. This is related to the work of Kac on the Ehrenfest model. Applications are made to the theory of imperfect gases and liquids, following ideas of H. S. Green. (Received March 5, 1956.)

592t. A. T. Bharucha-Reid: *On the fundamental theorem concerning branching processes.*

Let $F(s) = \sum_{n=0}^{\infty} p_n s^n$, $|s| < 1$, be the generating function of the probabilities p_n associated with the discrete branching process $\{X_n, n \geq 0\}$. The fundamental theorem can be stated as follows: Let $p_0 > 0$, and let $\omega = \Pr [X_n = 0]$ for some n. Then $\omega = 1$ if $\sum xp_x \leq 1$ while, if $\sum xp_x > 1$ ω, is given by the smallest non-negative root of the functional equation $F(s) = s$. Various authors have given proofs based on Markov chain theory, fixed point methods, etc. In this note a theorem due to Kantorovitch (Acta Math. vol. 71 (1939) pp. 63–97) is used to establish the existence of a solution $\omega \in [0, 1]$, and to show that ω can be obtained by the method of successive approximations. (Received March 7, 1956.)

TOPOLOGY

593. E. A. Michael: *Some new characterizations of paracompactness.*

Call a collection $\mathfrak{A}$ of subsets of a topological space *closure-preserving* if, for every subcollection $\mathfrak{B} \subset \mathfrak{A}$, the union of the closures is the closure of the union. (Any locally finite collection is closure-preserving, but not conversely.) It is shown that, both in the usual definition of a paracompact space, and in the characterizations obtained by the author in [Proc. Amer. Math. Soc. vol. 4 (1953) pp. 831–838], "locally finite" can be replaced by "closure-preserving." This implies that the image of a paracompact space, under a closed, continuous mapping, must be paracompact, thus settling the author's Research Problem [Bull. Amer. Math. Soc. Research Problem 61-6-29]. (Received February 16, 1956.)

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BOOK REVIEWS

Transfinite Zahlen. By H. Bachmann. (Ergebnisse der Mathematik und ihrer Grenzgebiete, New Series, vol. 1.) Berlin, Springer, 1955. 7+204 pp. 29.80 DM.

This first volume in a new series of the *Ergebnisse* continues the admirable tradition of the original series, of presenting a very valuable up-to-date guide to the results, problems, and literature of a special field—in this case the theory of transfinite numbers: ordinal numbers and powers, but not general order types. The author develops the theory from the beginning, in a clear and rigorous fashion, and presents proofs of most of the theorems cited, so that the book could well serve as the basis for an intensive course in transfinite numbers. The treatment is founded on the Zermelo-Fraenkel axiom system but is expressed in the language of naïve set theory, and although axiomatic questions are considered, the emphasis is definitely on the arithmetical aspects of the theory. It is shown how far it is possible to get in various branches of the subject before applying the axiom of choice to derive further results, and the use of this axiom is always expressly indicated.

The first chapter opens with a brief discussion of naïve set theory, the foundations problem, and the various schools of mathematics. The axioms are then introduced, as well as the fundamental notions of equivalence, similarity, well-ordering, transfinite induction, and transfinite number.

The next two chapters are devoted to the theory of ordinal numbers, mostly without the use of the axiom of choice. One finds here, in addition to standard material, a simple treatment of principal numbers, as well as recent results concerning normal functions, regressive functions, sequences of continuous functions, rarified classes of ordinal numbers, and permutations of sequences of ordinal numbers.

The fourth chapter is concerned with the arithmetic of powers without the aid of the axiom of choice, and then in the fifth chapter this axiom is used to develop the arithmetic of cardinal numbers. Much of what is known concerning alephs and beths (transfinite powers of alephs) is to be found here. The generalized continuum hypothesis, some consequences of it, and some propositions equivalent to it are also discussed.

Chapter six deals with the well-ordering and the cardinal number of the continuum considered as an order type, and with the second number class and the problem of distinguished sequences. A few of the point-set-theoretical consequences of the continuum hypothesis are noted. There is a brief remark on the formal representation of ordinal numbers, and a short section on alternatives to the axiom of choice.

The work closes with a chapter on inaccessible numbers.

In addition to an index, the book is supplied with a useful bibliography arranged according to groups of sections in the text.

Springer's printing is excellent, as usual.

F. BAGEMIHL

Experimental design, theory and application. By W. T. Federer. New York, Macmillan, 1955. 19+544+45 pp. $11.00.

This book is addressed exclusively to the experimenter and practical statistician and presents a thorough comprehensive discussion and description of all major types of designs and of the methods for their analysis. Many of the designs described are here for the first time incorporated in a book. The underlying mathematical models are not discussed, but references to the pertinent literature are given. Although the book is not addressed to mathematicians or mathematical statisticians, it will be useful for this group as a reference work.

H. B. MANN

Mathematical theory of elasticity. 2d ed. By I. S. Sokolnikoff. New York, McGraw-Hill, 1956. 11+476 pp. $9.50.

This book constitutes a welcome contribution to the field. It is well written, and is extensively documented, particularly in so far as work in Russia is concerned. This book fills a need which has been apparent for quite some time.

There are seven chapters. Chapters 1–4 and Chapter 7 contain the material which appeared in the first edition, except for minor modifications.

Chapter 5 is new. It deals with the two-dimensional problems of plane strain and of generalized plane stress, which are of course identical mathematically. The method of attack which is usually associated with the name N. I. Muskhelishvili is treated in considerable detail. It will be recalled that in this method the problem is reduced to the determination of two functions of a complex variable, which functions are determined by conformal mapping, together with either solution in series or the solution of certain integrodifferential equations. This material appeared in the author's Brown University

Notes, which were widely circulated but which have been unavailable for quite some time.

Chapter 6, which deals with three-dimensional problems, is also new. The basic approach involves the expression of the components of displacement in terms of four arbitrary harmonic functions. Treated here are cases of concentrated loading, the problem of Boussinesq, the equilibrium of the sphere, thermoelastic problems, vibration problems and others.

G. E. HAY

La géométrie des groupes classiques. By Jean Dieudonné. Berlin, Springer, 1955. 7+115 pp. 19.60 DM.

This book gives an excellent survey of recent work on classical groups, simplifying and unifying the results of many authors. No attempt is made to cover all of the voluminous literature on classical groups; the author deals with only that portion of the subject which can be handled by the methods of linear algebra. By thus restricting his scope, he is able to include proofs of most of the results described, thereby making the book more self-contained than most Ergebnisse tracts.

While the book is written on an advanced level, it presupposes only some familiarity with linear algebra. However, a reader with a minimum background will have to work hard to master this book, which cannot be skimmed lightly. By use of a highly-condensed method of presentation, and omission of many routine details of proofs, the author has succeeded in packing a large amount of information into relatively few pages. The average reader will want to keep pencil and paper handy, in order to work through most of the proofs. There were several places where this reviewer would have been grateful for a few extra lines of exposition.

Chapter I (Collineations and correlations, pp. 1–35). By a *collineation* of an n-dimensional vector space E over a skew-field K is meant a one-to-one semi-linear map of E onto itself. The group $\Gamma L_n(K)$ of all such collineations contains the group $GL_n(K)$ of linear one-to-one maps of E onto itself. The "projective" groups are defined as groups modulo their subgroups of homothetic maps ($x \to xa$, $a \in K$). The beginning sections take up the concepts of dilatations and transvections (these are collineations leaving a hyperplane pointwise fixed), involutions and semi-involutions (these are collineations u for which $u^2(x) = x$ or xc ($c \in K$), respectively), and their centralizers in $P\Gamma L_n(K)$, the group of projective collineations.

By a *correlation* is meant a one-to-one semi-linear map ϕ of E onto its dual E^*, relative to an anti-automorphism J of the skew-field K. A non-degenerate form f is introduced on $E\times E$ by means of $f(x, y) = \langle\phi(x), y\rangle$ (inner product), and it is assumed that f is *reflexive* (that is, $f(y, x)=0$ if and only if $f(x, y)=0$). The fundamental notions of hermitian and skew-hermitian forms, orthogonal complements, isotropic spaces, symplectic and orthogonal bases, and equivalence of forms follow next.

Call a collineation $u\in\Gamma L_n(K)$, with automorphism σ, a *unitary semi-similitude* relative to a form f if $f(u(x), u(y)) = r_u(f(x, y))^\sigma$, where $r_u\in K$. The group $\Gamma U_n(K, f)$ of all such u contains the subgroup of unitary transformations $U_n(K, f)$ consisting of those u for which $r_u=1$ and σ is the identity. For hermitian forms, it is shown that it suffices to consider only *trace* forms (forms f for which $f(x, x) = a+a^J$, $a=a(x)\in K$). Chevalley's proof of Witt's theorem is given for such forms.

There is next a discussion of the dilatations (called *quasi-symmetries* in this case) and transvections in $\Gamma U_n(K, f)$, and the problem of centralizers of semi-involutions in $P\Gamma U_n(K, f)$ is considered. (These sections are rather difficult reading.) The chapter concludes with a section on projective permutability of correlations, and one on orthogonal groups over fields of characteristic 2.

Chapter II (Structure of the classical groups, pp. 36–72). (The following results are established except for certain special cases, which are described in the book.) Let $SL_n(K)$ be the subgroup of $GL_n(K)$ generated by transvections. It is shown that $SL_n(K)$ is the commutator subgroup of $GL_n(K)$, and the structure of their quotient is determined. A proof of the simplicity of $PSL_n(K)$ is given. The analogous problems are then considered for the unitary and orthogonal groups. The author shows that $U_n(K, f)$ is generated by quasi-symmetries, and the structures of $U_n(K, f)$, $T_n(K, f)$ (subgroup of U_n generated by unitary transvections, assuming that f has positive index), and U_n/T_n are determined. The reader is reminded very briefly of what Clifford algebras are, and then these are used to derive results on the orthogonal group $O_n(K, f)$ and its commutator subgroup $\Omega_n(K, f)$. A new proof of the simplicity of $P\Omega_n(K, f)$ is given.

Chapter III (Geometric characterization of the classical groups, pp. 72–85). Let $G_r(E)$ be the Grasman space consisting of all $(r+1)$-dimensional subspaces of E. Let E' be a vector space over K', with $[E':K'] = [E:K]$; then any one-to-one semi-linear map u of E onto E' induces a map of $G_r(E)$ onto $G_r(E')$. The problem of characterizing such induced maps is solved here for $r=0$ by the fundamental theorem of projective geometry, and for $r>0$ by use of the concept of "adjacent" elements of the Grassman space. The same type of problem is also considered for $u\in U_n(K,f)$. The chapter concludes with a brief discussion of other characterizations of the classical groups.

Chapter IV (Automorphisms and isomorphisms of the classical groups, pp. 85–108). The automorphisms of $GL_n(K)$, $SL_n(K)$, $Sp_{2m}(K)$ (symplectic group), $U_n(K,f)$, and the corresponding projective groups are determined. The methods used are quite elegant and non-computational. Many details are omitted, but adequate references are given to cover these gaps. In each case, the structure of the proof is clearly described. There is also a discussion of the possible isomorphisms between the various classical groups.

Following these chapters is a table of notations used in the book, a table comparing these notations with those of Dickson (*Linear Groups*) and van der Waerden (*Gruppen von Linearen Transformationen*), an index of principal theorems and definitions, and a useful bibliography which brings up to date that given in the above-mentioned book of van der Waerden. The book is well-printed, although the style of using solid paragraphs of exposition with little displayed material is a bit trying.

The author has asked that the following corrections be pointed out. P. 11, line -9: $(f(y\xi, x))^J$. P. 13, lines -15 to -13, replace by the sentences: Un sous-espace V de E qui n'est pas isotrope est caractérisé par le propriété que le sous-espace orthogonal V^0 est supplémentaire de V; mais un tel sous-espace peut contenir des droites isotropes. On dit que V est *anisotrope* s'il ne contient pas de droites isotropes; un tel espace est évidemment non isotrope. P. 23, line -11: $U_\nu(K, f_1)$. P. 26, line 3: $vu=uv\cdot a$. P. 33, line 8: $2p+d+1\leqq i \leqq n \cdots \sum_{i=1}^{n}$. P. 36, middle of part (a): $B_{ij}(1)B_{ji}(-1)B_{ij}(1)$. P. 36, last sentence in part (a), and p. 37, lines 21–23: these are correct only when K is commutative. P. 37, lines -8 to -5, replace by the sentence: si $A=(a_{ij})$ et si $a_{i1}\neq 0$, en retranchant des lignes de A d'indice $\neq i$ des multiples à gauche de la i-ème ligne, on obtient une matrice B dont la première colonne n'a que a_{i1} comme term $\neq 0$; on pose alors $\det(A)=\phi((-1)^{i+1}a_{i1})$ det (B_{i1}), en désignant par B_{i1} la matrice obtenue en supprimant dans B la première colonne et la i-ème ligne. P. 72, line -14: $\cdots u$ de E sur $E' \cdots$. P. 72, line -7: $\cdots$ par $\phi \cdots$. P. 107, line -16: read "$m=n=1$" for "$m=n=2$." P. 111, insert before line -4: ABE, M.: [1] Projective transformation groups over non-commutative fields, Sijo-Sugaku-

Danwakai, 240 (1942). P. 114, line 13: read "J. Reine Angew. Math., 196" for "Math. Zeitschrift."

To summarize: in the reviewer's opinion, this is an important and well-written book which should help to stimulate research on the classical groups. The book not only gives a thorough exposition of the present state of the subject, but is also an excellent introduction to the modern techniques basic to further work in this field.

IRVING REINER

Relativity: The special theory. By J. L. Synge. New York, Interscience. 1956. 7+415+20 pp. $10.50.

In the preface of this book Synge states, "The basic idea of this book is to present the essentials of relativity from the Minkowskian point of view, that is, in terms of the geometry of space-time." This reviewer agrees that an exposition of the special theory of relativity based on such an idea is sorely needed and the author's "Ambition . . . to make space-time a real workshop for physicists, and not a museum visited occasionally with a feeling of awe" is laudable.

On the whole this is a well-written discussion of the following topics in special relativity: Kinematics, mechanics of single particles and systems of particles, mechanics of a continuum and electromagnetic theory. These topics are covered with varying degrees of thoroughness, completeness and quality of exposition.

The first chapter discusses the relationship between the metric of space-time and physical measurements, the latter being described in terms of ideal experiments. The author's intention is to lay a foundation strong enough to support both the special and general theories. This intention is fulfilled in a lively thought-provoking way.

The next four chapters are devoted to the geometry of flat space-time (Minkowski space), the group of this space (the Lorentz group), and the explanation of the classical experiments which were first satisfactorily accounted for by the Einstein special theory of relativity. The discussion of these topics is clarified greatly by using space-time diagrams in an effective manner.

Chapter IV which deals with the proper homogeneous Lorentz transformations which do not interchange the past and the future contains as one of its main theorems the incorrect statement: "Any finite Lorentz transformation (of the restricted class defined (above)) is equivalent to a 4-screw." A 4-screw is defined as "a rotation in a time-like 2-flat π, followed (or preceded by . . .) a rotation in a space-like 2-flat π^*, the 2-flats π and π^* being orthogonal to one another." The matrix representing a 4-screw may be taken to be

$$\begin{Vmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh x & \sinh x \\ 0 & 0 & \sinh x & \cosh x \end{Vmatrix}.$$

In this case the 2-flat π is the x^3, x^4 plane and the 2-flat π^* is the x^1, x^2 plane.

The parabolic Lorentz transformation which in real coordinates $(x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ct)$ has the form

$$L = \begin{Vmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1/2 & 1/2 \\ 1 & 0 & -1/2 & 3/2 \end{Vmatrix}$$

furnishes a counterexample to the theorem quoted. For, its only proper values are $+1$ and its elementary divisors are not simple. In fact if P, U and V are the vectors given by

$$P^i = \delta^i_3 + \delta^i_4, \qquad V^i = \delta^i_2, \quad \text{and} \quad U^i = \delta^i_1$$

then

$$\begin{aligned} LP &= P, \\ LV &= V, \\ LU &= U + P. \end{aligned}$$

The only null-vector left invariant by L is P. The three-space tangent to the light cone and spanned by the vectors P, U and V is left invariant under the transformation L and the two-space spanned by P and V is left pointwise invariant.

The author's discussion does not properly take into account the possibility of the two real null-vectors left invariant by the 4-screw coinciding. The theory of such parabolic Lorentz transformations can be given quite readily in terms of the two component theory of spinors which the author discusses too briefly for the reviewer's taste.

In Chapters VI and VII which deal with the mechanics of a particle and the mechanics of a discrete system the laws of conservation of energy and momentum, that is, the conservation of the momentum four vector, are treated in detail for a variety of problems including problems involving the interaction of material particles and photons. The notion of angular momentum is introduced and discussed by means of an antisymmetric second order tensor.

Chapter VIII deals with the mechanics of a continuum, the latter being considered as a limit of a discrete system described by a distribution function which in turn specifies the number of particles of various types being considered. Properties of the distribution functions are not considered; instead attention is directed to various average quantities. In the first part of the chapter density, pressure and temperature of a gas are defined and discussed. In the second part of the chapter the stress energy tensor of a continuum of particles interacting only via collisions is derived. For the case of the perfect fluid the time-like proper value of this tensor is said to be density and the three coincident space-like ones are said to be pressure. The relation between these definitions and those given at the first part of the chapter is not discussed. In particular it is not mentioned that the time-like proper value is not in general the proper density of proper mass as defined in the earlier sections but is energy density of the continuum as measured by an observer whose time axis is in the direction of the time-like proper vector λ_r associated with this proper value. In general the energy density associated with proper mass will be different from the energy density given by the time-like proper value. This difference has been called the internal energy density of the fluid described by the stress energy tensor and is the relativistic analogue of an important concept in classical hydrodynamics.

In discussing relativistic hydrodynamics the author deals only with four conservation laws, those contained in the statement that the divergence of the stress energy tensor vanishes. He does not remark on the fact that classical hydrodynamics of a single fluid involves five conservation laws: the conservation of energy, momentum and mass. The first four are treated relativistically by means of the stress energy tensor. The fifth is contained in the statement that the divergence of the vector, obtained by multiplying the proper density of proper mass by the mean velocity vector, vanishes.

The final two chapters of the book, Chapters IX and X are devoted to electromagnetic theory. In the first of these entitled the *Electromagnetic field in vacuo*, considerable space is devoted to the geometry of second order antisymmetric tensors and to that associated with the symmetric stress energy tensor associated with the antisymmetric tensor describing a Maxwell field. It is regrettable that the author does not point out that when the determinant of the stress energy tensor is nonvanishing, the coefficients of this tensor are proportional to those of a proper-Lorentz matrix of period two. Thus, the extensive geometrical discussions of Chapter IX have an important bearing on Chapter IV.

Chapter IX contains a discussion of solutions of the scalar wave equation which vanish at infinity and have no singularities. Solutions of the Maxwell field equations are generated from these. It is shown that the space integral of the stress energy tensor for these solutions can be made to be proportional to a time-like vector with a finite proportionality factor. It is suggested that these solutions correspond to particles. Discontinuous electromagnetic fields called electromagnetic shock waves are also discussed.

Chapter X deals with electromagnetic fields due to moving singularities and also with Maxwell's equations in moving matter. The author uses very effectively the four dimensional formulation and geometric properties of Minkowski space to give a concise and clear treatment of a variety of topics on the behavior of fields and charges.

It is hoped that the faults pointed out above will be corrected in some future edition of the book. If this is done the book will be a valuable addition to the literature for it has many virtues and the basic idea behind it is a very sound one.

A. H. TAUB

BRIEF MENTION

Convegno internazionale sulle equazioni lineari alle derivate parziali. Trieste, 25–28 Agosto 1954. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Rome, Cremonese, 1955. 13+233 pp. 3000 lire.

This volume contains, besides an introduction by Sansone, papers by Agmon, Bers and Nirenberg, Cimmino, Courant, Diaz, Fichera, Miranda, Pleijel, Synge, Tautz, Tricomi, and Weinstein.

Functional analysis. By F. Riesz and B. Sz.-Nagy. Trans. by L. F. Boron from the 2d French edition of *Leçons d'analyse fonctionelle.* New York, Ungar, 1955. 12+468 pp. $10.00.

For a review of the original edition, cf. this Bulletin, vol. 59, pp. 270–281.

Spheroidal wave functions. By J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little, and F. J. Corbató. New York, The Technology Press of M. I. T. and John Wiley and Sons, 1956. 13+613 pp. $12.50.

Tables of coefficients of the expansions of the radial and angular functions of the first kind in spherical Bessel functions and in associated Legendre functions, computed, tabulated, and printed automatically.

The mathematics of physics and chemistry. By H. Margenau and G. M. Murphy, 2d ed. Princeton, van Nostrand, 1956. 12+604 pp. $7.95.

For a review of the first edition, cf. this Bulletin, vol. 51, pp. 508–509.

The number system. By H. A. Thurston. New York, Interscience, 1956. 8+134 pp. $2.50.

This book is similar in purpose to Landau's *Grundlagen der Analysis.* It gives a detailed development of the real number system starting from the Peano axioms. It differs in the use of Cauchy sequences instead of cuts, in the explicit introduction of such algebraic notions as group, hemigroup, field and isomorphism, and in the inclusion of a great deal of informal explanation in addition to the formal development. It should prove useful in connection with a first rigorous course in analysis, in the same way as Landau's.

The meaning of relativity. By A. Einstein. 5th ed., including the relativistic theory of the non-symmetric field. Princeton, Princeton University Press, 1955. 169 pp. $3.75.

RESEARCH PROBLEMS

10. R. Bellman: *Probability theory.*

Consider the recurrence relation $x_{n+1}=ax_n-bx_n^2+z_n$, $x_0=c$, where a and b are given parameters and the z_k constitute a set of random variables drawn from a common distribution. Determine the asymptotic behavior of $E(x_n)$ under various assumptions concerning a, b and the distribution of the z_n. (Received February 27, 1956.)

11. R. Bellman: *Number theory.*

The relation $\sum_{k\leq N} 2^{\gamma(k)}\sim N\log N$ (an analogue of a classical device of S. Bernstein in probability theory), where $\gamma(k)$ represents the number of prime divisors of k, can be used to show that the number of numbers less than N which have more than $c\log\log N$ prime divisors is $O(N/(\log N)^{c\log 2})$ as $N\to\infty$. Can one substantially improve this estimate for large c? (Received February 27, 1956.)

12. A. D. Wallace: *A problem on minimax semi-groups.*

Let M_n be the set of all $n\times n$ matrices whose entries lie in the closed unit interval and define a multiplication in M_n by $(\alpha,\beta)_{ij}=\mathrm{Max}_k\,(\mathrm{Min}\,(\alpha_{ik},\beta_{kj}))$. Clearly M_n is a topological semi-group which is homeomorphic with the n^2-cell. If S is a topological semi-group which is homeomorphic with an n^2-cell is it possible to give an abstract description of S that will identify it with M_n? It would be useful to have such a description even at the expense of supplying S with a dual operation such as obviously exists in M_n. (Received April 27, 1956.)

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A PARTITION CALCULUS IN SET THEORY

P. ERDŐS AND R. RADO

1. **Introduction.** Dedekind's pigeon-hole principle, also known as the box argument or the chest of drawers argument (Schubfachprinzip) can be described, rather vaguely, as follows. *If sufficiently many objects are distributed over not too many classes, then at least one class contains many of these objects.* In 1930 F. P. Ramsey [12] discovered a remarkable extension of this principle which, in its simplest form, can be stated as follows. *Let S be the set of all positive integers and suppose that all unordered pairs of distinct elements of S are distributed over two classes. Then there exists an infinite subset A of S such that all pairs of elements of A belong to the same class.* As is well known, Dedekind's principle is the central step in many investigations. Similarly, Ramsey's theorem has proved itself a useful and versatile tool in mathematical arguments of most diverse character. The object of the present paper is to investigate a number of analogues and extensions of Ramsey's theorem. We shall replace the sets S and A by sets of a more general kind and the unordered pairs, as is the case already in the theorem proved by Ramsey, by systems of any fixed number r of elements of S. Instead of an unordered set S we consider an ordered set of a given order type, and we stipulate that the set A is to be of a prescribed order type. Instead of two classes we admit any finite or infinite number of classes. Further extension will be explained in §§2, 8 and 9.

The investigation centres round what we call *partition relations* connecting given cardinal numbers or order types and in each given case the problem arises of deciding whether a particular partition relation is true or false. It appears that a large number of seemingly unrelated arguments in set theory are, in fact, concerned with just such a problem. It might therefore be of interest to study such relations for their own sake and to build up a *partition calculus* which might serve as a new and unifying principle in set theory.

In some cases we have been able to find best possible partition relations, in one sense or another. In other cases the methods available to the authors do not seem to lead anywhere near the ultimate

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Part of this paper was material from an address delivered by P. Erdős under the title *Combinatorial problems in set theory* before the New York meeting of the Society on October 24, 1953, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors May 17, 1955.

truth. The actual description of results must be deferred until the notation and terminology have been given in detail. The most concrete results are perhaps those given in Theorems 25, 31, 39 and 43. Of the unsolved problems in this field we only mention the following question. *Is the relation* $\lambda\to(\omega_0 2, \omega_0 2)^2$ *true or false?* Here, λ denotes the order type of the linear continuum.

The classical, Cantorian, set theory will be employed throughout. In some arguments it will be advantageous to assume the continuum hypothesis $2^{\aleph_0}=\aleph_1$ or to make some even more general assumption. In every such case these assumptions will be stated explicitly.

The authors wish to thank the referee for many valuable suggestions and for having pointed out some inaccuracies.

2. **Notation and definitions.** Capital letters, except Δ, denote sets, small Greek letters, except possibly π, order types, briefly: *types*, and $k, l, m, n, \kappa, \lambda, \mu, \nu$ denote ordinal numbers (*ordinals*). The letters r, s denote non-negative integers, and a, b, d cardinal numbers (*cardinals*). No distinction will be made between finite ordinals and the corresponding finite cardinals. Union and intersection of A and B are $A+B$ and AB respectively, and $A\subset B$ denotes inclusion, in the wide sense. For any A and B, $A-B$ is the set of all $x\in A$ such that $x\notin B$. No confusion will arise from our using 0 to denote both zero and the empty set. If $p(x)$ is a proposition involving the general element x of a set A then $\{x:p(x)\}$ is the set of all $x\in A$ such that $p(x)$ is true.

η and λ are the types, under order by magnitude, of the set of all rational and of all real numbers respectively. λ will also be used freely as a variable ordinal in places where no confusion can arise. The relation $\alpha\leqq\beta$ means that every set, ordered according to β, contains a subset of type α, and $\alpha\not\leqq\beta$ is the negation of $\alpha\leqq\beta$. To every type α there belongs the *converse type* α^* obtained from α by replacing every order relation $x<y$ by the corresponding relation $x>y$. We put

$$[m, n] = \{\nu : m \leqq \nu < n\} \qquad \text{for } m \leqq n.$$

The symbol

$$\{x_0, x_1, \cdots\}_<$$

denotes the set $\{x_0, x_1, \cdots\}$ and, at the same time, expresses the fact that $x_0<x_1< \cdots$. Brackets $\{\ \}$ are only used in order to define sets by means of a list of their elements. For typographical convenience we write

$$\sum [x \in A] f(x)$$

instead of $\sum_{x\in A} f(x)$, and we proceed similarly in the case of products etc. or when the condition $x\in A$ is replaced by some other type of condition.

The cardinal of S is $|S|$, and the cardinal of α is $|\alpha|$. For every cardinal a, the symbol a^+ denotes the next larger cardinal. If $a=b^+$ for some b, then we put $a^-=b$, and if a is not of the form b^+, i.e. if a is zero or a limit cardinal, then we put $a^-=a$. Similarly, we put $k^-=l$, if $k=l+1$, and $k^-=k$, if $k=0$ or if k is a limit ordinal. If S is ordered by means of the order relation $x<y$, then the type of S is denoted by $\bar{S}_<$ and, if no confusion can arise, by $\bar{S}$. For any $a>1$ we denote by a' the least cardinal $|n|$ such that a can be represented in the form $\sum[\nu<n]a_\nu$ where $a_\nu<a$ for all $\nu<n$. This cardinal a', the *cofinality cardinal* belonging to a, is closely related to the cofinality ordinal $cf(\beta)$ of an ordinal β introduced by Tarski [17]. A *regular* cardinal is a cardinal a such that $a'=a$. The least ordinal of a given cardinal a is the *initial ordinal* belonging to a. Initial ordinals are the finite ordinals and the infinite ordinals ω_ν of cardinal $\aleph_\nu$. We put

$$[S]^a = \{X : X\subset S;\ |X| = a\}.$$

In particular, $[S]^a=0$ if $|S|<a$. The relation

$$A = \sum{}'[\nu<k]A_\nu = A_0 +' A_1 +' \cdots$$

means, by definition, that $A=\sum[\nu<k]A_\nu$ and, also,

$$A_\mu A_\nu = 0 \qquad (\mu<\nu<k).$$

Fundamental throughout this paper is the *partition relation*

$$a\to(b,d)^2$$

introduced in [6]. More generally, for any a, b_ν, k, r the relation

$$a\to(b_0,b_1,\cdots)^r_k \tag{1}$$

is said to hold if, and only if, the following statement is true. The cardinals b_ν are defined for $\nu<k$. Whenever

$$|S|=a;\qquad [S]^r=\sum[\nu<k]K_\nu,$$

then there are $B\subset S$; $\nu<k$ such that

$$|B|=b_\nu;\qquad [B]^r\subset K_\nu.$$

For $k<\omega_0$ we also write (1) in the form

$$a\to(b_0,b_1,\cdots,b_{k-1})^r,$$

and if k is arbitrary, and $b_\nu = b$ for all $\nu < k$, then we may write (1) in the form

$$a \to (b)^r_k.$$

We also introduce partition relations between types. By definition, the relation

$$\alpha \to (\beta_0, \beta_1, \cdots)^r_k \tag{2}$$

holds if, and only if, β_ν is defined for $\nu < k$ and if, whenever a set S is ordered and

$$\overline{S} = \alpha; \qquad [S]^r = \sum[\nu < k]K_\nu,$$

there are $B \subset S$; $\nu < k$ such that

$$\overline{B} = \beta_\nu; \qquad [B]^r \subset K_\nu.$$

If $k < \omega_0$, or if all β_ν are equal to each other, we use an alternative notation for (2) analogous to that relating to (1). The negation of (1), and similarly in the case of (2), is denoted by

$$a \nrightarrow (b_0, b_1, \cdots)^r_k.$$

We mention in passing that the gulf between (1) and (2) can be bridged by the introduction of more general partition relations referring to partial orders. These will, however, not be considered here.

If $a \geqq \aleph_0$ then, clearly, a' is the least cardinal $\aleph_n$ such that $a \nrightarrow (a)^1_{\omega_n}$. Also, $\aleph_m$ is regular if, and only if, $\aleph_m \to (\aleph_m)^1_{\omega_n}$ for all $n < m$. Finally, the relation $a \to (b_0^+, b_1^+, \cdots)^1_k$ is equivalent to $\sum[\nu < k]b_\nu < a$.

We now introduce some abbreviations. Let S be ordered. Then, for $x \in S$,

$$L(x) = \{y : \{y, x\}_< \subset S\}; \qquad R(x) = \{y : \{x, y\}_< \subset S\}.$$

If, in addition, $[S]^r = \sum[\nu < k]K_\nu$, then, for $B \subset S$; $\nu < k$,

$$F_\nu(B) = \{\overline{A} : A \subset B;\ [A]^r \subset K_\nu\},$$
$$[K_\nu] = F_\nu(S).$$

In the special case $r = 2$, we put, for $x \in S$; $\nu < k$, $L_\nu(x) = \{y : \{y, x\}_< \in K_\nu\}$; $R_\nu(x) = \{y : \{x, y\}_< \in K_\nu\}$; $U_\nu = L_\nu + R_\nu$. U_ν is independent of the order of S. If $0 \neq A \subset S$, and if $W(x)$ is any one of the functions L, R, L_ν, R_ν, U_ν, then we put

$$W(A) = \prod[x \in A]W(x).$$

Also, $W(0) = S$. If $n < \omega_0$, then we write $W(x_0, \cdots, x_{n-1})$ instead of

$W(\{x_0, \cdots, x_{n-1}\})$. It will always be clear from the context to which ordered set S and to which partition of $[S]^r$ these functions refer. We shall occasionally make use of the notation and the calculus of partitions (distributions) summarized in [5, p. 419]. The meaning of *canonical partition relations*

$$\alpha \to *(\beta)^r$$

and that of *polarised partition relations*

$$\begin{pmatrix} a_0 \\ \cdot \\ \cdot \\ \cdot \\ a_{t-1} \end{pmatrix} \to \begin{pmatrix} b_{00} & b_{01} & \cdots \\ \cdot & \cdot \cdot \cdot \cdot & \cdot \\ b_{t-1,0} & & \cdots \end{pmatrix}^{r_0, \cdots, r_{t-1}}_{k}$$

will be given in §§8 and 9 respectively. The relation $\alpha\to(\beta)^r_a$ will be defined in §4.

3. Previous results.

THEOREM 1. *If* $k<\omega_0$ *then* $\aleph_0\to(\aleph_0)^r_k$ [12, *Theorem* A].

THEOREM 2. *If* $k, n<\omega_0$, *then, for some* $f=f(k, n, r)<\aleph_0$,

$$f \to (n)^r_k$$

[12, *Theorem* B].

THEOREM 3. (i) *If* $a\geqq\aleph_0$, *then* $a\to(\aleph_0, a)^2$.
(ii) $\aleph_{\omega_0}\nrightarrow(\aleph_1, \aleph_{\omega_0})^2$.

(i) is proved in [2, 5.22]. This formula will be restated and proved as Theorem 44.

(ii) is in [3, p. 366] and will follow from Theorem 36 (iv).

THEOREM 4. (i) *If* $a\geqq\aleph_0$, *then* $(a^a)^+\to(a^+)^2_a$.
(ii) *If* $a\geqq\aleph_0$, *then* $a^a\nrightarrow(3)^2_a$.
(iii) *If* $2^{\aleph_n}=\aleph_{n+1}$, *then* $\aleph_{n+2}\to(\aleph_{n+1}, \aleph_{n+2})^2$.

(i) is given in [3] and will be deduced as a corollary of Theorem 39.

(ii) is in [3, p. 364], and (iii) is [3, Theorem II] and follows from Theorem 7(i).[1]

THEOREM 5. *If* $\phi\leqq\lambda$; $|\phi|>\aleph_0$, *then, for* $\alpha<\omega_0 2$; $\beta<\omega_0^2$; $\gamma<\omega_1$,
(i) $\phi\to(\omega_0, \gamma)^2$.
(ii) $\phi\to(\alpha, \beta)^2$.

---

[1] The partition relations occurring in (i) and (ii) are to be interpreted in the obvious way. Their formal definition is given in §4.

(i) is [5, Theorem 5], and (ii) is [5, Theorem 7]. Both results will follow from Theorem 31.

THEOREM 6. $\eta \rightarrow (\aleph_0, \eta)^2$.

This relation, a cross between (1) and (2), has, by definition, the following meaning. If $\bar{S} = \eta$; $[S]^2 = K_0 + K_1$, then there is $A \subset S$ such that either

$$|A| = \aleph_0; \ [A]^2 \subset K_0$$

or

$$\bar{A} = \eta; \ [A]^2 \subset K_1.$$

This result is [5, Theorem 4].

THEOREM 7. *If $a \geqq \aleph_0$, and if b is minimal such that $a^b > a$, then*
(i) $a^+ \rightarrow (b, a^+)^2$
(ii) $a^b \nrightarrow (b^+, a^+)^2$.

These results are contained in [6, p. 437]. (i) will follow from Theorem 34.[8]

THEOREM 8. *If $2^{\aleph_\nu} = \aleph_{\nu+1}$ for all ν, and if a is a regular limit number[9] then, for every $b < a$, $a \rightarrow (b, a)^2$.*

This result is [6, Lemma 3], and will follow from Theorem 34.

THEOREM 9. *If $\phi \leqq \lambda$; $|\phi| = |\lambda|$, then $\lambda \nrightarrow (\phi, \phi)^1$.*

This result is due to Sierpiński who kindly communicated it to one of us. It will follow from Theorem 29. Our proof of Theorem 29 uses some of Sierpiński's ideas.

THEOREM 10. *For any a, $a \nrightarrow (\aleph_0, \aleph_0)^{\aleph_0}$.*

This is in [5, p. 434]. The last result justifies our restriction to the case of finite "exponents" r.

4. Simple properties of partition relations.

THEOREM 11. *The two relations*

$$\text{(i) } \alpha \rightarrow (\beta_0, \beta_1, \cdots)^r_k \qquad \text{(ii) } \alpha^* \rightarrow (\beta_0^*, \beta_1^*, \cdots)^r_k$$

are equivalent.

---

[8] By methods similar to those used in [17] one can show that (i) $b \leqq a'$ for all $a > 1$, (ii) $b = a'$ for those $a > 1$ for which $d < a$ implies $2^d \leqq a$.

[9] It is not known if regular limit numbers $> \aleph_0$ exist or not. Cf. [13, p. 224].

PROOF. Let (i) hold; $\overline{S}_{<}=\alpha^*$; $[S]^r=\sum[\nu<k]K_\nu$. Then $\overline{S}_{>}=\alpha$. Hence, by hypothesis, there are $A\subset S$; $\nu<k$ such that $\overline{A}_{>}=\beta_\nu$; $[A]^r\subset K_\nu$. Then $\overline{A}_{<}=\beta_\nu^*$. This proves (ii), and the theorem follows by reasons of symmetry.

THEOREM 12. *Let* $\alpha\to(\beta_0,\beta_1,\cdots)^r_k$; $\alpha\leqq\alpha^{(1)}$; $k\geqq k^{(1)}$;

$$\beta_\nu\geqq\beta_\nu^{(1)} \qquad (\nu<k^{(1)}),$$
$$\beta_\nu\geqq r \qquad (k^{(1)}\leqq\nu<k).$$

Then

$$\alpha^{(1)}\to(\beta_0^{(1)},\beta_1^{(1)},\cdots)^r_{k^{(1)}}.$$

An analogous result holds when the types α,β_ν *are replaced by cardinals.*

PROOF. It suffices to consider the case of types. Let

$$\overline{S}^{(1)}=\alpha^{(1)};\qquad [S^{(1)}]^r=\sum[\nu<k^{(1)}]K_\nu^{(1)}.$$

Then there is $S\subset S^{(1)}$ such that $\overline{S}=\alpha$. Then

$$[S]^r=\sum[\nu<k]K_\nu,$$

where $K_\nu=K_\nu^{(1)}[S]^r$ for $\nu<k^{(1)}$, and $K_\nu=0$ otherwise. By hypothesis, there are $A\subset S$; $\nu<k$ such that

$$\overline{A}=\beta_\nu;\qquad [A]^r\subset K_\nu.$$

If $\nu\geqq k^{(1)}$, then $|A|=|\beta_\nu|\geqq r$; $0\neq[A]^r\subset K_\nu$ which is a contradiction. Hence $\nu<k^{(1)}$. There is $A^{(1)}\subset A$ such that $\overline{A}^{(1)}=\beta_\nu^{(1)}$. Then $[A^{(1)}]^r\subset[A]^r\subset K_\nu\subset K_\nu^{(1)}$, and the assertion follows.

THEOREM 13. *If* $\alpha\to(\beta_0,\beta_1,\cdots)^r_k$ *then*

$$|\alpha|\to(|\beta_0|,|\beta_1|,\cdots)^r_k.$$

PROOF. Let $|S|=|\alpha|$; $[S]^r=\sum[\nu<k]K_\nu$. We order S so that $\overline{S}=\alpha$. Then there are $A\subset S$; $\nu<k$ such that $\overline{A}=\beta_\nu$; $[A]^r\subset K_\nu$. Then $|A|=|\beta_\nu|$, and the theorem follows.

THEOREM 14. *If* β_ν *is an initial ordinal, for all* $\nu<k$, *then the two relations*

$$\mathfrak{m}\to(\beta_0,\beta_1,\cdots)^r_k, \tag{3}$$
$$|\mathfrak{m}|\to(|\beta_0|,|\beta_1|,\cdots)^r_k \tag{4}$$

are equivalent.

PROOF. By Theorem 13, (3) implies (4). Now suppose that (4) holds. Let $\overline{S}=m$; $[S]^r=\sum[\nu<k]K_\nu$. Then $|S|=|m|$, and hence, by (4), there are $A\subset S$; $\nu<k$ such that $|A|=|\beta_\nu|$; $[A]^r\subset K_\nu$. Then, as β_ν is an initial ordinal, $\overline{A}\geqq\beta_\nu$, and there is $B\subset A$ such that $\overline{B}=\beta_\nu$. This proves (3).

THEOREM 15. *If* $1+\alpha\rightarrow(1+\beta_0,\ 1+\beta_1,\ \cdots)_k^{1+r}$, *then*

$$\alpha\rightarrow(\beta_0,\ \beta_1,\ \cdots)_k^r.$$

In this proposition, $1+\alpha$ *and* $1+\beta_\nu$ *may be replaced by* $\alpha+1$ *and* $\beta_\nu+1$ *respectively. Also, the types* α, β_ν *may be replaced by cardinals.*

PROOF. Let $\overline{S}=\alpha$. Let x_0 be an object which is not an element of S, and put $S_0=S+\{x_0\}$. The order of S is extended to an order of S_0 by stipulating that $x_0\in L(S)$. Then $\overline{S}_0=1+\alpha$. Now let $[S]^r=\sum[\nu<k]K_\nu$. Then $[S_0]^{1+r}=\sum[\nu<k]K_{0\nu}$, where $K_{0\nu}=\{\{y_0,\ \cdots,\ y_r\}_<:\{y_1,\ \cdots,\ y_r\}\in K_\nu\}$. If $1+\alpha\rightarrow(1+\beta_0,\ 1+\beta_1,\ \cdots)_k^{1+r}$, then there are $A_0\subset S_0$; $\nu<k$ such that $\overline{A}_0=1+\beta_\nu$; $[A_0]^{1+r}\subset K_{0\nu}$. Then $A_0=\{y_0\}+A$; $y_0\in L(A)$;

$$\overline{A}=\beta_\nu;\qquad [A]^r\subset K_\nu.$$

This proves the first assertion.

Next, if $\alpha+1\rightarrow(\beta_0+1,\ \cdots)_k^{r+1}$, then, by Theorem 11 and the result just obtained, we conclude that

$$1+\alpha^*\rightarrow(1+\beta_0^*,\ 1+\beta_1^*,\ \cdots)_k^{1+r},$$

$$\alpha^*\rightarrow(\beta_0^*,\ \cdots)_k^r;\qquad \alpha\rightarrow(\beta_0,\ \cdots)_k^r.$$

Finally, let $1+a\rightarrow(1+b_0,\ 1+b_1,\ \cdots)_k^{1+r}$. Let α and β_ν be the initial ordinals belonging to a and b_ν respectively. Then, by Theorems 14 and 13, $1+\alpha\rightarrow(1+\beta_0,\ \cdots)_k^{1+r}$,

$$\alpha\rightarrow(\beta_0,\ \cdots)_k^r;\qquad a\rightarrow(b_0,\ \cdots)_k^r.$$

THEOREM 16. *If* $\alpha\rightarrow(\beta_0,\ \beta_1,\ \cdots)_{1+k}^r$; $\beta_0\rightarrow(\gamma_0,\ \gamma_1,\ \cdots)_l^r$, *then*

$$\alpha\rightarrow(\gamma_0,\ \gamma_1,\ \cdots,\ \beta_1,\ \beta_2,\ \cdots)_{l+k}^r.$$

In this proposition the types α, β_μ, γ_ν *may be replaced by cardinals.*

In formulating the last theorem we use an obvious extension of the symbol (2).

PROOF. We consider the case of types. Let $\overline{S}=\alpha$,

$$[S]^r=\sum[\lambda<l]K_{0\lambda}+\sum[0<\nu<1+k]K_\nu.$$

Put $K_0=\sum[\lambda<l]K_{0\lambda}$. Then, by hypothesis, there are $A\subset S$; $\nu<1+k$

such that $\bar{A}=\beta_\nu$; $[A]^r\subset K_\nu$. If $\nu>0$, then this is the desired conclusion. If $\nu=0$, then $\bar{A}=\beta_0$; $[A]^r\subset\sum[\lambda<l]K_{0\lambda}$ and so, by hypothesis, there are $B\subset A$; $\lambda<l$ such that $\bar{B}=\gamma_\lambda$; $[B]^r\subset K_{0\lambda}$ which, again, is a conclusion of the desired kind. This proves the theorem.

It is clear that, instead of replacing in the relation $\alpha\to(\beta_0, \beta_1, \cdots)^r_{1+k}$ a single type β_0 by a well-ordered system of types $\gamma_0, \gamma_1, \cdots$, we could have replaced simultaneously every type β_ν by a system of types and in this way obtained a more general form of the transitive property of the partition relation than that given in Theorem 16.

THEOREM 17. *If* $\alpha\to(\beta_0, \beta_1, \cdots)^r_k$; $\gamma_\lambda=\beta_{\rho_\lambda}$ $(\lambda<l)$, *where* $\lambda\to\rho_\lambda$ *is a one-one mapping of* $[0, l]$ *into* $[0, k]$ *such that* $\beta_\nu\geqq r$ *for* $\nu\in[0, k]-\{\rho_\lambda:\lambda<l\}$, *then*

$$\alpha\to(\gamma_0, \gamma_1, \cdots)^r_l.$$

In particular, the condition on the mapping $\lambda\to\rho_\lambda$ is satisfied whenever this mapping is on $[0, k]$.

The types α, β_ν may be replaced by cardinals.

PROOF. Let $N=\{\rho_\lambda:\lambda<l\}$, and let $\nu\to\sigma_\nu$ be the mapping of N on $[0, l]$ which is inverse to the given mapping $\lambda\to\rho_\lambda$. Now let $\bar{S}=\alpha$; $[S]^r=\sum[\lambda<l]K_\lambda$. Then

$$[S]^r=\sum[\nu\in N]K_{\sigma_\nu}+\sum[\nu\in[0, k]-N]0.$$

By hypothesis, there are $A\subset S$; $\nu<k$ such that $\bar{A}=\beta_\nu$ and either

(i) $\nu\in N;\quad [A]^r\subset K_{\sigma_\nu}$,

or

(ii) $\nu\notin N;\quad [A]^r=0.$

In case (ii), $r\nleqq\bar{A}=\beta_\nu$ which contradicts the hypothesis. Hence (i) holds, $\bar{A}=\gamma_\lambda$; $[A]^r\subset K_\lambda$, where $\lambda=\sigma_\nu<l$, and the result follows.

We note that the hypothesis relating to $\beta_\nu\geqq r$ cannot be omitted, as is seen from the pair of the obviously correct relations $4\to(1, 3, 3)^2$; $4\not\to(3, 3)^2$.

COROLLARY. *If* $\alpha\to(\beta)^r_k$; $|k|=|l|$, *then* $\alpha\to(\beta)^r_l$.

This shows that, as far as k is concerned, the truth of the relation $\alpha\to(\beta)^r_k$ depends only on $|k|$. We are therefore able to introduce the relation

$$\alpha\to(\beta)^r_d$$

which, by definition, holds if, and only if,

$$\alpha \rightarrow (\beta)_k^r$$

for some, and hence for all, k such that $|k| = d$. A similar remark applies to the relation

$$a \rightarrow (b)_d^r.$$

THEOREM 18. *Let* $k < \omega_0$; $\alpha \rightarrow (\beta_0, \beta_1, \cdots)_k^r$;

$$\bar{S} = \alpha; \qquad [S]^r = K_0 + \cdots + K_{k-1}.$$

Then there are sets $M, N \subset [0, k]$ *such that* $|M| + |N| > k$,

$$\beta_\mu \in [K_\nu] \qquad \textit{for } \mu \in M; \nu \in N.$$

In the special case $k = 2$ *we have either*

(i) $\beta_0 \in [K_0][K_1]$ *or* (ii) $\beta_1 \in [K_0][K_1]$

or

(iii) $\beta_0, \beta_1 \in [K_0]$ *or* (iv) $\beta_0, \beta_1 \in [K_1]$.

PROOF. Let

$$P_\nu = \{\mu : \mu < k; \beta_\mu \in [K_\nu]\},$$
$$Q_\nu = [0, k] - P_\nu \qquad (\nu < k).$$

We have to find a set $N \subset [0, k]$ such that $|\prod [\nu \in N] P_\nu| > k - |N|$ or, what is equivalent, $|\sum [\nu \in N] Q_\nu| < |N|$. If no such N exists, i.e. if $|\sum [\nu \in N] Q_\nu| \geqq |N|$ for all $N \subset [0, k]$, then, by a theorem of P. Hall [8], it is possible to choose numbers $\rho_\nu \in Q_\nu$ such that $\rho_\mu \neq \rho_\nu$ $(\mu < \nu < k)$. Then $\beta_{\rho_\nu} \notin [K_\nu]$ $(\nu < k)$. On the other hand, by Theorem 17 and the hypothesis, $\alpha \rightarrow (\beta_{\rho_0}, \beta_{\rho_1}, \cdots)_k^r$. This is the required contradiction.

THEOREM 19. *Let* $\alpha \rightarrow (\beta, \gamma)^2$, *and suppose that* m *is the initial ordinal belonging to* $|\alpha|$. *Then at least one of the following four statements holds.*[4]

(i) $\beta < \omega_0$ (ii) $\gamma < \omega_0$ (iii) $\beta, \gamma \leqq \alpha, m$ (iv) $\beta, \gamma \leqq \alpha, m^*$.

PROOF. Let S be a set ordered by means of the relation $x < y$ and also by means of the relation $x \ll y$, and let the corresponding order types of S be $\bar{S} = \alpha$; $\bar{S}_{<<} = m$. Then $[S]^2 = K_0 +' K_1$ where $K_0 = \{\{x, y\}_< : \{x, y\}_{<<} \subset S\}$. Then, by Theorem 18, we have at least one of the following four cases.

Case 1. $\beta \in [K_0]_<[K_1]_<$. Then there are sets $A, B \subset S$ such that $\bar{A}_< = \bar{A}_{<<} = \beta$; $\bar{B}_< = \bar{B}_{>>} = \beta$, and hence $\beta = \bar{A}_{<<} \leqq \bar{S}_{<<} = m$. Then β is an ordinal. If $\beta \geqq \omega_0$, then the contradiction $\omega_0^* \leqq \beta^* = \bar{B}_{<<} \leqq \bar{S}_{<<} = m$ follows. Hence $\beta < \omega_0$.

---

[4] (iii) means that $\beta \leqq \alpha$; $\beta \leqq m$; $\gamma \leqq \alpha$; $\gamma \leqq m$.

Case 2. $\gamma \in [K_0]_< [K_1]_<$. Then, by symmetry, $\gamma < \omega_0$.

Case 3. $\beta, \gamma \in [K_0]_<$. Then, for some sets $A, B \subset S$, $\bar{A}_< = \bar{A}_{<<} = \beta$; $\bar{B}_< = \bar{B}_{<<} = \gamma$, and $\beta, \gamma \leqq \alpha, m$.

Case 4. $\beta, \gamma \in [K_1]_<$. Then, similarly, $\bar{A}_< = \bar{A}_{>>} = \beta$; $\bar{B}_< = \bar{B}_{>>} = \gamma$; $\beta, \gamma \leqq \alpha, m^*$. This proves the theorem.

COROLLARY. *For every* α,

$$(r-2) + \alpha \not\rightarrow (\omega_0, (r-2) + \omega_0^*)^r \qquad (r \geqq 2). \tag{5}$$

For none of the relations (i)–(iv) of Theorem 19 holds if $\beta = \omega_0$; $\gamma = \omega_0^*$. Hence $\alpha \not\rightarrow (\omega_0, \omega_0^*)^2$, and Theorem 15 yields (5).

The method employed in the proof of Theorem 19, i.e. the definition of a partition of $[S]^2$ from two given orders of S, seems to have been first used by Sierpiński [15]. In that note Sierpiński proves $\aleph_1 \not\rightarrow (\aleph_1, \aleph_1)^2$. Cf. Theorem 30.

THEOREM 20. (i) *If* $\beta_0 \leqq \alpha$; $|\beta_0| < r$, *then* $\alpha \rightarrow (\beta_0, \beta_1, \cdots)^r_k$ *holds for any* $k, \beta_1, \beta_2, \cdots$.

(ii) *If* $\beta_\nu = r$ *for* $\nu < k$, *then the two relations*

$$\alpha \rightarrow (\beta_0, \beta_1, \cdots, \gamma_0, \gamma_1, \cdots)^r_{k+l}, \tag{6}$$

$$\alpha \rightarrow (\gamma_0, \gamma_1, \cdots)^r_l \tag{7}$$

are equivalent.

PROOF OF (i). If $\bar{S} = \alpha$; $[S]^r = \sum [\nu < k] K_\nu$, then there is $A \subset S$ such that $\bar{A} = \beta_0$. Then $[A]^r = 0 \subset K_0$.

PROOF OF (ii). By Theorem 12, (6) implies (7). Now suppose that (7) holds. Let $\bar{S} = \alpha$; $[S]^r = \sum [\nu < k+l] K_\nu$. If there is $\nu < k$ such that $K_\nu \neq 0$, then we can choose $A \in K_\nu$, and we shall have $A \subset S$; $\bar{A} = \beta_\nu$; $[A]^r \subset K_\nu$. If, on the other hand, $K_\nu = 0$ for all $\nu < k$, then $[S]^r = \sum [\lambda < l] K_{k+\lambda}$, and there exist, by (7), $B \subset S$; $\lambda < l$ such that $\bar{B} = \gamma_\lambda$; $[B]^r \subset K_{k+\lambda}$. This proves (6).

THEOREM 21. *Let*

$$\alpha \rightarrow (\beta_0, \beta_1, \cdots)^r_k. \tag{8}$$

Then either (i) *there is* $\nu_0 < k$ *such that* $\beta_{\nu_0} \leqq \alpha$; $|\beta_{\nu_0}| < r$, *or* (ii) $\beta_\nu \leqq \alpha$ *for all* $\nu < k$.

REMARK. If (i) holds then (8) is true trivially. For, let $\bar{S} = \alpha$; $[S]^r = \sum [\nu < k] K_\nu$. Then we can choose $B \subset S$ such that $\bar{B} = \beta_{\nu_0}$. Then $[B]^r = 0 \subset K_{\nu_0}$. We see, therefore, that the relation (8) need only be studied in the case in which $\beta_\nu \leqq \alpha$ for all $\nu < k$. In particular, if α is an an ordinal, then we may assume, if we wish, that every β_ν is an ordinal.

PROOF. Suppose that (ii) is false. Then there is $\nu_1 < k$ such that $\beta_{\nu_1} \nleqq \alpha$. Let $\overline{S} = \alpha$, and put $[S]^r = \sum' [\nu < k] K_\nu$, where $K_{\nu_1} = [S]^r$. Then, by (8), there are $B \subset S$; $\nu_0 < k$ such that $\overline{B} = \beta_{\nu_0}$; $[B]^r \subset K_{\nu_0}$. Then $\beta_{\nu_0} = \overline{B} \leqq \overline{S} = \alpha$; $\nu_0 \neq \nu_1$; $[B]^r = 0$; $|\beta_{\nu_0}| = |B| < r$. Hence (i) holds.

THEOREM 22. *The following two tables give information about a number of cases in which the truth or otherwise of any of the relations*

$$\alpha \rightarrow (\beta_0, \beta_1, \cdots)_k^r, \tag{9}$$

$$a \rightarrow (b_0, b_1, \cdots)_k^r \tag{10}$$

can be decided trivially.

$k = 0$:

$r \leqq \|\alpha\|$ $r \leqq a$	+
$r > \|\alpha\|$ $r > a$	−

$k > 0$:

	$\beta_\nu \leqq \alpha$ $b_\nu \leqq a$	$\beta_\nu = \alpha$ $b_\nu = a$	$\beta_\nu \geqq \alpha$; $\beta_0 \nleqq \alpha$ $b_\nu \geqq a$; $b_0 > a$	$\beta_\nu \nleqq \alpha$ $b_\nu > a$	$\beta_0 \ngeqq \alpha$ $b_0 < a$	$\beta_0 \leqq \alpha$ $b_0 \leqq a$	$\beta_0 \nleqq \alpha$ $b_0 > a$	$\beta_0 \ngeqq r$ $b_0 < r$
$r = 0$	+						−	
$0 < r < \|\alpha\|$ $0 < r < a$				−		±		+
$r = \|\alpha\| > 0$ $r = a > 0$		+	−		+			
$r > \|\alpha\|$ $r > a$				−		+		

The proofs may be omitted. When a row or column is headed by two lines of conditions the first line refers to (9) and the second line to (10). Every condition involving the suffix ν is meant to hold for every $\nu<k$. An entry + means that both, (9) and (10) are true, and an entry − means that both, (9) and (10) are false. The one entry ± marks the only case worth studying, i.e. the case in which (9) or (10) can be either true or false, and this for nontrivial reasons. In each row the entries are chosen in such a way that all possibilities are covered. In the column headings we may, of course, replace β_0 and b_0 by β_{ν_0} and b_{ν_0} respectively, for any choice of $\nu_0<k$. The case $k=0$ has, obviously, only curiosity value but is included for the sake of completeness.

5. Denumerable order types.

THEOREM 23. *If* $n<\omega_0$; $\alpha<\omega_0 2$, *then*

(11) $$\omega_0 n \to (n, \alpha)^2,$$

(12) $$\omega_0 n \not\to (n+1, \omega_0+1)^2.$$

PROOF. We may assume $n>0$.

(a) In order to prove (12), consider the set $S=\{(\nu, \lambda):\nu<n;\ \lambda<\omega_0\}$, ordered alphabetically: $(\nu, \lambda)<(\nu_1, \lambda_1)$ if either (i) $\nu<\nu_1$ or (ii) $\nu=\nu_1$; $\lambda<\lambda_1$. Then $[S]^2=K_0+'K_1$, where K_1 is the set of all sets $\{(\nu, \lambda), (\nu, \lambda_1)\}_<\subset S$. Then, clearly, $\bar{S}=\omega_0 n$; $n+1\notin[K_0]$; $\omega_0+1 \notin[K_1]$, and (12) follows.

(b) We now prove (11). Let the set $A=\sum[\nu<n]A_\nu$ be ordered, $\bar{A}_\nu=\omega_0$ for $\nu<n$, and $A_\nu\subset L(A_{\nu+1})$ for $\nu+1<n$. Suppose that $[A]^2=K_0+K_1$; $n\notin[K_0]$; $\alpha\notin[K_1]$. We want to deduce a contradiction.

By Theorem 1, there is, for every $\nu<n$, a set $B_\nu\in[A_\nu]^{\aleph_0}$ such that $[B_\nu]^2\subset K_{\rho_\nu}$, for some $\rho_\nu<2$. Since $n\notin[K_0]$, we have $\rho_\nu=1$. Let $\lambda<\mu<n$. We define an operator $p_{\lambda\mu}$ as follows. There is at least one set $B\subset B_\lambda+B_\mu$ such that $|BB_\lambda|=\aleph_0$; $[B]^2\subset K_1$. For instance, B_λ is such a set B. Since $\alpha\notin[K_1]$, we have, for every such B, $\omega_0\leqq\bar{B}<\alpha<\omega_0 2$. Hence we can choose B such that $\bar{B}$ is maximal. We fix such a B by any suitable convention and put

$$P_{\lambda\mu}(B_0, B_1, \cdots, B_{n-1}) = (C_0, C_1, \cdots, C_{n-1}),$$

where $C_\lambda=BB_\lambda$; $C_\mu=B_\mu-B$, and $C_\nu=B_\nu$ for $\nu\neq\lambda, \mu$. Then $\bar{C}_\nu=\omega_0$; $|C_\lambda L_1(x)|<\aleph_0$ for $\nu<n$; $x\in C_\mu$. We now apply, in turn, all

$$\binom{n}{2}$$

operators $p_{\lambda\mu}$, corresponding to all choices of λ, μ, to the system $(B_0, \cdots, B_{n-1})$, applying each one of the operators, from the second onwards, to the system obtained by the preceding operator, and obtain, as end product, the system $(D_0, \cdots, D_{n-1})$. Then $D_\nu \subset A_\nu$; $\overline{D}_\nu = \omega_0$ $(\nu < n)$; $|D_\nu L_1(x)| < \aleph_0$ for $\nu < n$; $x \in D_{\nu+1} + \cdots + D_{n-1}$. Hence it is possible to choose, in this order, elements $x_{n-1}, x_{n-2}, \cdots, x_0$ such that

$$x_\nu \in D_\nu L_0(x_{\nu+1}, x_{\nu+2}, \cdots, x_{n-1}) \qquad (\nu < n).$$

Then, putting $D = \{x_\nu : \nu < n\}$, we have $\overline{D} = n$; $[D]^2 \subset K_0$ and therefore $n \in [K_0]$ which is the required contradiction.

THEOREM 24. *If* $\alpha < \omega_0 4$, *then*

$$\alpha \nrightarrow (3, \omega_0 2)^2, \tag{13}$$

$$\omega_0 4 \rightarrow (3, \omega_0 2)^2. \tag{14}$$

This theorem is a special case of the following theorem.

THEOREM 25. *Let* $2 \leq m$, $n < \omega_0$, *and denote by* $l_0 = l_0(m, n)$ *the least finite number* l *possessing the following property.*[5]

Property P_{mn}. *Whenever* $\rho(\lambda, \mu) < 2$ *for* $\{\lambda, \mu\}_{\neq} \subset [0, l]$, *then there is either* $\{\lambda_0, \cdots, \lambda_{m-1}\}_{\neq} \subset [0, l]$ *such that*

$$\rho(\lambda_\alpha, \lambda_\beta) = 0 \qquad \text{for } \alpha < \beta < m,$$

or there is $\{\lambda_0, \cdots, \lambda_{n-1}\}_{\neq} \subset [0, l]$ *such that*

$$\rho(\lambda_\alpha, \lambda_\beta) = 1 \qquad \text{for } \{\alpha, \beta\}_{\neq} \subset [0, n].$$

Then

$$\omega_0 l_0 \rightarrow (m, \omega_0 n)^2, \tag{15}$$

$$\gamma \nrightarrow (m, \omega_0 n)^2 \qquad \textit{for } \gamma < \omega_0 l_0. \tag{16}$$

Moreover, if $l_1 \rightarrow (m, m, n)^2$, *then* $l_0 \leq l_1$.

Deduction of Theorem 24 *from Theorem* 25. We have to prove that $l_0(3, 2) = 4$. (i) By considering the function ρ defined by

$$\rho(0, 1) = \rho(1, 2) = \rho(2, 0) = 0; \quad \rho(2, 1) = \rho(1, 0) = \rho(0, 2) = 1,$$

we deduce that 3 does not possess the property P_{32}. (ii) Let us assume that 4 does not possess P_{32}. Then there is $\rho(\lambda, \mu)$ such that the condition stipulated for P_{32} does not hold, with $l = 4$. If

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[5] The existence of such a number l follows from Theorem 2. It will follow from Theorem 39 that we may take $l = (1 + 3^{2m+n-5})/2$.

$$\{\alpha, \beta, \gamma\}_{\neq} \subset [0, 4]; \qquad \rho(\alpha, \beta) = \rho(\alpha, \gamma) = 0,$$

then the assumption $\rho(\beta, \gamma) = 0$ would lead to

$$\rho(\alpha, \beta) = \rho(\beta, \gamma) = \rho(\alpha, \gamma) = 0,$$

i.e. to a contradiction. Hence $\rho(\beta, \gamma) = 1$ and, by symmetry, $\rho(\gamma, \beta) = 1$. This, again, is a contradiction. This argument proves that

$$\text{(17)} \qquad \text{if } \{\alpha, \beta, \gamma\}_{\neq} \subset [0, 4]; \quad \rho(\alpha, \beta) = 0, \quad \text{then } \rho(\alpha, \gamma) = 1.$$

Since at least one of the numbers $\rho(0, 1)$, $\rho(1, 0)$ is zero, there is a permutation $\alpha, \beta, \gamma, \delta$ of 0, 1, 2, 3 such that $\rho(\alpha, \beta) = 0$. Then repeated application of (17) yields $\rho(\alpha, \gamma) = \rho(\alpha, \delta) = 1$; $\rho(\gamma, \alpha) = 0$; $\rho(\gamma, \delta) = 1$; $\rho(\delta, \alpha) = \rho(\delta, \gamma) = 0$, which contradicts (17). This proves $l_0(3, 2) \leq 4$ and, in conjunction with (i), $l_0(3, 2) = 4$.

PROOF OF THEOREM 25. 1. We begin by proving the last clause. Let

$$\text{(18)} \qquad l_1 \to (m, m, n)^2.$$

Suppose that $\rho(\lambda, \mu) < 2$ for $\{\lambda, \mu\}_{\neq} \subset [0, l_1]$. Then

$$[S]^2 = K_0 + K_1 + K_2,$$

where $S = [0, l_1]$, and K_ν is the set of all $\{\lambda, \mu\}_< \subset [0, l_1]$ such that

$$\rho(\lambda, \mu) = 0 \qquad (\nu = 0),$$

$$\rho(\lambda, \mu) > \rho(\mu, \lambda) \qquad (\nu = 1),$$

$$\rho(\lambda, \mu) = \rho(\mu, \lambda) = 1 \qquad (\nu = 2).$$

By (18), there is $S_1 = \{\lambda_0, \cdots, \lambda_{k-1}\}_< \subset S$ such that one of the following three statements holds.

$$\text{(19)} \qquad k = m; \qquad [S_1]^2 \subset K_0,$$

$$\text{(20)} \qquad k = m; \qquad [S_1]^2 \subset K_1,$$

$$\text{(21)} \qquad k = n; \qquad [S_1]^2 \subset K_2.$$

(19) implies that $\rho(\lambda_\alpha, \lambda_\beta) = 0$ for $\alpha < \beta < m$;
(20) implies that $\rho(\lambda_{m-1-\alpha}, \lambda_{m-1-\beta}) = 0$ for $\alpha < \beta < m$;
(21) implies that $\rho(\lambda_\alpha, \lambda_\beta) = 1$ for $\{\alpha, \beta\}_{\neq} \subset [0, n]$.

This shows that $l_0(m, n) \leq l_1$.

2. We now prove (15). Let $l = l_0(m, n)$; $A = [0, \omega_0 l]$; $N = [0, \omega_0]$;

$$[A]^2 = K_0 + K_1 \qquad \text{(partition } \Delta\text{)}.$$

We use the notation of the partition calculus given in detail in [4, p. 419] which can be summarized as follows. If Δ is an equivalence relation on a set M or a partition of M into disjoint classes then $|\Delta|$ denotes the cardinal of the set of nonempty classes, and the relation

$$x \equiv y(\cdot\Delta)$$

expresses the fact that x and y belong to M and lie in the same class of Δ. If, for $\rho \in R$, Δ_ρ is a partition of M, and if $t \rightarrow f_\rho(t)$ is a mapping of a set T into M, then the formula

$$\Delta'(t) = \prod [\rho \in R] \Delta_\rho(f_\rho(t)) \qquad (t \in T)$$

defines that partition Δ' of T for which

$$s \equiv t(\cdot\Delta')$$

if, and only if,

$$f_\rho(s) \equiv f_\rho(t)(\cdot\Delta) \qquad \text{for } \rho \in R.$$

We continue the proof of (15) by putting

$$\Delta'(\{\sigma, \tau\}) = \prod [\lambda, \mu < l] \Delta(\{\omega_0\lambda + \sigma, \omega_0\mu + \tau\}) \qquad (\sigma < \tau < \omega_0).$$

By Theorem 1 there is $N' \in [N]^{\aleph_0}$ such that $|\Delta'| = 1$ in $[N']^2$. Then, by definition of Δ', there is $\rho(\lambda, \mu) < 2$ such that

$$\{\omega_0\lambda + \sigma, \omega_0\mu + \tau\} \in K_{\rho(\lambda,\mu)} \qquad \text{for } \lambda, \mu < l;\ \{\sigma, \tau\}_< \subset N'.$$

By definition of l this implies that there is a set $\{\lambda_0, \cdots, \lambda_{k-1}\}_{\neq} \subset [0, l]$ such that either

$$k = m; \quad \rho(\lambda_\alpha, \lambda_\beta) = 0 \qquad \text{for } \alpha < \beta < m \tag{22}$$

or

$$k = n; \quad \rho(\lambda_\alpha, \lambda_\beta) = 1 \qquad \text{for } \{\alpha, \beta\}_{\neq} \subset [0, n]. \tag{23}$$

If (22) holds, then we put

$$A' = \{\omega_0\lambda_\alpha + \sigma_\alpha : \alpha < m\},$$

where σ_α is chosen such that $\{\sigma_0, \sigma_1, \cdots, \sigma_{m-1}\}_< \subset N'$. Then $[A']^2 \subset K_0$, so that the desired conclusion $m \in [K_0]$ is reached.

If we now assume that $m \notin [K_0]$, then $\rho(\lambda, \lambda) = 1$ for $\lambda < l$, and, furthermore, (23) holds. Then we put $N' = \{\sigma_0, \sigma_1, \cdots\}_<$; $\overline{A}'' = \{\omega_0\lambda_\alpha + \sigma_{\alpha+tn} : \alpha < n;\ t < \omega_0\}$. We find that $[A'']^2 < K_1$; $\omega_0 n = A'' \in [K_1]$. This proves (15).

3. Finally, let $\gamma < \omega_0 l_0$. Then there is $l < l_0$ such that $\omega_0 l \leqq \gamma < \omega_0(l+1)$. Then, by definition of l_0, there is $\rho(\lambda, \mu) < 2$ for $\{\lambda, \mu\}_{\neq} \subset [0, l]$ such that, whenever $\{\lambda_0, \cdots, \lambda_{m-1}\}_{\neq} \subset [0, l]$, then

$$\rho(\lambda_\alpha, \lambda_\beta) \neq 0 \tag{24}$$

for some $\{\alpha, \beta\}_< \subset [0, m]$, and, whenever $\{\lambda_0, \cdots, \lambda_{n-1}\}_{\neq} \subset [0, l]$, then

$$\rho(\lambda_\alpha, \lambda_\beta) \neq 1 \tag{25}$$

for some $\{\alpha, \beta\}_{\neq} \subset [0, n]$. Then, if $A = [0, \gamma]$, we have $[A]^2 = K_0 +' K_1$, where K_0 is the set of all $\{\omega_0\lambda + \sigma, \omega_0\mu + \tau\}$ such that $\{\lambda, \mu\}_{\neq} \subset [0, l]$; $\sigma < \tau < \omega_0$; $\rho(\lambda, \mu) = 0$. If, now, $A' \in [A]^m$; $[A']^2 \subset K_0$, then

$$A' = \{\omega_0\lambda_\alpha + \sigma_\alpha : \alpha < m\}; \qquad \sigma_0 < \cdots < \sigma_{m-1} < \omega_0;$$

$$\{\lambda_0, \cdots, \lambda_{m-1}\}_{\neq} \subset [0, l];$$

$$\rho(\lambda_\alpha, \lambda_\beta) = 0 \qquad \text{for } \alpha < \beta < m,$$

which contradicts (24). If, on the other hand,

$$A'' \subset A; \qquad \overline{A''} = \omega_0 n; \qquad [A'']^2 \subset K_1,$$

then there is $\{\lambda_0, \cdots, \lambda_{n-1}\}_< \subset [0, l]$ such that $\overline{B}_\alpha = \omega_0$ for $\alpha < n$, where $B_\alpha = A''[\omega_0\lambda_\alpha, \omega_0(\lambda_\alpha + 1))$ $(\alpha < n)$. Then $\rho(\lambda_\alpha, \lambda_\beta) = 1$ for $\{\alpha, \beta\}_{\neq} \subset [0, n]$, which contradicts (25). Hence neither A' nor A'' exist, with the properties stated, so that (16) follows. This completes the proof of Theorems 24 and 25.

6. **The linear continuum.** Our object is to investigate relations of the form

$$\lambda \rightarrow (\alpha_0, \alpha_1, \cdots)_k^r$$

and their negatives. It turns out[6] that every positive relation we were able to prove holds not only for the particular type λ of the set of all real numbers but for every type ϕ such that

$$|\phi| > \aleph_0; \quad \omega_1, \omega_1^* \not\leq \phi. \tag{26}$$

This fact seems to suggest that, given any type ϕ satisfying (26), there always exists λ_1 such that

$$\lambda_1 \leqq \lambda, \phi; \qquad |\lambda_1| > \aleph_0,$$

i.e., that every nondenumerable type which does not "contain" ω_1 or ω_1^* contains a nondenumerable type which is embeddable in the real continuum. This conjecture has, as far as the authors are aware, neither been proved nor disproved.[7]

Throughout this section S denotes the set of all real numbers x such that $0 < x < 1$, ordered by magnitude. The letters x, y, z denote elements of S, and $\lambda = \overline{S}$.

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[6] Cf. Theorems 31, 32.

[7] Since this paper was submitted E. Specker has disproved this conjecture.

THEOREM 26.

(i) $$\lambda \not\rightarrow (\omega_1)^r_k \qquad \textit{for } r \geqq 0;\ k > 0.$$

(ii) $$\lambda \not\rightarrow (r+1)^r_{\omega_0} \qquad \textit{for } r \geqq 2.$$

PROOF. (i) is trivial, in view of $\omega_1 \not\leqq \lambda$. In order to prove (ii) it suffices, by Theorem 15, to consider the case $r=2$. Let $\{x_\nu : \nu < \omega_0\}$ be the set of all rational numbers in S, and denote, for $n < \omega_0$, by K_n the set of all $\{x, y\}_<$ such that the least ν satisfying $x < x_\nu < y$ is $\nu = n$. Then $[S]^2 = \sum [\nu < \omega_0] K_\nu$. Also, if $[\{x, y, z\}_<]^2 \subset K_n$, then the contradiction $x < x_n < y < x_n < z$ follows. Hence $3 \notin [K_n]$, and Theorem 26 is proved.

THEOREM 27. *$\lambda \not\rightarrow (\omega_0, \omega_0 + 2)^r$ for $r \geqq 3$.*

PROOF. By Theorem 15, we need only consider the case $r=3$. We have $[S]^3 = K_0 +' K_1$, where $K_0 = \{\{x, y, z\}_< : y - x < z - y\}$.

ASSUMPTION 1. Let $[\{x_0, x_1, \cdots\}_<]^3 \subset K_0$. Then $\lim x_\nu = u$ as $\nu \rightarrow \infty$, and we have, for $0 < m < \omega_0$,

$$\{x_0, x_m, x_{m+1}\} \in K_0; \qquad x_m - x_0 < x_{m+1} - x_m.$$

If $m \rightarrow \infty$, then the contradiction $u - x_0 \leqq u - u$ follows.

ASSUMPTION 2. Let $A \subset S$; $\overline{A} = \omega_0 + 2$; $[A]^3 \subset K_1$. Then $A = B + \{y, z\}_<$; $B = \{x_0, x_1, \cdots\}_< \subset L(y)$; $\lim x_\nu = u$ as $\nu \rightarrow \infty$, and we have, for $m < \omega_0$, $\{x_m, x_{m+1}, z\} \in K_1$; $x_{m+1} - x_m \geqq z - x_{m+1}$. If $m \rightarrow \infty$, then the contradiction $u - u \geqq z - u$ follows. This proves Theorem 27.

THEOREM 28. *$\lambda \not\rightarrow (r+1, \omega_0 + 2)^r$ for $r \geqq 4$.*

PROOF. It suffices to consider the case $r=4$. We have $[S]^4 = K_0 +' K_1$, where $K_0 = \{\{x_0, x_1, x_2, x_3\}_< : x_2 - x_1 < x_3 - x_2, x_1 - x_0\}$.

ASSUMPTION 1. Let $[\{x_0, x_1, x_2, x_3, x_4\}_<]^4 \subset K_0$. Then $\{x_0, x_1, x_2, x_3\} \in K_0$, and hence $x_2 - x_1 < x_3 - x_2$. Also, $\{x_1, x_2, x_3, x_4\} \in K_0$, and hence $x_3 - x_2 < x_2 - x_1$. This is a contradiction.

ASSUMPTION 2. Let $A \subset S$; $\overline{A} = \omega_0 + 2$; $[A]^4 \subset K_1$. We define B, y, z, x_ν, u as in the proof of Theorem 27. Then there is $m_0 < \omega_0$ such that, for $m_0 \leqq m < \omega_0$, $u - x_m < x_m - x_0$. Then, for $m_0 \leqq m < \omega_0$, $\{x_0, x_m, x_{m+1}, z\} \in K_1$; $x_{m+1} - x_m < u - x_m < x_m - x_0$; $x_{m+1} - x_m \geqq z - x_{m+1}$, and if $m \rightarrow \infty$, then the contradiction $u - u \geqq z - u$ follows. This proves Theorem 28.

The next two theorems are extensions of results due to Sierpiński.

THEOREM 29. *If $2 \leqq |k| \leqq |\lambda|$; $|\alpha_\nu| \geqq |\lambda|$ $(\nu < k)$, then*

$$\lambda \not\rightarrow (\alpha_0, \alpha_1, \cdots)^1_k.$$

Sierpiński proved that $\lambda \not\rightarrow (\alpha, \alpha)^1$ if $\alpha \leqq \lambda$; $|\alpha| = |\lambda|$ (Theorem 9).

PROOF.

Case 1. There is $\mu<k$ such that $\alpha_\mu \not\leq \lambda$. We consider the partition $S=\sum' [\nu<k]K_\nu$, where $K_\mu=S$. We have $\alpha_\mu\not\leq\lambda=\overline{K}_\mu$ and, for $\nu\neq\mu$, $\alpha_\nu\not\leq 0=\overline{K}_\nu$. Hence $\alpha_\nu\not\leq\overline{K}_\nu$ $(\nu<k)$.

Case 2. $\alpha_\nu\leq\lambda$ $(\nu<k)$. We choose a fixed set $A_0\subset S$ such that $\overline{A}_0=\alpha_0$. Generally, the letter A denotes sets such that $A\subset S$; $\overline{A}=\alpha_0$. Corresponding to every A, there is a real function $f_A(x)$, defined and strictly increasing for $x\in A_0$, such that $x\to f_A(x)$ is a mapping of A_0 on A. We extend the definition of f_A by putting $f_A(x)=0$ for $x\in L(A_0)$ and

$$f_A(x)=\sup\,[y\leq x;\, y\in A_0]f_A(y) \qquad \text{for } x\notin L(A_0).$$

Then $f_A(x)$ is nondecreasing in S. The set A is uniquely determined by the function f_A and the set A_0. Let $D(A)$ be the set of those x_0 for which $f_A(x)$ is discontinuous at $x=x_0$. Then $|D(A)|\leq\aleph_0$. The function f_A is uniquely determined by (i) the set $D(A)$ and (ii) the values of $f_A(x)$ for $x\in D(A)$ and (iii) the values of $f_A(x)$ for all rational x. Therefore

$$\left|\sum\{A\}\right|=\left|\sum\{f_A\}\right|\leq|\lambda|^{\aleph_0}=|\lambda|\leq\left|\sum\{A\}\right|,$$

and $\left|\sum\{A\}\right|=|\lambda|=\aleph_n$, say. Now we can write $\sum\{A\}=\{A_{0\rho}: \rho<\omega_n\}$. By symmetry, we have, for every $\nu<k$, a set $\{A_{\nu\rho}:\rho<\omega_n\}$ whose elements are all subsets of S of type α_ν.

The set $N=\{(\nu,\rho):\nu<k;\ \rho<\omega_n\}$ satisfies $|N|=|k|\aleph_n=\aleph_n$. We order N in such a way that $\overline{N}=\omega_n$. Then we can find, inductively, $x_{\nu\rho}$ such that, for $(\nu,\rho)\in N$,

$$x_{\nu\rho}\in A_{\nu\rho}-\{x_{\mu\sigma}:(\mu,\sigma)<(\nu,\rho)\}.$$

For, $\left|\{(\mu,\sigma):(\mu,\sigma)<(\nu,\rho)\}\right|<|N|\leq|A_{\nu\rho}|$. We have $x_{\nu\rho}\neq x_{\mu\sigma}$ for $(\nu,\rho)\neq(\mu,\sigma)$. Now, $S=\sum[\nu<k]K_\nu$, where $K_\nu=S-\{x_{\nu\rho}:\rho<\omega_n\}$. For, if $x\in S$, then, since $k\geq 2$, there is $\nu<k$ such that $x\in K_\nu$. If, now, $\alpha_\nu\leq\overline{K}_\nu$ for some $\nu<k$, then there is ρ such that $x_{\nu\rho}\in A_{\nu\rho}\subset K_\nu$, which is the required contradiction. This proves Theorem 29.

THEOREM 30. $|\lambda|\not\to(\aleph_1,\aleph_1)^r$ *for* $r\geq 2$.

PROOF. The substance of this theorem is due to Sierpiński [15]. By Theorem 15 we need only consider the case $r=2$. Let $x<y$ be, as throughout this section, the order of S by magnitude, and let $x\ll y$ be an order of S such that $\overline{S}_{<<}=\omega_n$, where $|\lambda|=\aleph_n$. Then $[S]^2=K_0+'K_1$, where $K_0=\{\{x,y\}_<:x\ll y\}$. Now let $A\subset S$; $[A]^2\subset K_\nu$.

If $\nu=0$, then $\overline{A}_<\leq\overline{S}_<=\lambda$; $\overline{A}_<=\overline{A}_{<<}\leq\overline{S}_{<<}=\omega_n$, and hence $\overline{A}_<<\omega_1$; $|A|<\aleph_1$. If $\nu=1$, then $\overline{A}_>\leq\overline{S}_>=\lambda^*$; $\overline{A}_>=\overline{A}_{<<}\leq\overline{S}_{<<}=\omega_n$; $\overline{A}_><\omega_1$; $|A|<\aleph_1$.

This proves Theorem 30. We note that this theorem is, in fact, an easy corollary of [5, Example 4A].

7. The general case. We shall consider relations involving certain types of cardinal $\aleph_1$ as well as relations between types of any cardinal. We begin by proving a lemma. We establish this lemma in a form which is more general than will later be required, but in this form it seems to possess some interest of its own. We recall that a' denotes the cofinality cardinal belonging to a which was defined in §2.

LEMMA 1. *Let S be an ordered set, and $|S|' = \aleph_\kappa$; $\omega_\kappa, \omega_\kappa^* \not\leq \overline{S}$. Then, corresponding to every rational number t, there is $S_t \subset S$ such that $|S_t| = |S|$; $S_t \subset L(S_u)$ for $t < u$.*

Sierpiński, in a letter to one of us, had already noted the weaker result that, *if $|S| = \aleph_1$; $\omega_1, \omega_1^* \not\leq \overline{S}$, then $\eta \leq \overline{S}$.*

PROOF. *Case* 1. There is $A \subset S$ such that

$$|AL(x)| < |A| = |S| \qquad (x \in A).$$

Then we define x_ν for $\nu < \omega_\kappa$ inductively as follows. Let $\nu_0 < \omega_\kappa$; $x_\nu \in A\ (\nu < \nu_0)$. Then, by definition of κ,

$$\left| \sum [\nu < \nu_0](AL(x_\nu) + \{x_\nu\}) \right| < |A|,$$

and hence there is $x_{\nu_0} \in A - \sum[\nu < \nu_0](L(x_0) + \{x_\nu\})$. Then $x_\mu < x_\nu$ $(\mu < \nu < \omega_\kappa)$ and so $\omega_\kappa \leq \overline{S}$, which is false.

Case 2. There is $A \subset S$ such that

$$|AR(x)| < |A| = |S| \qquad (x \in A).$$

Then, by symmetry, the contradiction $\omega_\kappa^* \leq \overline{S}$ follows.

Case 3. There is $A \subset S$ such that

$$\min(|AL(x)|, |AR(x)|) < |A| = |S| \qquad (x \in A).$$

Then we put

$$A_0 = \{x : x \in A;\ |AL(x)| < |A|\},$$
$$A_1 = \{x : x \in A;\ |AR(x)| < |A|\}.$$

Then $A = A_0 + A_1$.

Case 3.1. $|A_0| = |S|$. Then $|A_0 L(x)| \leq |AL(x)| < |S| = |A_0|$ $(x \in A_0)$, and hence, by Case 1, we find a contradiction.

Case 3.2. $|A_0| \neq |S|$. Then $|A_1| = |S|$ and, by symmetry, a contradiction follows.

We have so far proved that, if $A \subset S$; $|A| = |S|$, there is $z \in A$ such that $|AL(z)| = |AR(z)| = |S|$. Then $A = A' + A''$, where $A' = AL(z)$;

$|A'| = |A''| = |S|$; $A' \subset L(A'')$. By applying this result to A' we find a partition $A = A(0) + A(1) + A(2)$ such that

$$|A(\nu)| = |S| \ (\nu < 3); \quad A(\nu) \subset L(A(\nu + 1)) \qquad (\nu < 2).$$

Repeated application leads to sets

$$A(\lambda_0, \lambda_1, \cdots, \lambda_{k-1}) \qquad (k < \omega_0; \lambda_\nu < 3)$$

such that

$$|A(\lambda_0, \cdots, \lambda_{k-1})| = |S|;$$
$$A(\lambda_0, \cdots, \lambda_{k-1}) = \sum [\nu < 3] A(\lambda_0, \cdots, \lambda_{k-1}, \nu);$$
$$A(\lambda_0, \cdots, \lambda_{k-1}, \nu) \subset L(A(\lambda_0, \cdots, \lambda_{k-1}, \nu + 1)) \qquad (\nu < 2).$$

Let N be the set of all systems $(\lambda_0, \cdots, \lambda_k)$ such that $k < \omega_0$; $\lambda_\nu \in \{0, 2\} (\nu < k)$; $\lambda_k = 1$, ordered alphabetically. More accurately, if $p = (\lambda_0, \cdots, \lambda_k)$ and $q = (\mu_0, \cdots, \mu_l)$ are elements of N, then we put $p < q$ if $\sum [\nu < k] \lambda_\nu 3^{-\nu} < \sum [\nu < l] \mu_\nu 3^{-\nu}$. Then we have $A(\lambda_0, \cdots, \lambda_k) \subset L(A(\mu_0, \cdots, \mu_l))$, if $(\lambda_0, \cdots) < (\mu_0, \cdots)$. It now suffices to show that N is dense in itself. In fact, if $\{(\lambda_0, \cdots, \lambda_k), (\mu_0, \cdots, \mu_l)\}_< \subset N$, then

$$(\lambda_0, \cdots, \lambda_k) < (\mu_0, \cdots, \mu_{l-1}, 0, 2, 2, \cdots, 2, 1) < (\mu_0, \cdots, \mu_l),$$

provided only that the inner bracket contains a sufficiently large number of two's. Lemma 1 is proved.

THEOREM 31. *Suppose that ϕ is a type such that*

$$|\phi| > \aleph_0; \qquad \omega_1, \omega_1^* \not\leq \phi.$$

Let $\alpha < \omega_0 2$; $\beta < \omega_0^2$; $\gamma < \omega_1$. Then

$$\phi \to (\alpha, \alpha, \alpha)^2, \tag{27}$$

$$\phi \to (\alpha, \beta)^2, \tag{28}$$

$$\phi \to (\omega_0, \gamma)^2, \tag{29}$$

$$\phi \to (4, \alpha)^3. \tag{30}$$

THEOREM 32. *Let ϕ, α, γ be as in Theorem 31. Let S be an ordered set, $\bar{S} = \phi$, and $[S]^2 = K_0 + K_1$. Then*

(a) *there is $V \subset S$ such that either*

$$\text{(i) } \bar{V} = \alpha; \quad [V]^2 \subset K_0,$$

or

$$\text{(ii) } \bar{V} = \gamma; \quad [V]^2 \subset K_1,$$

or

$$\text{(iii)}\ \bar{V} = \omega_0\gamma^*; \quad [V]^2 \subset K_1,$$

and

(b) *there is* $W \subset S$ *such that either*

$$\text{(i)}\ \bar{W} = \omega_0 + \omega_0^*; \quad [W]^2 \subset K_0,$$

or

$$\text{(ii)}\ \bar{W} = \gamma; \quad [W]^2 \subset K_1,$$

or

$$\text{(iii)}\ \bar{W} = \gamma^*; \quad [W]^2 \subset K_1.$$

In proving Theorems 31 and 32 we may assume that $|\phi| = \aleph_1$. There is m such that

$$4 \leqq m < \omega_0; \qquad \alpha \leqq \omega_0 + m; \qquad \beta \leqq \omega_0 m.$$

Let $\bar{S} = \phi$. The letters A, B, P, Q denote subsets of S, and we shall always suppose, in the proofs of the last two theorems, that

$$|A| = |B| = \aleph_1; \qquad \bar{P} = \bar{Q} = \omega_0.$$

PROOF OF THEOREM 31, (29). Let $[S]^2 = K_0 + K_1$, and

$$\omega_0 \notin [K_0]. \tag{31}$$

We want to deduce that

$$\gamma \in [K_1]. \tag{32}$$

There is B such that

$$|BR_0(x)| \leqq \aleph_0 (x \in B). \tag{33}$$

For otherwise there would be elements x_ν such that

$$x_0 \in S; \qquad |R_0(x_0)| = \aleph_1,$$
$$x_1 \in R_0(x_0); \qquad |R_0(x_0, x_1)| = \aleph_1,$$

generally, $x_\nu \in R_0(x_0, \cdots, x_{\nu-1})$,

$$|R_0(x_0, \cdots, x_\nu)| = \aleph_1 \qquad (\nu < \omega_0).$$

Then $[\{x_0, x_1, \cdots\}_<]^2 \subset K_0$, and hence $\omega_0 \in [K_0]$ which contradicts (31).

By hypothesis, $\omega_1, \omega_1^* \not\leqq \bar{B}$. Hence, by Lemma 1,

$$\sum[t \text{ rational}]B(t) \subset B$$

for some sets $B(t)$ such that $B(t)\subset L(B(u))$ $(t<u)$.

There is a set T of rational numbers which, if ordered by magnitude, is of type γ. Let $T=\{t_\mu:\mu<\gamma\}$ where $t_\mu<t_\nu$ for $\mu<\nu<\gamma$. We define inductively elements $x_\nu(\nu<\gamma)$ as follows. Let $\nu_0<\gamma$, and suppose that $x_\nu\in B(\nu<\nu_0)$. Then, by (33),

$$\left|\sum[\nu<\nu_0]BR_0(x_\nu)\right| \leq \aleph_0 < \left|B(t_{\nu_0})\right|,$$

and therefore we can choose $x_{\nu_0}\in B(t_{\nu_0})-\sum[\nu<\nu_0]R_0(x_\nu)$. Put $X=\{x_\nu:\nu<\gamma\}$. Then $\overline{X}=\gamma$; $[X]^2\subset K_1$. Hence (32) holds, and (29) follows.

PROOF OF THEOREM 32 (a). Let the hypotheses be satisfied but suppose that (a) is false, i.e. that

$$(34) \qquad \alpha \notin [K_0]; \qquad \gamma \notin [K_1]; \qquad \omega_0\gamma^* \notin [K_1].$$

ASSUMPTION. If $A\subset S$, then there is $x_0\in A$ such that

$$|AL_0(x_0)| > \aleph_0.$$

Then there are x_ν, $A_\nu(\nu\leq m)$ such that

$$x_0\in A_0 \triangleq S; \qquad A_0L_0(x_0)=A_1; \qquad x_1\in A_1; \qquad A_1L_0(x_1)=A_2$$

and so on, up to

$$A_m = A_{m-1}L_0(x_{m-1}) = A_0L_0(x_0, x_1, \cdots, x_{m-1}).$$

Then, by (29), $\overline{A}_m\to(\omega_0, \gamma)^2$; $\gamma\notin F_1(A_m)$, and hence $\omega_0\in F_0(A_m)$. There is $P\subset A_m$ such that $[P]^2\subset K_0$. Then $[P+\{x_0, \cdots, x_{m-1}\}]^2\subset K_0$ which contradicts (34). Hence our assumption is false, and there is A such that

$$(35) \qquad |AL_0(x)| \leq \aleph_0 \qquad (x\in A).$$

By Lemma 1, there is $B(t)\subset A$, for rational t, such that $B(t)\subset L(B(u))$ $(t<\mu)$. There are rational numbers t_ν $(\nu<\gamma)$ such that $t_\mu>t_\nu$ $(\mu<\nu<\gamma)$. We define sets P_ν $(\nu<\gamma)$ as follows. Let $\nu_0<\gamma$, and suppose that $P_\nu\subset B(t_\nu)$ $(\nu<\nu_0)$. Then we may put, by (35),

$$B' = B(t_{\nu_0})L_1(\sum[\nu<\nu_0]P_\nu).$$

Then, by (29), $\overline{B}'\to(\alpha, \omega_0)^2$; $\alpha\notin F_0(B')$; $\omega_0\in F_1(B')$, and there is $P_{\nu_0}\subset B'$ such that $[P_{\nu_0}]^2\subset K_1$. This defines P_ν for $\nu<\gamma$. Put $\sum[\nu<\gamma]P_\nu = X$. Then $\overline{X}=\omega_0\gamma^*$; $[X]^2\subset K_1$. But this contradicts (34), and so (a) is proved.

PROOF OF THEOREM 32 (b). Let the hypotheses be satisfied but (b) be false. Then

$$(36) \qquad \omega_0+\omega_0^* \notin [K_0]; \qquad \gamma \notin [K_1]; \qquad \gamma^* \notin [K_1].$$

Choose any A.

ASSUMPTION. $|AR_0(x)| \leq \aleph_0$ $(x\in A)$.

Then, by Lemma 1, there are sets $B(t)\subset A$, for rational t, such that $B(t)\subset L(B(u))$ $(t<u)$. There are rational numbers t_ν $(\nu<\gamma)$ such that $t_\mu<t_\nu$ $(\mu<\nu<\gamma)$. We define x_ν $(\nu<\gamma)$ as follows. Let $\nu_0<\gamma$, and suppose that $x_\nu\in B(t_\nu)$ $(\nu<\nu_0)$. Then, by our assumption, there is

$$x_{\nu_0} \in B(t_{\nu_0}) - \sum[\nu<\nu_0]R_0(x_\nu).$$

Then the set $X=\{x_\nu:\nu<\gamma\}$ satisfies $\bar{X}=\gamma$; $[X]^2\subset K_1$ which is a contradiction against (36). Hence our assumption is false, i.e., given any A, there is $x\in A$ such that $|AR_0(x)|=\aleph_1$. By symmetry, it follows that there also is $y\in A$ such that $|AL_0(y)|=\aleph_1$. By alternate applications of these two results we obtain elements x_ν, y_ν and sets A_ν, $B_\nu (\nu<\omega_0)$ such that the following conditions are satisfied.

$$x_0\in S; \quad y_0\in R_0(x_0)=B_0; \quad x_1\in B_0L_0(y_0)=A_1; \quad y_1\in A_1R_0(x_1)=B_1;$$

generally, for $\nu<\omega_0$,

$$x_{\nu+1}\in B_\nu L_0(y_\nu)=A_{\nu+1}; \qquad y_{\nu+1}\in A_{\nu+1}R_0(x_{\nu+1})=B_{\nu+1}.$$

Then the set $\sum[\nu<\omega_0]\{x_\nu, y_\nu\}=D$ satisfies $\bar{D}=\omega_0+\omega_0^*$; $[D]^2\subset K_0$. This contradiction against (36) completes the proof of Theorem 32.

PROOF OF THEOREM 31, (27). Let $[S]^2=K_0+K_1+K_2$,

$$(37) \qquad \alpha\notin[K_\nu] \qquad (\nu<3).$$

Our aim is to deduce a contradiction. We shall reduce the general case to more and more special cases. For the sake of convenience of notation we shall use the same notation for the sets in question at each stage.

We put $K_{12}=K_1+K_2$. The functions F_{12}, L_{12}, R_{12} refer to K_{12} in the same way as the functions F_ν, L_ν, R_ν refer to K_ν.

Let $A\subset S$. By Lemma 1, there are sets A_0, $A_1\subset A$ such that $A_0\subset L(A_1)$. Let $x_0\in A_1$. Then $|AL(x_0)|=\aleph_1$, and there is $\nu_0<3$ such that $|AL_{\nu_0}(x_0)|=\aleph_1$. By repeating this argument we find numbers $\nu_\rho<3$ and elements x_ρ $(\rho<\omega_0)$ such that

$$x_\rho\in SL_{\nu_0}(x_0)L_{\nu_1}(x_1)\cdots L_{\nu_{\rho-1}}(x_{\rho-1}),$$
$$|SL_{\nu_0}(x_0)\cdots L_{\nu_\rho}(x_\rho)|=\aleph_1 \qquad (\rho<\omega_0).$$

There are $\rho_0<\rho_1<\cdots\rho_{m-1}<\omega_0$ such that $\nu_{\rho_0}=\cdots=\nu_{\rho_{m-1}}$. We may assume $\nu_{\rho_0}=0$. Put $L_0(x_{\rho_0},\cdots,x_{\rho_{m-1}})=A_0$.

ASSUMPTION 1. $\omega_0\in F_0(A_0)$.

Then there is $P \subset A_0$ such that $[P]^2 \subset K_0$. Then $\alpha \leqq \bar{C}$; $[C]^2 \subset K_0$, where $C = P + \{x_{\mu_\nu} : \nu < m\}$, which contradicts (37). Hence the Assumption 1 is false, and we have $\omega_0 \notin F_0(A_0)$. We may assume that

$$\omega_0 \notin [K_0]. \tag{38}$$

For a later application we remark that in what follows we may replace S by any nondenumerable subset of S without any of the conclusions becoming invalid.

Now let $A \subset S$. Then, by (29), $\bar{A} \to (\omega_0, \alpha)^2$. Also, $\omega_0 \to (\omega_0, \omega_0)^2$. Therefore, by Theorem 16, $\bar{A} \to (\omega_0, \omega_0, \alpha)^2$. Hence at least one of the following three relations holds.

$$\text{(i) } \omega_0 \in F_0(A), \qquad \text{(ii) } \omega_0 \in F_1(A), \qquad \text{(iii) } \alpha \in F_2(A).$$

Since (i) and (iii) are false, it follows that

$$\omega_0 \in F_1(A) \qquad (A \subset S). \tag{39}$$

By symmetry,

$$\omega_0 \in F_2(A) \qquad (A \subset S). \tag{40}$$

Assumption 2. There are x_ν, A_ν $(\nu < \omega_0)$ such that $x_0 \in A_0$; $A_0 R_0(x_0) = A_1$; $x_1 \in A_1$; $A_1 R_0(x_1) = A_2$; $x_2 \in A_2$, etc.

Then $[\{x_0, x_1, \cdots\}_<]^2 \subset K_0$ which contradicts (38). Hence the Assumption 2 is false, and there are $\nu_0 < \omega_0$; $x_\nu \in S$ $(\nu < \nu_0)$ such that we may put $A = R_0(x_0, \cdots, x_{\nu_0 - 1})$ and we then have $|AR_0(x)| \leqq \aleph_0$ $(x \in A)$. We may assume that

$$|R_0(x)| \leqq \aleph_0 \qquad (x \in S). \tag{41}$$

By Lemma 1, there are sets A, B such that $A \subset L(B)$. By (39), there is $P \subset A$ such that $[P]^2 \subset K_1$. For a later application we remark that at this stage we might have applied (40) in place of (39) and in this way could have interchanged the roles of K_1 and K_2. By (41), $|\sum [x \in P] R_0(x)| \leqq \aleph_0$, and hence $|BR_{12}(P)| = \aleph_1$. Therefore we may assume

$$[P]^2 \subset K_1; \qquad P \subset L_{12}(S - P). \tag{42}$$

Assumption 3. If $Q \subset P$; $A \subset S$, then there is $x \in A$ such that

$$|QL_1(x)| = \aleph_0.$$

Now we argue as follows. By Lemma 1, there are sets $A_\nu \subset S - P$ such that $A_\mu \subset L(A_\nu)$ $(\mu < \nu < \omega_0 m)$. We define inductively x_ν, P_ν $(\nu < \omega_0 m)$ as follows. There is $x_0 \in A_0$ such that, if $P_0 = P$, we have $|P_0 L_1(x_0)| = \aleph_0$. Let $0 < \nu_0 < \omega_0 m$, and suppose that

$$x_\nu \in A_\nu; \qquad P_\nu \subset P \qquad (\nu < \nu_0),$$
$$|P_\nu - P_\mu| < \aleph_0 \qquad (\mu < \nu < \nu_0).$$

Then we can write $[0, \nu_0] = \{\rho_\lambda : \lambda < \omega_0\}$. We can choose y_λ such that

$$y_\lambda \in P_{\rho_0} P_{\rho_1} \cdots P_{\rho_\lambda} - \{y_0, \cdots, y_{\lambda-1}\} \qquad (\lambda < \omega_0).$$

By (41) and Assumption 3, there is $x_{\nu_0} \in A_{\nu_0} - \sum [\nu < \nu_0] R_0(x_\nu)$ such that $|\{y_0, y_1, \cdots\} L_1(x_{\nu_0})| = \aleph_0$. We put $P_{\nu_0} = \{y_0, y_1, \cdots\} L_1(x_{\nu_0})$. Then, if $\nu < \nu_0$, there is $\lambda < \omega_0$ such that $\nu = \rho_\lambda$. Then $|P_{\nu_0} - P_\nu| \leqq |\{y_0, y_1, \cdots\} - P_{\rho_\lambda}| < \aleph_0$. This completes the definition of x_ν, P_ν $(\nu < \omega_0 m)$.

We have $|P_\nu - P_\mu| < \aleph_0$ $(\mu < \nu < \omega_0 m)$; $P_\nu \subset PL_1(x_\nu)$ $(\nu < \omega_0 m)$. Put $X = \{x_\nu : \nu < \omega_0 m\}$. Then, by (11), $[X]^2 \subset K_1 + K_2$; $X = \omega_0 m \rightarrow (m, \alpha)^2$.

Case 1. There is $D = \{x_{\mu_0}, \cdots, x_{\mu_{m-1}}\}_< \subset X$ such that $[D]^2 \subset K_1$. Then we put $P' = P_{\mu_{m-1}}$ and have, for $\tau < m$,

$$|P' - L_1(x_{\mu_\tau})| \leqq |P' - P_{\mu_\tau}| + |P_{\mu_\tau} - L_1(x_{\mu_\tau})| < \aleph_0 + 0.$$

By summing over τ we obtain $|P' - L_1(D)| < \aleph_0$. Hence we may put $P'L_1(D) = Q$, and we then have $\overline{Q+D} \geqq \alpha$; $[Q+D]^2 \subset K_1$ which contradicts (37).

Case 2. There is $D \subset X$ such that $\bar{D} = \alpha$; $[D]^2 \subset K_2$. This, again, contradicts (37). Hence the Assumption 3 is false, i.e., there are $P' \subset P$; $A' \subset S$ such that

$$|P'L_1(x)| < \aleph_0 \qquad (x \in A').$$

Then there is $A'' \subset A'$ such that the set $P'L_1(x)$ is constant for $x \in A''$. Then there is P'' such that $P'L_1(x) = P''$ $(x \in A'')$. We have therefore proved that there is P'', A'' such that

$$(43) \qquad [P'']^2 \subset K_1; \qquad P'' \subset L_1(A'').$$

The whole argument from (38) onwards remains valid if S is replaced by any set A. Hence it follows from (43) that if $A \subset S$, then there are P, $A' \subset A$ such that

$$(44) \qquad [P]^2 \subset K_1; \qquad P \subset L_1(A').$$

By Lemma 1, there are A_0, B_0 such that $A_0 \subset L(B_0)$. By repeated application of (44) we obtain sets P_ν, A'_ν $(\nu < \omega_0)$ such that

$$P_0 + A'_0 \subset A_0; \qquad [P_0]^2 \subset K_1; \qquad P_0 \subset L_1(A'_0),$$
$$P_1 + A'_1 \subset A'_0; \qquad [P_1]^2 \subset K_1; \qquad P_1 \subset L_1(A'_1),$$

generally, $P_\nu + A'_\nu \subset A'_{\nu-1}$; $[P_\nu]^2 \subset K_1$; $P_\nu \subset L_1(A'_\nu)$ $(0 < \nu < \omega_0)$. Then $P_\nu \subset A_0 \subset L(B_0)$ $(\nu < \omega_0)$,

$$P_\mu \subset L_2(A'_\mu) \subset L_2(A'_{\nu-1}) \subset L_2(P_\nu) \qquad (\mu < \nu < \omega_0).$$

We put $B_1 = B_0 R_{12}(P_0 + P_1 + \cdots)$. Then we have the result that there are sets P_ν, B_1 $(\nu < \omega_0)$ such that

$$(45) \qquad \begin{cases} [P_\nu]^2 \subset K_1; \quad P_\nu \subset L_{12}(B_1) & (\nu < \omega_0), \\ P_\mu \subset L_2(P_\nu) & (\mu < \nu < \omega_0). \end{cases}$$

Now let $\nu_0 < \omega_0$; $B_2 \subset B_1$; $P' \subset P_{\nu_0}$.

ASSUMPTION 4. $|P'L_2(x)| < \aleph_0$ $(x \in B_2)$.

Then there is $B_3 \subset B_2$ such that the set $D = P'L_2(x)$ is constant for $x \in B_3$. By (39), there is $Q \subset B_3$ such that $[Q]^2 \subset K_1$. Then $[(P'-D)+Q]^2 \subset K_1$; $\omega_0 2 \in [K_1]$ which contradicts (37). Hence the Assumption 4 is false, i.e.

$$(46) \qquad \begin{gathered} \text{if} \quad \nu_0 < \omega_0; \qquad P' \subset P_{\nu_0}, \text{ then} \\ |\{x : x \in B_1;\ |P'L_2(x)| < \aleph_0\}| \leqq \aleph_0. \end{gathered}$$

To B_1 the same argument applies as to S, from (38) onwards. The only change we make is that, after (41), we apply (40) instead of (39), so that now the roles of K_1 and K_2 are interchanged. We find sets Q_ν, $B_2 \subset B_1$ such that, in analogy to (45), (46), the following statements are true.

$$(47) \qquad \begin{aligned} [Q_\nu]^2 \subset K_2; \quad Q_\nu &\subset L_{12}(B_2) && (\nu < \omega_0), \\ Q_\mu &\subset L_1(Q_\nu) && (\mu < \nu < \omega_0). \end{aligned}$$

$$(48) \qquad \begin{gathered} \text{If } \nu_0 < \omega_0;\ Q' \subset Q_{\nu_0}, \text{ then} \\ |\{x : x \in B_2;\ |Q'L_1(x)| < \aleph_0\}| \leqq \aleph_0. \end{gathered}$$

By Lemma 1, there is $B'_\nu \subset B_2$ $(\nu < \omega_0)$ such that $B'_\mu \subset L(B'_\nu)$ $(\mu < \nu < \omega_0)$. Let $P'_\nu \subset P_\nu$; $Q'_\nu \subset Q_\nu$ $(\nu < \omega_0)$. Then, by (46), (48), there are at most $\aleph_0$ elements $x \in B_2$ such that at least one of the relations

$$|P'_\nu L_2(x)| < \aleph_0; \qquad |Q'_\nu L_1(x)| < \aleph_0$$

holds. By using this result repeatedly we find elements x_λ $(\lambda < \omega_0)$ such that, for all $\nu < \omega_0$,

$$\begin{aligned} x_0 \in B'_0; &\qquad |P_\nu L_2(x_0)| = |Q_\nu L_1(x_0)| = \aleph_0, \\ x_1 \in B'_1; &\qquad |P_\nu L_2(x_0, x_1)| = |Q_\nu L_1(x_0, x_1)| = \aleph_0, \end{aligned}$$

generally, $x_\lambda \in B'_\lambda$;

$$|P_\nu L_2(x_0, \cdots, x_\lambda)| = |Q_\nu L_1(x_0, \cdots, x_\lambda)| = \aleph_0 \qquad (\nu, \lambda < \omega_0).$$

Since $\omega_0 \to (\omega_0)^2_3$, there is a number $\nu < 3$ and a sequence $\lambda_0 < \lambda_1 < \cdots$;

$\lambda_\rho < \omega_0$, such that $[\{x_{\lambda_0}, x_{\lambda_1}, \cdots\}_<]^2 \subset K_\nu$. By (38), $\nu \neq 0$. We can choose y_μ, z_μ such that, for $\mu < \omega_0$,

$$y_\mu \in P_\mu L_2(x_0, x_1, \cdots, x_{\lambda_{m-1}}); \qquad z_\mu \in Q_\mu L_1(x_0, \cdots, x_{\lambda_{m-1}}).$$

Put $X = \{x_{\lambda_\rho} : \rho < m\}$; $Y = \{y_\mu : \mu < \omega_0\}$; $Z = \{z_\mu : \mu < \omega_0\}$.

Case 1. $\nu = 1$. Then $[Z+X]^2 \subset K_1$; $\alpha \in [K_1]$.

Case 2. $\nu = 2$. Then $[Y+X]^2 \subset K_2$; $\alpha \in [K_2]$. In either case, a contradiction against (37) follows. This proves (27).

PROOF OF THEOREM 31, (28). If $[S]^2 = K_0 + K_1$ and if we put $\gamma = \omega_0 m$ then we have, by Theorem 32 (a), either (i) $\alpha \in [K_0]$ or (ii) $\beta \leqq \gamma \in [K_1]$ or (iii) $\beta \leqq \omega_0 m \leqq \omega_0 \gamma^* \notin [K_1]$. This proves (28).

PROOF OF THEOREM 31, (30). Let $[S]^2 = K_0 +' K_1$,

$$4 \notin [K_0]; \qquad \alpha \notin [K_1]. \tag{49}$$

We shall deduce a contradiction.

By Theorem 2, there is $n < \omega_0$ such that $n \rightarrow (m, m)^2$, and p such that

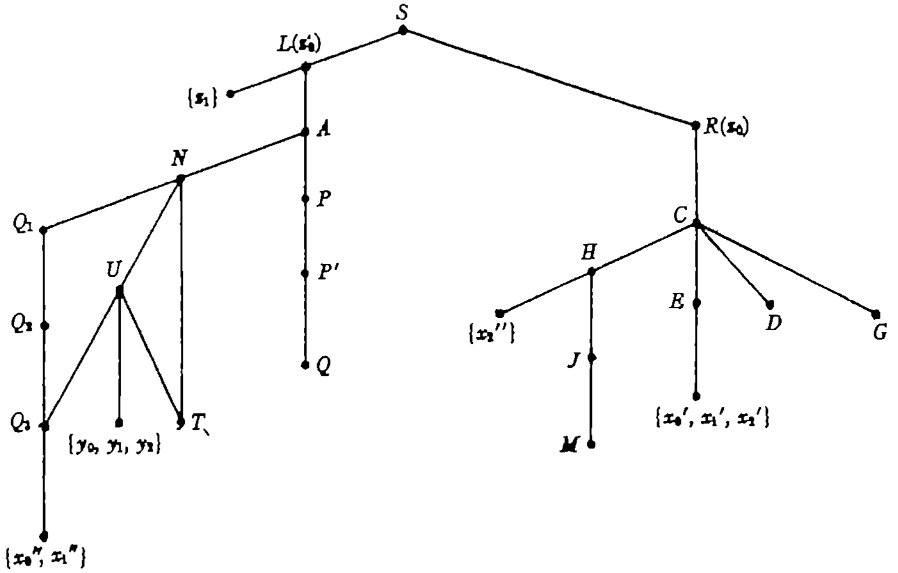
$$(n-1)(1 + m + m(m-1)/2) < p < \omega_0. \tag{50}$$

By Lemma 1, there is $z_0 \in S$ such that

$$|L(z_0)|, \ |R(z_0)| > \aleph_0$$

and then there is $C \subset R(z_0)$ such that $\overline{C} = \eta$.

The following diagram shows the relative position in S and the inclusion relations between the various sets to be considered in the argument that follows. It might be of help to the reader.



ASSUMPTION. If $D\in[C]^p$, then $|\prod[x_1, x_2\in D]\{x_0:x_0<z_0;\ \{x_0, x_1, x_2\}\notin K_0\}|\leqq\aleph_0$. Then there is $z_1<z_0$ such that

$$\text{(51)}\quad \text{if}\quad D\in[C]^p,\quad\text{then}\quad \{z_1, x_1, x_2\}\in K_0 \qquad \text{for some } x_1, x_2\in D.$$

Then $[C]^2=K_0'+K_1'$, where

$$K_\nu'=\{\{x_1, x_2\}:x_1, x_2\in C;\ \{z_1, x_1, x_2\}\in K_\nu\}\qquad(\nu<2).$$

By (11), $\overline{C}=\eta\geqq\omega_0p\rightarrow(\omega_0+m, p)^2$. Hence there are two cases.

Case 1. There is $E\subset C$ such that $\overline{E}=\omega_0+m$; $[E]^2\subset K_0'$. Then, since, by (49), $\overline{E}=\omega_0+m\notin[K_1]$, there are $x_0', x_1', x_2'\in E$ such that $\{x_0', x_1', x_2'\}\in K_0$. Then $[\{z_1, x_0', x_1', x_2'\}_<]^3\subset K_0$ which contradicts (49).

Case 2. There is $G\in[C]^p$ such that $[G]^2\subset K_1'$. Then $\{z_1, x_1, x_2\}\notin K_0$ for all $x_1, x_2\in G$, which is a contradiction against (51).

It follows that our assumption is false, and that there are $H\in[C]^p$ and $A\subset L(z_0)$ such that

$$\{x_0, x_1, x_2\}\notin K_0\qquad\text{for } x_0\in A;\ x_1, x_2\in H.$$

Put

$$V(x_0, x_1)=\{x_2:x_2\in H;\ \{x_0, x_1, x_2\}\in K_1\}\qquad\text{for } x_0, x_1\in A.$$

Then $[A]^2=K_0''+'K_1''$, where K_0'' is the set of all $\{x_0, x_1\}_<\subset A$ such that $|V(x_0, x_1)|\geqq n$. By (28), $\overline{A}\rightarrow(\omega_0, \omega_0+m)^2$. Hence there are two cases.

Case 1. There is $P\subset A$ such that $[P]^2\subset K_0''$. Then

$$|V(x_0, x_1)|\geqq n\qquad\text{for } \{x_0, x_1\}_<\subset P,$$

$$[P]^2=\sum[W\subset H]K_W^{(2)},$$

where $K_W^{(2)}=\{\{x_0, x_1\}_<:x_0, x_1\in P;\ V(x_0, x_1)=W\}$. The number k of sets W is finite, and $\omega_0\rightarrow(\omega_0)_k^2$, by Theorem 1. Hence there are $P'\subset P$; $J\subset H$ such that $[P']^2\subset K_J^{(2)}$,

$$V(x_0, x_1)=J\qquad\text{for } \{x_0, x_1\}_<\subset P'.$$

Then

$$|J|\geqq n,$$

$$\{x_0, x_1, x_2\}\in K_1\qquad\text{for } \{x_0, x_1\}_<\subset P';\ x_2\in J.$$

Since $[P']^3\subset K_0+K_1$ and, by Theorem 1, $\omega_0\rightarrow(\omega_0, \omega_0)^3$, there are $Q\subset P'$; $\nu<2$ such that $[Q]^3\subset K_\nu$. By (49), $\omega_0\notin[K_0]$. Hence $\nu=1$; $[Q]^3\subset K_1$.

Furthermore, $[J]^3\subset K_0+K_1$; $\overline{J}=n\rightarrow(m, m)^3$. Hence there are

$M \in [J]^m$; $\rho < 2$ such that $[M]^3 \subset K_\rho$. Since $m \geq 4 \notin [K_0]$, we have $\rho = 1$. Then, in view of $Q \subset P' \subset P \subset A$; $M \subset J \subset H$,

$$[Q + M]^3 \subset K_1; \qquad \omega_0 + m = \overline{Q + M} \in [K_1]$$

which contradicts (49).

Case 2. There is $N \subset A$ such that $\overline{N} = \omega_0 + m$; $[N]^3 \subset K_1''$. Then

$$|V(x_0, x_1)| \leq n - 1 \qquad \text{for } x_0, x_1 \in N.$$

Then

$$N = Q_1 + T; \qquad Q_1 \subset L(T); \qquad |T| = m.$$

We have $[Q_1]^2 = \sum' [\kappa < k_1] K_\kappa^{(1)}$, where $K_\kappa^{(1)} \neq 0$ $(\kappa < k_1)$, and two elements Z_0, Z_1 of $[Q_1]^2$ belong to the same $K_\kappa^{(1)}$ if, and only if, for every $x_2 \in H$, the sets $Z_0 + \{x_2\}$ and $Z_1 + \{x_2\}$ belong to the same class K_ν. Then $k_1 < \omega_0$ and, by $\omega_0 \rightarrow (\omega_0)^2_{k_1}$, there are $Q_2 \subset Q_1$; $\kappa_2 < k_1$ such that $[Q_2]^2 \subset K_{\kappa_2}^{(1)}$. This means that, for some $\rho(x_2) < 2$,

$$\{x_0, x_1, x_2\} \in K_{\rho(x_2)} \qquad \text{for } \{x_0, x_1\}_< \subset Q_2;\ x_2 \in H.$$

Similarly, we have

$$Q_2 = \sum{}' [\kappa < k_2] K_\kappa^{(2)}, \quad \text{where} \quad K_\kappa^{(2)} \neq 0 \quad (\kappa < k_2),$$

and two elements x_{00} and x_{01} of Q_2 belong to the same $K_\kappa^{(2)}$ if, and only if, for every $x_1 \in T$; $x_2 \in H$, the two sets $\{x_{00}, x_1, x_2\}$ and $\{x_{01}, x_1, x_2\}$ belong to the same class K_ν. Then $k_2 < \omega_0$ and, by $\omega_0 \rightarrow (\omega_0)^1_{k_2}$, there are $Q_3 \subset Q_2$; $\kappa_3 < k_2$ such that $Q_3 \subset K_{\kappa_3}^{(2)}$. This means that, for some $\sigma(x_1, x_2) < 2$,

$$\{x_0, x_1, x_2\} \in K_{\sigma(x_1, x_2)} \qquad \text{for } x_0 \in Q_3;\ x_1 \in T;\ x_2 \in H.$$

Put $U = Q_3 + T$, and choose $\{x_0'', x_1''\}_< \subset Q_3$.

Consider any $x_2 \in H$ such that there are $x_0, x_1 \in U$ satisfying $\{x_0, x_1, x_2\}_< \in K_1$. If $x_0, x_1 \in Q_3$, then $\rho(x_2) = 1$; $\{x_0'', x_1'', x_2\} \in K_1$; $x_2 \in V(x_0'', x_1'')$. If $x_0 \in Q_3$; $x_1 \in T$, then $\sigma(x_1, x_2) = 1$; $\{x_0'', x_1, x_2\} \in K_1$; $x_2 \in V(x_0'', x_1)$. If $x_0, x_1 \in T$, then $\{x_0, x_1, x_2\} \in K_1$; $x_2 \in V(x_0, x_1)$. Hence, in any case,

$$x_2 \in V(x_0'', x_1'') + \sum [x \in T) V(x_0'', x) + \sum [\{x, y\}_< \subset T] V(x, y)$$

and therefore, in view of the definition of x_2 and the relations $|T| = m$ and (50),

$$\left| \sum [x_0, x_1 \in U] \{x_2 : x_2 \in H;\ \{x_0, x_1, x_2\}_< \in K_1\} \right|$$
$$\leq (n - 1) + (n - 1)\binom{m}{1} + (n - 1)\binom{m}{2} < p = |H|.$$

We deduce the existence of $x_2'' \in H$ such that

$$\{x_0, x_1, x_2''\} \notin K_1 \qquad \text{for all } x_0, x_1 \in U.$$

Since $\overline{U} = \omega_0 + m \notin [K_1]$, there are $y_0, y_1, y_2 \in U$ such that $\{y_0, y_1, y_2\} \in K_0$. But then $[\{y_0, y_1, y_2, x_2''\}_{\neq}]^3 \subset K_0$ which contradicts (49). This proves (30) and thus completes the proof of Theorems 31 and 32.

THEOREM 33. *Let* $\alpha < \omega_0 2$. *Then* $\omega_1 \to (\alpha, \alpha)^2$.

PROOF. Let $\overline{S} = \omega_1$; $[S]^2 = K_0 +' K_1$; $2 \leqq m < \omega_0$; $\alpha \leqq \omega_0 + m$,

$$(52) \qquad \alpha \notin [K_\nu] \qquad (\nu < 2).$$

We have to deduce a contradiction. Let the conventions concerning the use of the letters A, B, P, Q be the same as in the proofs of Theorems 31 and 32. Choose any P.

ASSUMPTION. Let $[P]^2 \subset K_0$. Suppose that, if $P' \subset P$, then there is A such that

$$|P' L_0(x)| = \aleph_0 \qquad (x \in A).$$

Then we define x_ν, P_ν $(\nu < \omega_1)$ as follows. There is x_0 such that $|PL_0(x_0)| = \aleph_0$. Put $P_0 = PL_0(x_0)$. Now let $0 < \nu_0 < \omega_1$, and suppose that $x_\nu \in S$; $P_\nu \subset PL_0(x_\nu)$ $(\nu < \nu_0)$;

$$|P_\nu - P_\mu| < \aleph_0 \qquad (\mu < \nu < \nu_0).$$

Then we can write $[0, \nu_0] = \{\mu_\lambda : \lambda < \omega_0\}$. We can choose elements $y_\lambda (\lambda < \omega_0)$ such that $y_\lambda \in P_{\mu_0} P_{\mu_1} \cdots P_{\mu_\lambda} - \{y_\rho : \rho < \lambda\}$ $(\lambda < \omega_0)$. Put $P' = \{y_\lambda : \lambda < \omega_0\}$. Then, by our assumption, there is A such that $|P' L_0(x)| = \aleph_0$ $(x \in A)$. We can choose

$$x_{\nu_0} \in A - \sum [\nu < \nu_0](\{x_\nu\} + L(x_\nu)).$$

We put $P_{\nu_0} = P' L_0(x_{\nu_0})$. If, now, $\nu_1 < \nu_0$, then there is $\lambda < \omega_0$ such that $\nu_1 = \mu_\lambda$. Then

$$|P_{\nu_0} - P_{\nu_1}| \leqq |\{y_0, y_1, \cdots\} - P_{\mu_\lambda}| < \aleph_0.$$

Also, $P_{\nu_0} \subset PL_0(x_{\nu_0})$. This completes the inductive definition of x_ν, $P_\nu (\nu < \omega_1)$ such that

$$P_\mu \subset PL_0(x_\mu); \qquad |P_\nu - P_\mu| < \aleph_0 \qquad (\mu < \nu < \omega_1).$$

Put $X = \{x_\nu : \nu < \omega_1\}$. Then, by Theorem 23, $\overline{X} = \omega_1 > \omega_0 m \to (m, \omega_0 + m)^2$. Since, by (52), $\omega_0 + m \notin F_1(X)$, we have $m \in F_0(X)$, and there is $D \in [X]^m$ such that $[D]^2 \subset K_0$. Let $x_\rho = \max [x \in D] x$. Then, for any $x_\nu \in D$, $|P_\rho - L_0(x_\nu)| \leqq |P_\rho - P_\nu| + |P_\nu - L_0(x_\nu)| < \aleph_0$. Hence we may put $Q = P_\rho L_0(D)$, and then we have $\overline{Q + D} = \omega_0 + m \geqq \alpha$; $[Q + D]^2$

$\subset K_0$. This is a contradiction against (52). Therefore our assumption is false.

Now let $A \subset S$. Then, by Theorem 3, $|A| = \aleph_1 \to (\aleph_0, \aleph_1)^2$ and hence, by Theorem 14, $\overline{A} = \omega_1 \to (\omega_0, \omega_1)^2$. Since $\omega_1 \notin F_1(A)$, we conclude that $\omega_0 \in F_0(A)$, so that there is $P \subset A$ such that $[P]^2 \subset K_0$. As the assumption made above is false, there is $P' \subset P$ such that there are at most $\aleph_0$ elements x such that $|P'L_0(x)| = \aleph_0$. Then there is $A' \subset A$ such that $|P'L_0(x)| < \aleph_0$ $(x \in A')$. Hence there are at most $\aleph_0$ distinct sets $P'L_0(x)$ for varying values $x \in A'$, and there is $A'' \subset A'$ and E such that

$$P'L_0(x) = E \qquad (x \in A'').$$

Since $|E| < \aleph_0$, we may put $P'' = P' - E$, and since $\overline{A''} = \omega_1$; $\overline{P''} = \omega_0$, we may put $A''' = A''R(P'')$. Now let $x \in A'''$; $y \in P''$. Then

$$x \in A''' \subset A''; \qquad y \notin E = P'L_0(x); \qquad y \notin L_0(x).$$

Also,

$$x \in A''' \subset R(P'') \subset R(y); x > y.$$

Hence

$$y \in L_1(x); \qquad P'' \subset L_1(A''').$$

So far we have proved that, given any A, there are sets P''', $A'' \subset A$ such that $P'' \subset L_1(A''')$; $[P'']^2 \subset K_0$, and, moreover, there are at most $\aleph_0$ elements x such that $|P''L_0(x)| = \aleph_0$.

By applying the last result repeatedly, starting with $A = S$, we obtain sets $P_\nu (\nu < \omega_0)$ such that

$$[P_\mu]^2 \subset K_0; \qquad P_\mu \subset L_1(P_\nu) \qquad (\mu < \nu < \omega_0).$$

There is Q_ν such that

$$|P_\nu L_0(x)| < \aleph_0 \qquad (\nu < \omega_0; x \in S - Q_\nu).$$

We can choose $B \subset S - \sum[\nu < \omega_0]Q_\nu$ such that $P_\nu \subset L(B)$ $(\nu < \omega_0)$. Then, by Theorem 23, $\overline{B} = \omega_1 > \omega_0 m \to (\omega_0 + m, m)^2$;

$$\omega_0 + m \notin F_0(B); \qquad m \in F_1(B),$$

and there is $D \in [B]^m$ such that $[D]^2 \subset K_1$. Then, for every $\nu < \omega_0$, $|\sum[x \in D]P_\nu L_0(x)| < \aleph_0$. Therefore we can choose $y_\nu \in P_\nu L_1(D)$ $(\nu < \omega_0)$. If we put $Q = \{y_\nu : \nu < \omega_0\}$ then $\overline{Q + D} = \omega_0 + m \geqq \alpha$; $[Q + D]^2 \subset K_1$. This contradiction against (52) completes the proof of Theorem 33.

The next theorem, while perhaps appearing to be of a rather special and complicated nature, is of interest in that it implies Theorem 7 (i) and Theorem 8. It may well be capable of further worthwhile applications.

THEOREM 34. *Let* α, β, γ *be ordinals, and* $\alpha \nrightarrow (\beta, \gamma)^2$. *Then there are ordinals* α_λ $(\lambda < \beta^-)$ *such that, if*

$$|k_\mu| = \prod[\lambda < \mu] \, |\alpha_\lambda| \qquad (\mu < \beta^-),$$

then

$$\alpha \nrightarrow (\alpha_0 + 1, \alpha_1 + 1, \cdots)^1_{\beta^-}; \qquad \alpha_\lambda \nrightarrow (\gamma)^1_{k_\lambda} \qquad (\lambda < \beta^-).$$

We begin by deducing (i) of Theorem 7 or, rather, a slightly stronger proposition, from Theorem 34.

COROLLARY 1. *Let* m *and* n *be such that* $\aleph_n^d \leq \aleph_n$ $(d < \aleph_m)$. *Then* $\omega_{n+1} \rightarrow (\omega_m + 1, \omega_{n+1})^2$.

This implies, a fortiori, $\omega_{n+1} \rightarrow (\omega_m, \omega_{n+1})^2$ which, in its turn, by Theorem 14, is equivalent to Theorem 7 (i).

Deduction of Corollary 1 *from Theorem* 34. Let us suppose that $\omega_{n+1} \nrightarrow (\omega_m + 1, \omega_{n+1})^2$. Then, by Theorem 34, there are ordinals α_λ, k_λ such that $|k_\mu| = \prod[\lambda < \mu] \, |\alpha_\lambda| \; (\mu < \omega_m)$;

$$\omega_{n+1} \nrightarrow (\alpha_0 + 1, \alpha_1 + 1, \cdots)^1_{\omega_m}, \tag{53}$$

$$\alpha_\mu \nrightarrow (\omega_{n+1})^1_{k_\mu} \qquad (\mu < \omega_m). \tag{54}$$

Then, for $\lambda < \omega_m$,

$$\alpha_\lambda < \omega_{n+1}. \tag{55}$$

For, let $\mu < \omega_m$, and suppose that (55) holds for $\lambda < \mu$. Then, using $|\mu| < \aleph_m$ and the hypothesis, we find that $|k_\mu| \leq \aleph_n^{|\mu|} \leq \aleph_n$. Now, by (54), we can write $|\alpha_\mu| = \sum[\nu < k_\mu] \, |\rho_\nu|$, where $\rho_\nu < \omega_{n+1}$ $(\nu < k_\mu)$. Hence $|\alpha_\mu| \leq \aleph_n |k_\mu| \leq \aleph_n$. This proves (55) for all $\lambda < \omega_m$. Now, by (53) and the obvious relation $m \leq n$,

$$|\omega_{n+1}| \leq \sum[\lambda < \omega_m] \, |\alpha_\lambda| \leq \aleph_n \aleph_m = \aleph_n$$

which is the required contradiction.

COROLLARY 2. *Let* $\aleph_n' = \aleph_n$; $2^{\aleph_\nu} < \aleph_n$ *for all* $\nu < n$. *Then*

$$\omega_n \rightarrow (\beta, \omega_n)^2 \qquad (\beta < \omega_n).$$

By Theorem 14, this proposition implies Theorem 8.

Deduction of Corollary 2 *from Theorem* 34. Let $\beta < \omega_n$. Suppose that $\omega_n \nrightarrow (\beta, \omega_n)^2$. Then, by Theorem 34, there are ordinals α_λ, k_λ such that $|k_\mu| = \prod[\lambda < \mu] \, |\alpha_\lambda| \; (\mu < \beta^-)$;

$$\omega_n \nrightarrow (\alpha_0 + 1, \cdots)^1_{\beta^-}; \qquad \alpha_\mu \nrightarrow (\omega_n)^1_{k_\mu} \qquad (\mu < \beta^-).$$

Let us assume that, for some $\mu<\beta^-$, we have $\alpha_\lambda<\omega_n$ $(\lambda<\mu)$. Then, putting $d=\sum[\lambda<\mu]|\alpha_\lambda|$, $d<\aleph_n$; $|k_\mu|\leqq d^{|\mu|}\leqq 2^{d|\mu|}<\aleph_n$; $|\alpha_\mu| = \sum[\nu<k_\mu]|\rho_\nu|$, where $\rho_\nu<\omega_n$. Then $|\alpha_\mu|<\aleph_n$. This proves, by induction, that $\alpha_\lambda<\omega_n$ $(\lambda<\beta^-)$. Now, $|\omega_n| = \sum[\lambda<\beta^-]|\sigma_\lambda|$, where $|\sigma_\lambda|\leqq|\alpha_\lambda|$. Therefore $|\omega_n|<\aleph_n$ which is the required contradiction.

The proof of Theorem 34 depends on a lemma.

LEMMA 2. *Let T be a well ordered set, and $[T]^2=K_0+K_1$. Then there is*[8] *a set $B=B(T)\subset T$ which has the following properties. We have $[B]^2\subset K_1$. If $x\in T-B$, then there is $y\in B$ such that $\{y, x\}_<\in K_0$.*

PROOF. We may assume $T\neq 0$. Choose l such that $|l|>|T|$. We define, inductively, y_λ $(\lambda<l)$ as follows. Let $\mu<l$; $y_\lambda\in T$ $(\lambda<\mu)$.

Case 1. There is $y\in T$ such that $\{y_\lambda, y\}\in K_1$ $(\lambda<\mu)$. Then we take as y_μ the first element y of this kind.

Case 2. If $y\in T$, then there is $\lambda<\mu$ such that $\{y_\lambda, y\}\notin K_1$. Then we have $\mu>0$. We put $y_\mu=y_0$.

Let $B=\{y_\lambda:\lambda<l\}$. Then there is $m<l$ such that

$$B = \{y_\lambda:\lambda < m\}; \qquad \{y_\lambda, y_\mu\}_< \in K_1 \quad (\lambda < \mu < m).$$

For, m is the least μ such that $0<\mu<l$; $y_\mu=y_0$. We have $[B]^2\subset K_1$. Now let $x\in T-B$. Then, by definition of y_m, there is a least $\mu<m$ such that $\{y_\mu, x\}\in K_0$. Then $\{y_\lambda, x\}\in K_1$ $(\lambda<\mu)$ and hence, by definition of y_μ, $x>y_\mu$. This proves Lemma 2.

PROOF OF THEOREM 34. There is an ordered set S such that

$$\overline{S} = \alpha; \qquad [S]^2 = K_0 + K_1; \qquad \beta \notin [K_0]; \qquad \gamma \notin [K_1].$$

We choose an ordinal ρ such that $|\rho|>|\alpha|$. Let $x\in S$. We define $f_\mu(x)$ $(\mu<\rho)$ as follows. Let $\nu<\rho$, and suppose that

$$f_\mu(x) \in S \ (\mu < \nu),$$
$$\{ f_\mu(x), x\}_< \in K_0 \quad \text{if} \quad \mu < \nu; \ f_\mu(x) \neq x.$$

Then we define $f_\nu(x)$ by the following rule. If $f_\mu(x)=x$ for some $\mu<\nu$, then put $f_\nu(x)=x$. Now let $f_\mu(x)\neq x$ $(\mu<\nu)$. Let T be the set of all $y\in S$ such that $\{f_\mu(x), y\}_<\in K_0$ $(\mu<\nu)$. Then $x\in T$. Let $B=B(T)$ be the set given in Lemma 2. Then $B\subset T$; $[B]^2\subset K_1$; $\overline{B}<\gamma$. If $x\in B$, then put $f_\nu(x)=x$. Now let $x\notin B$. Then, by Lemma 2, there is a first element $z\in B$ such that $\{z, x\}_<\in K_0$. We put $f_\nu(x)=z$. We have now defined $f_\nu(x)$ for $\nu<\rho$; $x\in S$, and we have

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[8] In fact, there is exactly one such set.

$$\{f_\mu(x), f_\nu(x)\}_< \in K_0 \qquad (\mu < \nu < \rho; f_\mu(x) \neq x);$$
$$f_\nu(x) \leqq x \qquad (\nu < \rho; x \in S).$$

If, for some $x, f_\nu(x) < x$ $(\nu < \rho)$, then the contradiction

$$|\rho| = |\{f_\nu(x) : \nu < \rho\}| \leqq |S| = |\alpha|$$

follows. Hence, given $x \in S$, there is $\sigma(x) < \rho$ such that

$$f_\nu(x) < x \quad (\nu < \sigma(x)); \qquad f_{\sigma(x)}(x) = x \quad (x \in S).$$

Then, for fixed x, $[\{f_\nu(x) : \nu \leqq \sigma(x)\}]^2 \subset K_0$,

$$\sigma(x) + 1 < \beta; \qquad \sigma(x) < \beta^-.$$

Put $M_\nu = \{f_\nu(x) : \sigma(x) \geqq \nu\}$ $(\nu < \rho)$. Then $M_0 \subset B(S)$; $\overline{M}_0 < \gamma$;

$$S = \sum[\nu < \beta^-]M_\nu; \qquad \alpha \nrightarrow (\overline{M}_0 + 1, \overline{M}_1 + 1, \cdots)^1_{\beta^-}.$$

Let $0 < \nu < \beta^-$. Then

$$M_\nu = \sum[y_\mu \in M_\mu \text{ for } \mu < \nu]\{f_\nu(x) : \sigma(x) \geqq \nu; f_\lambda(x) = y_\lambda \text{ for } \lambda < \nu\}.$$

Now, for every choice of $y_\mu \in M_\mu$ $(\mu < \nu)$, the corresponding term in the last sum is a set contained in some set of the form $B(T)$. In order to see this, consider an element x such that $\sigma(x) \geqq \nu; f_\mu(x) = y_\mu$ $(\mu < \nu)$. We shall have $f_\mu(x) < x$ $(\mu < \sigma(x))$ and hence $f_\mu(x) < x$ $(\mu < \nu)$. Let T be the set used above in the inductive definition of $f_\nu(x)$, i.e. the set of all $y \in S$ such that $\{f_\mu(x), y\}_< \in K_0$ $(\mu < \nu)$. Then, by definition of $f_\nu(x)$, $x \in T$ and either

$$x \in B(T); \qquad f_\nu(x) = x \in B(T)$$

or

$$x \notin B(T); \qquad f_\nu(x) = z \in B(T).$$

In either case, $f_\nu(x) \in B(T)$. In fact, the set T does not depend on x since T is the set of all $y \in S$ such that

$$\{y_\mu, y\}_< \in K_0 \qquad (\mu < \nu).$$

All this proves that, given $y_\mu \in M_\mu$ $(\mu < \nu)$, there is T such that, whenever

$$x \in S; \qquad \sigma(x) \geqq \nu; \qquad f_\mu(x) = y_\mu \qquad (\mu < \nu),$$

then

$$f_\nu(x) \in B(T).$$

By definition of $B(T)$, we have $[B(T)]^2 \subset K_1$ and therefore $\overline{B(T)} < \gamma$.

Hence M_ν is a sum of $\prod[\mu<\nu]|M_\mu|$ sets each of type less than γ. This shows that $\bar{M}_\nu \nrightarrow (\gamma)^1_{k_\nu}$, when k_ν is any ordinal such that $|k_\nu| = \prod[\mu<\nu]|M_\mu|$. It now follows that the conclusion of Theorem 34 holds for $\alpha_\lambda = \bar{M}_\lambda$ $(\lambda<\beta^-)$.

THEOREM 35. *Suppose that* $\beta \geqq r \geqq 3$; $\beta, \beta^* \not\leqq \alpha$; $s>(r-1)^2$. *Then, for any type* ϕ *such that* $|\phi| = |\alpha|$,

$$\phi \nrightarrow (s, \beta)^r. \tag{56}$$

COROLLARY. *If* $r \geqq 3$; $s>(r-1)^2$, *then*

$$\eta \nrightarrow (s, \omega_0 + 1)^r, \tag{57}$$

$$\phi \nrightarrow (s, \omega_1)^r, \tag{58}$$

where ϕ *is any type such that* $|\phi| = |\lambda|$.

The negative results (57) and (58) are not too far from the ultimate truth as is seen by comparing them with the following positive results. By Theorem 1,

$$\omega_0 \rightarrow (\omega_0, \omega_0, \cdots, \omega_0)^r_k \qquad (k < \omega_0). \tag{59}$$

By Theorem 31,

$$\phi \rightarrow (\omega_0, \gamma)^2 \qquad (\gamma < \omega_1), \tag{60}$$

where ϕ is any type such that $|\phi| > \aleph_0$; $\omega_1, \omega_1^* \not\leqq \phi$.

PROOF OF THEOREM 35. The corollary follows by applying the theorem to the following two cases.

(i) $\beta = \omega_0+1$; $\alpha = \omega_0$; $\phi = \eta$,

(ii) $\beta = \omega_1$; $\alpha = \lambda$.

The proof of the theorem depends on the following lemma due to Erdős and Szekeres [7]. Throughout, we put

$$s = (r-1)^2 + 1.$$

LEMMA 3. *If* S *is an ordered set,* $r>0$, *and if* $x(\sigma) \in S$ $(\sigma<s)$, *then there is* $\{\sigma_0, \sigma_1, \cdots, \sigma_{r-1}\}_< \subset [0, s]$ *such that either*

$$x(\sigma_\rho) \leqq x(\sigma_{\rho+1}) \qquad (\rho+1<r)$$

or

$$x(\sigma_\rho) \geqq x(\sigma_{\rho+1}) \qquad (\rho+1<r).$$

We now prove the theorem. Let $\bar{S}_< = \phi$; $\bar{S}_{<<} = \alpha$. Then

$$[S]^r = K_0 + {}'K_1; \qquad K_1 = K_{10} + K_{11},$$

where

$$K_{10} = \{\{x_0, \cdots, x_{r-1}\}_<: \{x_0, \cdots, x_{r-1}\}_{<<} \subset S\},$$
$$K_{11} = \{\{x_0, \cdots, x_{r-1}\}_<: \{x_0, \cdots, x_{r-1}\}_{>>} \subset S\}.$$

Case 1. There is $A \in [S]^s$ such that $[A]^r \subset K_0$. Then, if $A = \{z(\sigma): \sigma < s\}$, an application of Lemma 3 shows the existence of $B \in [A]^r$ such that $B \in K_1$, which is a contradiction.

Case 2. There is $A \subset S$ such that $\overline{A}_< = \beta$; $[A]^r \subset K_1$. We shall prove that one of the two relations

$$[A]^r \subset K_{10}, \tag{61}$$
$$[A]^r \subset K_{11} \tag{62}$$

holds. If both (61) and (62) are false, then there are sets X, $Y \subset A$ such that

$$X = \{x_0, \cdots, x_{r-1}\}_< = \{x_0, \cdots, x_{r-1}\}_{<<}, \tag{63}$$
$$Y = \{y_0, \cdots, y_{r-1}\}_< = \{y_0, \cdots, y_{r-1}\}_{>>}. \tag{64}$$

Then there is $\sigma < r$ such that $x_\rho = y_\rho$ $(\rho < \sigma)$; $x_\sigma \neq y_\sigma$. We choose X and Y such that σ is as large as possible. We may assume that $x_\sigma < y_\sigma$. Then, if we suppose that $\sigma + 1 < r$, we find that $\{y_0, \cdots, y_{\sigma-1}, x_\sigma, y_\sigma, \cdots, y_{r-2}\}_<$ and therefore, by the maximum property of σ, $\{y_0, \cdots, y_{\sigma-1}, x_\sigma, y_\sigma, \cdots, y_{r-2}\}_{<<}$. Hence $\{y_0, \cdots, y_{\sigma-1}, y_\sigma, \cdots, y_{r-2}\}_{<<}$ and therefore, since $r-2>0$, $y_0 \ll y_{r-2}$ which contradicts (64). Therefore $\sigma+1=r$, and $x_0 = y_0 \gg y_1 = x_1$. But this contradicts (63). This shows that at least one of the relations (61), (62) holds. Now (61) implies $\beta = \overline{A}_< = \overline{A}_{<<} \leqq \overline{S}_{<<} = \alpha$, and (62) implies $\beta^* = \overline{A}_> = \overline{A}_{<<} \leqq \overline{S}_{<<} = \alpha$. Both conclusions contradict the hypothesis. We have proved that neither Case 1 nor Case 2 is possible, so that (56) is established. This proves Theorem 35.

THEOREM 36. (i) *If* $a_\nu < d$ $(\nu < n)$, *then* $\sum[\nu < n]a_\nu \nrightarrow (|n|^+, d)^2$.
(ii) $a \nrightarrow (a'^+, a)^2$ *for* $a > 1$.
(iii) $ab \nrightarrow (a^+, b^+)^2$ *for any* a, b.
(iv) $\aleph_n \nrightarrow (|n|^+, \aleph_n)^2$, *if* $n = n^- > 0$.

PROOF OF (i). Let $|A_\nu| = a_\nu (\nu < n)$; $S = \sum' [\nu < n]A_\nu$;

$$[S]^2 = K_0 + {}'K_1; \qquad K_1 = \sum[\nu < n][A_\nu]^2.$$

Then $[X]^2 \subset K_0$ implies $|XA_\nu| \leqq 1$ $(\nu < n)$; $|X| \leqq |n|$, and $[X]^2 \subset K_1$ implies the existence of $\nu < n$ such that $X \subset A_\nu$; $|X| \leqq |A_\nu| < d$. This proves (i).

PROOF OF (ii). By definition of a', the hypothesis of (i) holds for some a_ν, n, with $|n| = a'$; $d = a = \sum a_\nu$. Hence (ii) follows from (i).

PROOF OF (iii). Let $|n| = a$; $a_\nu = b(\nu < n)$. Then, by (i),

$$ab = \sum a_\nu \nrightarrow (a^+, b^+)^2.$$

PROOF OF (iv). Since $\aleph_n = \sum [\nu < n] \aleph_\nu$, (iv) follows from (i). This proves Theorem 36.

THEOREM 37. (i) *Let* $a \geqq \aleph_0$, *and let* b *be minimal such that* $a^b > a$. *Let* $a < \aleph_k' \leqq \aleph_k \leqq a^b$. *Then*

$$\aleph_{\omega_k} \nrightarrow (b^+, \aleph_{\omega_k})^2. \tag{65}$$

A possible value for $\aleph_k$ *is* a^+.

(ii) $\aleph_{\omega_{m+1}} \nrightarrow (\aleph_{m+1}, \aleph_{\omega_{m+1}})^2$ *for all* m.

(iii) *If* $\aleph_0 < \aleph_k' \leqq \aleph_k \leqq 2^{\aleph_0}$, *then* $\aleph_{\omega_k} \nrightarrow (\aleph_1, \aleph_{\omega_k})^2$. *A possible value is* $k = 1$.

Deduction of (ii) *from* (i). Let $a = \aleph_m$, and let b be minimal such that $a^b > a$. Then, if $k = m+1$, we have $a < \aleph_k' = \aleph_k \leqq a^b$ and therefore, by (i), $\aleph_{\omega_{m+1}} \nrightarrow (b^+, \aleph_{\omega_{m+1}})^2$. This implies (ii).

Deduction of (iii) *from* (i). Put, in (i), $a = b = \aleph_0$. Then (iii) follows.

Before proving (i) we establish a lemma. For the sake of further applications later on the lemma is more general than is needed for the present purpose.

LEMMA 4. *If* $s \geqq 2$; $\beta_0, \beta_1^* \nleqq \alpha_0$; $|\alpha_0| = |\alpha_1|$, *then*

$$\alpha_1 \nrightarrow (\beta_0, \beta_1, s+1, s+1, \cdots, s+1)^s_{s!}.$$

PROOF. Let $\bar{S}_< = \alpha_1$; $\bar{S}_{<<} = \alpha_0$. Then to every set $X \in [S]^s$ there belongs a permutation $\pi(X): \lambda \to \sigma(\lambda)$ defined by

$$X = \{x_0, x_1, \cdots, x_{s-1}\}_< = \{x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(s-1)}\}_{<<}.$$

Let π_λ $(\lambda < s!)$ be all permutations of $[0, s]$ and, in particular,

$$\pi_0: \lambda \to \lambda; \qquad \pi_1: \lambda \to s - 1 - \lambda \qquad (\lambda < s).$$

Then $[S]^s = \sum [\nu < s!] K_\nu$, where $K_\nu = \{X : X \in [S]^s;\ \pi(X) = \pi_\nu\}$. Now suppose that $A \subset S$; $\nu < s!$; $[A]^s \subset K_\nu$. We shall deduce a contradiction in each of the three cases that follow and so establish the lemma.

Case 1. $\nu = 0$; $\bar{A}_< = \beta_0$. Then the contradiction $\beta_0 = \bar{A}_{<<} \leqq \bar{S}_{<<} = \alpha_0$ follows.

Case 2. $\nu = 1$; $\bar{A}_< = \beta_1$. Then the contradiction $\beta_1^* = \bar{A}_{<<} \leqq \bar{S}_{<<} = \alpha_0$ follows.

Case 3. $2 \leqq \nu < s!$; $\bar{A}_< = s+1$. Let $\pi_\nu: \lambda \to \sigma(\lambda)$. Then $A = \{x_0, x_1, \cdots, x_s\}_<$ and therefore, putting $y_\lambda = x_{1+\lambda}$ $(\lambda < s)$, we have

$$(66)\qquad \{x_0, x_1, \cdots, x_{s-1}\}_< = \{x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(s-1)}\}_{<<},$$

$$\begin{aligned} \{x_1, x_2, \cdots, x_s\}_< &= \{y_0, y_1, \cdots, y_{s-1}\}_< \\ (67)\qquad &= \{y_{\sigma(0)}, y_{\sigma(1)}, \cdots, y_{\sigma(s-1)}\}_{<<} \\ &= \{x_{1+\sigma(0)}, x_{1+\sigma(1)}, \cdots, x_{1+\sigma(s-1)}\}_{<<}. \end{aligned}$$

If $x_0 \ll x_1$, then alternate applications of (66) and (67) lead to $x_0 \ll x_1 \ll x_2 \ll \cdots \ll x_s$ and so to the contradiction $\pi_s = \pi_0$, while, similarly, the assumption $x_0 \gg x_1$ leads to the contradiction $\pi_s = \pi_1$. This proves the lemma.

PROOF OF THEOREM 37, (i). Let $a = \aleph_m$; $b = \aleph_l$, and let F be the set of all mappings $\lambda \rightarrow h(\lambda)$ of $[0, \omega_l]$ into $[0, \omega_m]$. We order F by putting, for $h_0, h_1 \in F$, $h_0 \ll h_1$ if, and only if, there is $\lambda_0 < \omega_l$ such that

$$h_0(\lambda) = h_1(\lambda) \quad \text{for} \quad \lambda < \lambda_0; \qquad h_0(\lambda_0) < h_1(\lambda_0).$$

Then $|F| = a^b$, and we have, by[8] Lemma 2 of [6], if $a = \aleph_m$; $b = \aleph_l$,

$$(68)\qquad \omega_{m+1}, \omega_{l+1}^* \nleq \bar{F}_{<<} = \bar{F}, \quad \text{say.}$$

We can choose a set $X \in [F]^{\aleph_k}$. Let $x \rightarrow f(x)$ be a one-one mapping of X on $[0, \omega_k]$, and $S = \{(x, \nu) : x \in X;\ \nu < \omega_{f(x)}\}$. We order S alphabetically, by means of a relation $u < v$, and put $\bar{S}_< = \bar{S} = \phi$. Then

$$|S| = \sum[x \in X]\aleph_{f(x)} \leqq \sum[\lambda < \omega_k]\aleph_{\omega_k} = \aleph_{\omega_k}.$$

On the other hand, if $d < \aleph_{\omega_k}$, then $d < \aleph_n$, for some $n < \omega_k$, and there is $x_0 \in X$ such that $f(x_0) > n$. Then $|S| \geqq \aleph_{f(x_0)} > d$. Hence

$$(69)\qquad |\phi| = |S| = \aleph_{\omega_k}.$$

1. Let $S_1 \subset S$, and suppose that $\bar{S}_1$ is an ordinal. Put $X_1 = \sum[\nu < \omega_k] \cdot \{x : (x, \nu) \in S_1\}$. Then $\bar{X}_1$ is an ordinal, and $\bar{X}_1 \leqq \bar{F}$. Hence, by (68), $\bar{X}_1 < \omega_{m+1}$; $|X_1| \leqq \aleph_m = a < \aleph_k'$.

$$\begin{aligned} \sum[x \in X_1]\,|f(x)| < \aleph_k; \qquad \nu_1 = \sum[x \in X_1]f(x) < \omega_k; \\ |S_1| = \sum[x \in X_1]\,|\{\nu : (x, \nu) \in S_1\}| \leqq \sum[x \in X_1]\aleph_{f(x)} \\ \leqq \aleph_{\nu_1}|X_1| \leqq \aleph_{\nu_1}\aleph_m < \aleph_{\omega_k}; \end{aligned}$$

$$(70)\qquad \omega_{\omega_k} \nleq \phi.$$

2. Let $S_2 \subset S$, and suppose that $(\bar{S}_2)^*$ is an ordinal. Put $X_2 = \sum[\nu < \omega_k]\{x : (x, \nu) \in S_2\}$. Then $(\bar{X}_2)^*$ is an ordinal, and $(\bar{X}_2)^* \leqq \bar{F}$. Hence, by (68), $(\bar{X}_2)^* < \omega_{l+1}$; $|X_2| \leqq \aleph_l$. Put, for $x \in X_2$, $N(x) = \{\nu : (x, \nu) \in S_2\}$. Then $\overline{N(x)}$ is an ordinal. On the other hand,

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[8] The authors are indebted to G. Kurepa for pointing out that the result of this lemma had already been obtained by F. Hausdorff, [9, Satz 14].

$(\overline{N(x)})^*$ is an ordinal, since $(\overline{N(x)})^* \leqq (\bar{S}_2)^*$. Hence $\overline{N(x)} < \omega_0$;

$$|S_2| = \sum [x \in X_2] |N(x)| \leqq \aleph_0 |X_2| \leqq \aleph_l; \qquad (\bar{S}_2)^* < \omega_{l+1};$$

$$\overset{*}{\omega}_{l+1} \nleqq \phi. \tag{71}$$

3. We now apply Lemma 4 to the case

$$s = 2; \qquad \alpha_0 = \phi; \qquad \alpha_1 = \omega_{\omega_b}; \qquad \beta_0 = \omega_{\omega_b}; \qquad \beta_1 = \omega_{l+1}.$$

Its hypotheses are satisfied, by (70), (71), (69). We obtain $\omega_{\omega_b} \nrightarrow (\omega_{\omega_b}, \omega_{l+1})^2$. This implies (65), by Theorem 14, and completes the proof of Theorem 37.

Remark. If $a \geqq \aleph_0$ and $\aleph_b = a^+$, then (i) of Theorem 37 yields a stronger result then (ii) of Theorem 36. For, first of all, we note that the hypothesis of (i) of Theorem 37 holds, since $a < a^+ = a^{+'} = \aleph_b' = \aleph_b \leqq a^b$. Hence the latter theorem gives

$$\aleph_{\omega_b} \nrightarrow (b^+, \aleph_{\omega_b})^2. \tag{72}$$

On the other hand, (ii) of Theorem 36 gives

$$\aleph_{\omega_b} \nrightarrow (\aleph_{\omega_b}^{\prime +}, \aleph_{\omega_b})^2. \tag{73}$$

It is known that, for any m,

$$\aleph_{\omega_m}' = \aleph_m'. \tag{74}$$

Hence $\aleph_{\omega_b}^{\prime +} > \aleph_{\omega_b}' = \aleph_b' = a^{+'} = a^+ \geqq b^+$, and (72) is stronger than (73).

Since we were not able to find a reference for (74) we give, for the sake of completeness, a proof now.

Case 1. Let $\aleph_{\omega_m}' = \aleph_n < \aleph_m'$. Then $\aleph_{\omega_m} = \sum [\nu < \omega_n] \aleph_{\lambda_\nu}$, for some $\lambda_\nu < \omega_m$. Put $\lambda = \sum [\nu < \omega_n] \lambda_\nu$. Then, since $|\omega_n| < \aleph_m'$; $|\lambda_\nu| < \aleph_m$ we conclude that

$$|\lambda| = \sum [\nu < \omega_n] |\lambda_\nu| < \aleph_m; \qquad \aleph_n < \aleph_m' \leqq \aleph_m \leqq \aleph_{\omega_m};$$

$$\aleph_{\omega_m} \leqq \sum [\nu < \omega_n] \aleph_\lambda = \aleph_\lambda \aleph_n < \aleph_{\omega_m}$$

which is the desired contradiction.

Case 2. Let $\aleph_{\omega_m}' > \aleph_n = \aleph_m'$. Then $m > 0$, and $\aleph_m = \sum [\nu < \omega_n] \aleph_{\lambda_\nu}$ for some $\lambda_\nu < m$. Then $\sum [\nu < \omega_n] \aleph_{\omega_{\lambda_\nu}} = \aleph_l$ for some $l < \omega_m$; $\aleph_{\lambda_\nu} = |\omega_{\lambda_\nu}| \leqq |l|$; $\aleph_m = \sum [\nu < \omega_n] \aleph_{\lambda_\nu} \leqq |l| \aleph_n$; $m \leqq n$; $\aleph_{\omega_m} = \sum [\mu < \omega_m] \aleph_\mu$; $\aleph_{\omega_m}' \leqq |\omega_m| \leqq \aleph_n$ which, again, is a contradiction. This proves (74).

Theorem 38. *If* $\aleph_m \nrightarrow (|\beta_0|, |\beta_1|, \cdots)_k^r$, *then*

$$\omega_{m+1} \nrightarrow (\beta_0 + 1, \beta_1 + 1, \cdots)_k^{r+1}. \tag{75}$$

We give some applications of this theorem.

(i) If $|\beta| = |\gamma| = \aleph_{m+1}$, then

$$\omega_{m+2} \nrightarrow (\beta + 1, \gamma + 1)^2. \tag{76}$$

For, let $a = \aleph_m$, and let b be minimal such that $a^b > a$. Then, by Theorem 7, $a^b \nrightarrow (a^+, b^+)^2$ and therefore $\aleph_{m+1} \nrightarrow (|\beta|, |\gamma|)^2$. Now (76) follows from Theorem 38.

(ii) If $\aleph_n' = \aleph_m$; $|\beta| = \aleph_{m+1}$; $|\gamma| = \aleph_{\omega_n}$, then

$$\omega_{\omega_n+1} \nrightarrow (\beta + 1, \gamma + 1)^2. \tag{77}$$

In order to prove (77), we apply Theorem 36 (ii) to $a = \aleph_{\omega_n}$. We note that, by (74), $a' = \aleph_n' = \aleph_m$; $a'^+ = \aleph_{m+1}$. Hence, by Theorem 36, $\aleph_{\omega_n} \nrightarrow (|\beta|, |\gamma|)^2$, and (77) follows from Theorem 38.

(iii) If $|\beta| = \aleph_{k+1}$; $|\gamma| = \aleph_{\omega_{k+1}}$, then

$$\omega_{\omega_{k+1}+1} \nrightarrow (\beta + 1, \gamma + 1)^2. \tag{78}$$

This follows immediately from Theorem 37 (ii) and an application of Theorem 38.

We note that on putting $n = k+1$ in (ii) above one obtains a result which is weaker than (78). For, (ii) becomes: if $\aleph_{k+1}' = \aleph_m$; $|\beta| = \aleph_{k+2}$; $|\gamma| = \aleph_{\omega_{k+1}}$, then (78) holds.

The proof of Theorem 38 depends on a lemma.

LEMMA 5. *Let α be an ordinal. Suppose that β_ν $(\nu < k)$ and r are such that, whenever $\beta < \alpha$, then $\beta \nrightarrow (\beta_0, \beta_1, \cdots)_k^r$. Then*

$$\alpha \nrightarrow (\beta_0 + 1, \beta_1 + 1, \cdots)_k^{r+1}. \tag{79}$$

PROOF. Let $\bar{S} = \alpha$; $x \in S$. Then $\overline{L(x)} < \alpha$, and hence, by hypothesis, there is a partition $[L(x)]^r = \sum [\nu < k] K_\nu(x)$ such that, whenever $X \subset L(x)$; $[X]^r \subset K_\nu(x)$, then $\bar{X} < \beta_\nu$. Put $K_\nu = \{A + \{x\} : x \in S; A \in K_\nu(x)\}$ $(\nu < k)$. Then $[S]^{r+1} = \sum [\nu < k] K_\nu$. If we now assume that

$$S' \subset S; \qquad [S']^{r+1} \subset K_\nu; \qquad \bar{S}' = \beta_\nu + 1, \tag{80}$$

then $S' = S'' + \{x'\}$; $S'' \subset L(x')$; $\bar{S}'' = \beta_\nu$; $[S'']^r \subset K_\nu(x')$ which contradicts the definition of $K_\nu(x')$. Hence (80) is impossible, and (79) follows.

PROOF OF THEOREM 38. If $\beta < \omega_{m+1}$, then $|\beta| \leq \aleph_m$, $|\beta| \nrightarrow (|\beta_0|, |\beta_1|, \cdots)_k^r$. By Theorem 13, this implies $\beta \nrightarrow (\beta_0, \beta_1, \cdots)_k^r$. Now (75) follows from Lemma 5.

THEOREM 39. (i) *If*

$$l \rightarrow (\alpha_0, \alpha_1, \cdots)_k^r, \tag{81}$$

$$|m| > \sum[\lambda < l]\,|k|^{|\lambda|^r}, \tag{82}$$

then

$$m \rightarrow (\alpha_0 + 1, \alpha_1 + 1, \cdots)_k^{r+1}. \tag{83}$$

(ii) *If* $r>0$; $\omega_n \rightarrow (\alpha_0, \alpha_1, \cdots)_k^r$, *and*

$$2^{\aleph_\nu} \leq \aleph_n \qquad \textit{for } \nu < n, \tag{84}$$

then

$$\omega_{n+1} \rightarrow (\alpha_0 + 1, \alpha_1 + 1, \cdots)_k^{r+1}.$$

(iii) *If* $|k| < \aleph'_m$; $r \geq 0$, *and if* $2^{\aleph_\nu} \leq \aleph_n$ *for* $m \leq n < m+r$; $\nu < n$, *then*

$$\omega_{m+r} \rightarrow (\omega_m + r)_k^{r+1}, \tag{85}$$

$$\omega_{m+r} \rightarrow (\omega_m + r)_k^{r}. \tag{86}$$

Deduction of (ii) *from* (i). Let the hypothesis of (ii) hold. If $|\alpha_\nu| < r$ for some $\nu < k$, then the assertion is trivial, by Theorem 22. Hence we may assume that $|\alpha_\nu| \geq r$ $(\nu < k)$. Next, suppose that $|\alpha_\nu| = r$ for some $\nu < k$. Then, by Theorem 17, we may apply a suitable permutation to the system $\alpha_0, \alpha_1, \cdots$ so that for the new system, again denoted by $\alpha_0, \alpha_1, \cdots, \alpha_\nu, \cdots$ $(\nu < k)$, we have

$$\alpha_\nu = r \quad (\nu < k_0); \qquad |\alpha_\nu| > r \qquad (k_0 \leq \nu < k).$$

Here k_0 is some ordinal, $0 < k_0 \leq k$, and we can write $k = k_0 + k_1$. Then, by Theorem 20, (ii), the hypothesis implies $\omega_n \rightarrow (\alpha_{k_0}, \alpha_{k_0+1}, \cdots)_{k_1}^r$, and the assertion is implied by

$$\omega_{n+1} \rightarrow (\alpha_{k_0} + 1, \alpha_{k_0+1} + 1, \cdots)_{k_1}^{r+1}.$$

This shows that we may assume, without loss of generality, that $|\alpha_\nu| > r$ for $\nu < k$. Let us, now, suppose that $|k| \geq \aleph_n$. Then, if $A = [0, \omega_n]$, we can write $[A]^r = \sum[\nu < k]K_\nu$, where $|K_\nu| \leq 1$ for $\nu < k$. Since $\overline{A} \rightarrow (\alpha_0, \cdots)_k^r$, there are $X \subset A$, $\nu < k$ such that $\overline{X} = \alpha_\nu$; $[X]^r \subset K_\nu$. Then $|X| = |\alpha_\nu| > r$; $1 < |[X]^r| \leq |K_\nu|$ which is a contradiction. This proves that $|k| < \aleph_n$.

Put

$$\sum[\lambda < \omega_n]\,|k|^{|\lambda|^r} = a.$$

It suffices to show that $|\omega_{n+1}| > a$.

If $n = 0$, then $a < \aleph_0$. If $n > 0$, then

$$a \leq \sum[\lambda < \omega_n]2^{|k||\lambda|^r} \leq \sum[\lambda < \omega_n]2^{\aleph_{\nu_\lambda}},$$

for some $\nu_\lambda < n$. Hence, by (84), $a \leq \sum[\lambda < \omega_n]\aleph_n = \aleph_n$.

Deduction of (iii) *from* (ii). By definition of $\aleph'_m$, $\omega_m \to (\omega_m)^1_k$. Hence, by r applications of (ii), (85) follows. Now (86) follows from Theorem 15.

Proof of (i). Let $\bar{S} = m$; $[S]^{r+1} = \sum' [\nu < k] K_\nu$. We choose n such that $|n| > |m|$. Throughout this proof the letters κ, λ, ρ, σ denote ordinals less than n, and x, y and z elements of S. The relation

$$\{x_0, \cdots, x_r\} \equiv \{y_0, \cdots, y_r\}$$

expresses, by definition, the fact that, for some $\nu < k$,

$$\{x_0, \cdots, x_r\}, \{y_0, \cdots, y_r\} \in K_\nu.$$

We define $f_\kappa(x) \in S$ as follows. Let x be fixed, and suppose that, for some fixed λ, the elements $f_\kappa = f_\kappa(x)$ have already been defined for all $\kappa < \lambda$. Then we put $f_\lambda(x) = x$, if $f_\kappa = x$ for some $\kappa < \lambda$. If, on the other hand, $f_\kappa \neq x$ for $\kappa < \lambda$, then we define f_λ to be the first element y of $S - \{f_\kappa : \kappa < \lambda\}$ such that

(87) $\{f_{\kappa_0}, \cdots, f_{\kappa_{r-1}}, y\} \equiv \{f_{\kappa_0}, \cdots, f_{\kappa_{r-1}}, x\}$ for $\kappa_0 < \cdots < \kappa_{r-1} < \lambda$.

This defines f_κ for all κ. We now prove that

$$f_\lambda < f_\mu \tag{88}$$

if

$$\lambda < \mu; \qquad f_\lambda \neq x. \tag{89}$$

First of all, (87) holds for $y = x$. Hence, by (89) and the definition of f_λ, we have $f_\lambda < x$. This proves (88) in the case when $f_\mu = x$. Now suppose that $f_\mu \neq x$. Then (87) holds for $y = f_\mu$ and, again, (88) follows. By (88) and $|n| > |m|$, there is $\rho(x)$ such that

$$f_\kappa(x) < f_\lambda(x) = x, \qquad \text{if } \kappa < \rho(x) \leq \lambda.$$

Let, for $\kappa_0 < \cdots < \kappa_{r-1} < \rho(x)$,

$$\{f_{\kappa_0}(x), \cdots, f_{\kappa_{r-1}}(x), x\} \in K_{\nu(\kappa_0, \cdots, \kappa_{r-1}, x)} = K'(\kappa_0, \cdots, x).$$

We now show that if x and z are such that

$$\rho(x) = \rho(z), \tag{90}$$

$$\begin{aligned} &K'(\kappa_0, \cdots, \kappa_{r-1}, x) \\ &\qquad = K'(\kappa_0, \cdots, \kappa_{r-1}, z) \quad \text{for } \kappa_0 < \cdots < \kappa_{r-1} < \rho(x), \end{aligned} \tag{91}$$

then $x = z$. Let $\lambda \leq \rho(x)$, and suppose that

$$f_\kappa(x) = f_\kappa(z) \qquad \text{for } \kappa < \lambda. \tag{92}$$

Then $f_\lambda(x)$ is the first element y of $S-\{f_\kappa(x):\kappa<\lambda\}$ such that (87) holds, i.e.

$$\{f_{\kappa_0}(x), \cdots, f_{\kappa_{r-1}}(x), y\}\in K'(\kappa_0, \cdots, \kappa_{r-1}, x) \quad \text{for } \kappa_0< \cdots <\kappa_{r-1}<\lambda.$$

Now, $f_\lambda(z)$ is defined by the same property, with z in place of x, and (90) and (91) show that $f_\lambda(x)=f_\lambda(z)$. We have thus proved, by induction, that $f_\kappa(x)=f_\kappa(z)$ for all $\kappa\leqq\rho(x)$. In particular, by (90),

$$x = f_{\rho(x)}(x) = f_{\rho(z)}(z) = z.$$

We next prove that $\rho(x_0)\geqq l$ for at least one x_0. Let us suppose, on the contrary, that $\rho(x)<l$ for all x. Let $\sigma<l$. Then the cardinal $a(\sigma)$ of the set $\{x:\rho(x)=\sigma\}$ is at most equal to the cardinal of the set of all functions $h(\kappa_0, \cdots, \kappa_{r-1})<k$, defined for $\kappa_0< \cdots <\kappa_{r-1}<\sigma$. Hence

$$a(\sigma) \leqq |k|^{|\sigma|^r}; \qquad |m| = |S| = \left|\sum[\sigma < l]\{x:\rho(x) = \sigma\}\right|$$
$$\leqq \sum[\sigma < l]\,|k|^{|\sigma|^r},$$

which contradicts (82). This proves that $\rho(x_0)\geqq l$ for some suitable x_0. Put $S_0=\{f_\kappa(x_0):\kappa<l\}$. Then $\overline{S}_0=l$; $[S_0]^r=\sum[\nu<k]K_\nu''$, where $K_\nu''=\{A:A\in[S_0]^r;\ A+\{x_0\}\in K_\nu\}$ $(\nu<k)$. By (81), there are $S_1\subset S_0$; $\nu<k$ such that $\overline{S}_1=\alpha_\nu$; $[S_1]^r\subset K_\nu''$. Then the set $S_2=S_1+\{x_0\}$ satisfies $\overline{S}_2=\alpha_\nu+1$; $[S_2]^{r+1}\subset K_\nu$, and (83) follows. This completes the proof of Theorem 39.

Corollary of Theorem 39. *Given any r and any ordinals k, β_ν, there always exists an ordinal α such that*

$$\alpha \to (\beta_0, \beta_1, \cdots)^r_k.$$

For, if $r\leqq 1$, then any α can be taken such that $|\alpha| > \sum[\nu<k]|\beta_\nu|$, and the result for $r=2, 3, \cdots$ is obtained by applications of Theorem 39. The relation (5) shows that the last proposition becomes false if α and β_ν, instead of being ordinals, are allowed to be any order types. Later on (Theorem 45) the corollary will be extended, for $r=1$, to the case of arbitrary types β_ν.

We mention, without proof, the following further applications of Theorem 39 (i).

(a) $(2^{a})^{+}\to(a)^2_k$ if $|k|<a'$.

(b) $a^{+}\to(a)^2_k$; $\omega_{n+1}\to(\omega_n+1)^2_k$,
if $a=\aleph_n$; $|k|<a'$, and $2^b\leqq a$ for all $b<a$.

(c) $(k^{\alpha_0+\cdots+\alpha_{k-1}-2k+1}+k-2)(k-1)^{-1}\to(\alpha_0, \cdots, \alpha_{k-1})^2_k$, if $2\leqq k$, $\alpha_0, \cdots, \alpha_{k-1}<\omega_0$.

If in Theorem 39 we take as k, l, α_ν finite numbers we obtain a result which implies Theorem 1 of [5]. Denote, in this special case, by

$$\rho_k(r;\alpha_0,\alpha_1,\cdots,\alpha_{k-1})$$

the least number n such that

$$n\to(\alpha_0,\cdots,\alpha_{k-1})^r_k.$$

Without loss of generality, we restrict ourselves to the case $k\geqq 2$; $0<r\leqq\alpha_\nu$. In [5, Theorem 1], an explicit upper estimate was given for the number $\rho_k(r;\alpha,\alpha,\cdots,\alpha)$, which, in that paper, was denoted by $R(k,r,\alpha)$.

Clearly, $\rho_k(1;\alpha_0,\cdots,\alpha_{k-1})=1+\alpha_0+\cdots+\alpha_{k+1}-k$. By Theorem 39,

$$\begin{aligned}\rho_k(r+1;\alpha_0+1,\alpha_1+1,\cdots,\alpha_{k-1}+1)&\\ \leqq 1+\sum[\lambda<\rho_k(r;\alpha_0,\cdots,\alpha_{k-1})]k^{\lambda^r}.&\end{aligned}\tag{93}$$

It is easily proved that, for $l<\omega_0$,

$$1+\sum[\lambda<l]k^{\lambda^r}\leqq k^{l^r}.\tag{94}$$

For, (94) holds for $l=0$, and if $0<m<\omega_0$, and (94) holds for $l=m-1$, then

$$1+\sum[\lambda<m]k^{\lambda^r}\leqq k^{(m-1)^r}+k^{(m-1)^r}\leqq k^{1+(m-1)^r}\leqq k^{m^r},$$

so that (94) holds for $l=m$. We have thus proved the following recurrence relation.

THEOREM 40. *If* $2\leqq k<\omega_0$; $0<r\leqq\alpha_\nu<\omega_0$ $(\nu<k)$, *and if* ρ_k *is defined as above, then*

$$\rho_k(r+1;\quad\alpha_0+1,\cdots,\alpha_{k-1}+1)\leqq k^{\rho_k(r;\alpha_0,\cdots,\alpha_{k-1})^r}.$$

In particular, we have, using the notation of [5],

$$R(k,r+1,\alpha+1)\leqq k^{R(k,r,\alpha)^r}\ (k\geqq 2;0<r\leqq\alpha).$$

This is precisely the recurrence relation established in [5], from which the explicit estimate is deduced at once. This is no coincidence, as the method of proof of the present Theorem 39 is related to that used for proving Theorem 1 of [5].

Theorem 39 implies Theorem 4 (i), i.e.

$$(2^{\aleph_n})^+\to(\aleph_{n+1})^2_{\omega_n}.\tag{95}$$

For, clearly, $\aleph_{n+1}\to(\aleph_{n+1})^1_{\omega_n}$, and therefore $\omega_{n+1}\to(\omega_{n+1})^1_{\omega_n}$. Also,

$$\sum[\lambda<\omega_{n+1}]\,|\omega_n|^{|\lambda|}\leqq\aleph_n^{\aleph_n}\aleph_{n+1}=2^{\aleph_n}=\aleph_m,$$

say. Hence, by Theorem 39 (i), $\omega_{m+1}\to(\omega_{n+1}+1)^2_{\omega_n}$, and (95) follows.

THEOREM 41. *If* $r \geqq 3$, *then, for all* n,

$$\omega_{n+1} \nrightarrow (\omega_n + 2, \omega_0 + 1, r + 1, r + 1, \cdots, r + 1)^r_{(r-1)!}. \tag{96}$$

As an application, consider the case $r=3$; $n=0$:

$$\omega_1 \nrightarrow (\omega_0 + 2, \omega_0 + 1)^3. \tag{97}$$

This should be compared with:

$$\omega_1 \rightarrow (\omega_0 + 1)^r_k \qquad (k < \omega_0;\ r \geqq 0)$$

which follows from Theorem 39 (ii) and Theorem 1.

PROOF OF THEOREM 41. Let $\omega_n \leqq \beta < \omega_{n+1}$. We apply Lemma 4 to

$$s = r - 1; \quad \alpha_0 = \omega_n; \quad \alpha_1 = \beta; \quad \beta_0 = \omega_n + 1; \quad \beta_1 = \omega_0$$

and obtain

$$\beta \nrightarrow (\omega_n + 1, \omega_0, r, \cdots, r)^{r-1}_{(r-1)!}.$$

This holds, a fortiori, if $\beta < \omega_n$. Now Lemma 5 proves (96).

A type β is called *indecomposable* if the equation $\beta = \gamma + \delta$ implies that either $\gamma \geqq \beta$ or $\delta \geqq \beta$. It is known[10] that the indecomposable ordinals are those of the form ω_0^λ. The types η and λ are indecomposable. The next theorem asserts that in Lemma 4 the $s!-2$ classes corresponding to the entries $s+1$ in the partition relation may be suppressed in the special case when both β_0 and β_1 are indecomposable, at the cost, however, of raising the remaining entries slightly.

THEOREM 42. *Let* $s \geqq 3$; $|\alpha_0| = |\alpha_1|$; $\beta_0, \beta_1^* \nleqq \alpha_0$, *and suppose that* β_0 *and* β_1 *are indecomposable. Then*

$$(s - 3) + \alpha_1 \nrightarrow ((s - 3) + \beta_0, (s - 3) + \beta_1)^s. \tag{98}$$

PROOF. *Case* 1. $s=3$. Consider a set S with two orders such that $\overline{S}_< = \alpha_1$; $\overline{S}_{<<} = \alpha_0$. Then $[S]^3 = K_0 +' K_1$, where K_0 is the set of all sets $\{x_0, x_1, x_2\}_< = \{y_0, y_1, y_2\}_{<<} \subset S$ for which $x_\lambda \rightarrow y_\lambda$ is an even permutation of $[0, 3]$, i.e. one of the permutations 012, 120, 201. Now let us assume that

$$\alpha_1 \rightarrow (\beta_0, \beta_1)^3. \tag{99}$$

It suffices to deduce a contradiction in each of the two cases that follow.

Case 1.1. There is $A \subset S$ such that $\overline{A}_< = \beta_0$; $[A]^3 \subset K_0$. Let x, y, z denote elements of A. Then $\{x, y, z\}_< = \{x_1, y_1, z_1\}_{<<}$ implies that x_1, y_1, z_1 is a cyclic permutation of x, y, z. Put $B = \{x : y \ll x$, whenever

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[10] [13, §§75–78].

$y<x\}$; $C=A-B$. We shall prove three propositions about the two orders of A showing their effects on the partition $A=B+C$.

1. Let $x<y\in B$; $x\in C$. Then there is z such that $z<x\ll z$; $\{z, x, y\}_< = \{x, y, z\}_{<<}$; $z<y\ll z$; $y\in C$ which is false. Hence $x<y\in B$ implies $x\in B$, as well as $x\ll y$. Therefore $x>y\in C$ implies $x\in C$.

2. Let $x\in B$; $y\in C$; $x\ll y$. Then $x<y$. There is z such that $z<y\ll z$. Then $\{x, y, z\}_{<<} = \{z, x, y\}_<$; $z<x\ll z$; $x\in C$ which is false. Hence $x\in B$; $y\in C$ implies $y\ll x$.

3. Let $x, y\in C$; $x<y\ll x$. Then there is z such that $z<x\ll z$; $\{z, x, y\}_< = \{y, x, z\}_{<<}$ which contradicts the definition of K_0. Hence $x, y\in C$; $x<y$ implies $x\ll y$.

The results of 1, 2 and 3 show that, if we put $\overline{B}_< = \gamma_0$; $\overline{C}_< = \gamma_1$, then $\beta_0 = \overline{A}_< = \gamma_0+\gamma_1$; $\overline{A}_{<<} = \gamma_1+\gamma_0$. Since β_0 is indecomposable, it follows that $\gamma_\nu \geqq \beta_0$ for some $\nu<2$. Then

$$\beta_0 \leqq \gamma_\nu \leqq \overline{A}_{<<} \leqq \overline{S}_{<<} = \alpha_0 \tag{100}$$

which is a contradiction.

Case 1.2. There is $A\subset S$ such that $\overline{A}_< = \beta_1$; $[A]^3\subset K_1$. Then $\{x, y, z\}_< = \{x_1, y_1, z_1\}_{<<}\subset A$ implies that x_1, y_1, z_1 is an odd permutation of x, y, z. This is equivalent to saying that $\{x, y, z\}_< = \{x_2, y_2, z_2\}_{>>}\subset A$ implies that x_2, y_2, z_2 is an even permutation of x, y, z. Hence the result of Case 1.1 holds if β_0 is replaced by β_1, and "$\ll$" by "$\gg$". We note that β_1^* is indecomposable. Hence, in place of (100) we have

$$\beta_1 \leqq \overline{A}_{>>} \leqq \overline{S}_{>>} = \alpha_0^*$$

which is a contradiction. This shows that the assumption (99) was false, i.e. that (98) holds.

Case 2. $s>3$. Then, by the result of Case 1, we have $\alpha_1 \nrightarrow (\beta_0, \beta_1)^3$. By Theorem 15, this implies (98). This completes the proof of Theorem 42.

REMARK. If, in particular, β_0 and β_1 are ordinals, not zero, then $(s-3)+\beta_\nu = \beta_\nu$, so that (98) can be replaced by

$$(s-3) + \alpha_1 \nrightarrow (\beta_0, \beta_1)^s.$$

We may also mention here the following corollary of two of our lemmas, in which λ is the type of the continuum.

(101) *If* $2^{\aleph_0} = \aleph_n$, *then* $\omega_{n+1} \nrightarrow (\omega_1+1, \omega_1+1)^3$.

PROOF. Let $\omega_n \leqq \alpha_1 < \omega_{n+1}$. Then, by Lemma 4, with $\alpha_0 = \lambda$, we have $\alpha_1 \nrightarrow (\omega_1, \omega_1)^2$. By Lemma 5 this leads to (101).

THEOREM 43. *If* $r < s \leqq \beta_0$; $\alpha \nrightarrow (\beta_0)^r_k$; $\beta_1 \rightarrow (s)^r_k$, *then*

$$\alpha \rightarrow (\beta_0, \beta_1)^s.$$

This proposition remains valid if the types α, β_0, β_1 *are replaced by cardinals.*

PROOF. Let $r < s \leqq \beta_0$; $\alpha \rightarrow (\beta_0, \beta_1)^s$; $\beta_1 \rightarrow (s)^r_k$. We have to deduce that

$$\alpha \rightarrow (\beta_0)^r_k. \tag{102}$$

Let $\bar{S} = \alpha$; $[S]^r = \sum' [\nu < k] K_\nu$. Then $[S]^s = K_0' +' K_1'$, where

$$K_0' = \sum [\nu < k] \{A : A \in [S]^s; \qquad [A]^r \subset K_\nu\}.$$

Then there are $B \subset S$; $\lambda < 2$ such that $[B]^s \subset K_\lambda'$; $\bar{B} = \beta_\lambda$. If $\lambda = 1$, then $\bar{B} \rightarrow (s)^r_k$, and therefore there are $A \in [B]^s$; $\nu < k$ such that $[A]^r \subset K_\nu$. Then $A \in K_0'$; $A \notin K_1'$, which is false. Hence $\lambda = 0$. Let $\{X, Y\}_{\neq} \subset [B]^r$. Then we can write $X = \{x_0, \cdots, x_{r-1}\}$; $Y = \{x_m, \cdots, x_{m+r-1}\}$, where $1 \leqq m \leqq r$; $\{x_0, \cdots, x_{m+r-1}\}_{\neq}$. Put

$$X_\mu = \{x_\mu, \cdots, x_{\mu+r-1}\} \qquad (\mu \leqq m).$$

Now let $\mu < m$. Then, since $|B| = |\beta_0| \geqq s > r$, there is $Y_\mu \in [B]^s$ such that $X_\mu + X_{\mu+1} \subset Y_\mu$. But $Y_\mu \in K_0'$, so that $X_\mu, X_{\mu+1} \in [Y_\mu]^r \subset K_{\nu_\mu}$, for some $\nu_\mu < k$. Then $\nu_0 = \nu_1 = \cdots = \nu_{m-1}$; $X = X_0 \in K_{\nu_0}$; $Y = X_m \in K_{\nu_{m-1}} = K_{\nu_0}$. Since X and Y are arbitrary, it follows that ν_0 is independent of X and Y, and that $[B]^r \subset K_{\nu_0}$. This proves (102). The analogous theorem, with cardinals in place of types, is proved by means of the obvious modifications of the above argument.

Applications of Theorem 43. (a) Let λ be the type of the continuum and $|\lambda| = \aleph_\alpha$. Then, by Theorem 30, $\aleph_\alpha \nrightarrow (\aleph_1)^2_2$. Also, as is easily verified,

$$6 \rightarrow (3)^2_2. \tag{103}$$

Hence, by Theorem 43, $\aleph_\alpha \nrightarrow (\aleph_1, 6)^3$, and therefore $\omega_\alpha \nrightarrow (\omega_1, 6)^3$. Now, by Theorem 15, $\omega_\alpha \nrightarrow (\omega_1, r+3)^r$ $(r \geqq 3)$ follows and therefore, finally,

$$2^{\aleph_0} \nrightarrow (\aleph_1, r + 3)^r \qquad (r \geqq 3). \tag{104}$$

(b) By (97),

$$\omega_1 \nrightarrow (\omega_0 + 2)^3_2. \tag{105}$$

By (103) and Theorem 39, we have

$$m \rightarrow (4)^3_2, \tag{106}$$

where $m = 1 + \sum [\mu < 6] 2^{\mu^2} < 2^{26}$. It now follows from (105) and (106), by Theorem 43, that $\omega_1 \nrightarrow (\omega_0+2, 2^{\aleph_0})^4$ and therefore, by Theorem 15, that

$$\omega_1 \nrightarrow (\omega_0 + 2, 2^{\aleph_0} + r - 4)^r \qquad (r \geqq 4). \tag{107}$$

We now give a new proof of the theorem of Dushnik and Miller [2].[11] Our proof bears some resemblance to the original proof but can, we think, be followed more easily.

THEOREM 44. *If* $a \geqq \aleph_0$, *then* $a \rightarrow (\aleph_0, a)^2$.

PROOF. We use induction with respect to a. By Theorem 1, the assertion is true for $a = \aleph_0$. We assume that $n > 0$ and that the assertion is true for $a < \aleph_n$, and we let

$$|S| = b = \aleph_n > \aleph_0; \qquad [S]^2 = K_0 + K_1.$$

We suppose that

$$\text{if} \quad X \in [S]^{\aleph_0}, \quad \text{then} \quad [X]^2 \not\subset K_0,$$

and we want to find $Y \in [S]^b$ such that $[Y]^2 \subset K_1$.

There is a maximal set $A = \{x_\nu : \nu < l\} \subset S$ such that $l < \omega_0$,[12]

$$\begin{aligned} &x_\nu \in U_0(x_0, \cdots, x_{\nu-1}), \\ &|U_0(x_0, \cdots, x_\nu)| = b \end{aligned} \qquad (\nu < l). \tag{108}$$

For, the relations (108) imply that $[A]^2 \subset K_0$. Put $B = U_0(A)$. Then

$$|BU_0(x)| < |B| = b \qquad (x \in B). \tag{109}$$

Case 1. $b' = b$. Then we define x_ν $(\nu < \omega_n)$ as follows. Let $\nu < \omega_n$, and suppose that $x_\mu \in B$ $(\mu < \nu)$. Then, by (109),

$$\left| \sum [\mu < \nu](\{x_\mu\} + BU_0(x_\mu)) \right| < |B|,$$

and therefore there is $x_\nu \in B - \sum[\mu < \nu](\{x_\mu\} + U_0(x_\mu))$. We may put $Y = \{x_\nu : \nu < \omega_n\}$.

Case 2. $b' = \aleph_m < b$. Then $b = \sum[\mu < \omega_m] b_\mu$, where $b_\mu < b$. Let $x \in B$. Then there is a first ordinal $\rho(x) < \omega_m$ such that $|BU_0(x) < b_{\rho(x)}$, since otherwise we would obtain the contradiction $b = \sum[\mu < \omega_m] b_\mu \leqq |BU_0(x)| \aleph_m < b$. Now put

$$B(\tau) = \{x : x \in B; \rho(x) = \tau\} \qquad (\tau < \omega_m).$$

We define, by induction, τ_μ, X_μ $(\mu < \omega_m)$ as follows. Let, for some $\nu < \omega_m$,

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[11] Theorem 3, (I).

[12] The symbol U_0 was defined in §2.

$$\tau_\mu < \omega_m; \qquad X_\mu \in [B(\tau_\mu)]^{b_\mu} \qquad (\mu < \nu).$$

Then, by definition of m,

$$\left| \sum [\mu < \nu; x \in X_\mu](\{x\} + BU_0(x)) \right| \leqq \sum [\mu < \nu; x \in X_\mu](1 + b_{\tau_\mu})$$
$$= \sum [\mu < \nu](1 + b_{\tau_\mu})b_\mu < b.$$

Hence $|D| = b$, where $D = B - \sum[\mu<\nu;\ x\in X_\mu](\{x\} + U_0(x))$. There is a first ordinal $\tau_\nu < \omega_m$ such that $|DB(\tau_\nu)| < b_\nu$, since otherwise we would have the contradiction

$$b = |D| = \sum [\mu < \omega_m]\,|DB(\mu)| \leqq b_\nu \aleph_m < b.$$

Now we can choose $X_\nu \in [DB(\tau_\nu)]^{b_\nu}$. Then $X_\nu \subset U_1(X_\mu)$ $(\mu<\nu<\omega_m)$. By the induction hypothesis there is $Y_\mu \in [X_\mu]^{b_\mu}$ such that $[Y_\mu]^2 \subset K_1$ $(\mu<\omega_m)$. Then we may put $Y = \sum[\mu<\omega_m]Y_\mu$. This proves Theorem 44.

Our next theorem may be considered as providing, in the case $r=1$, an extension of the Corollary of Theorem 39 to the case of arbitrary types β_ν, not necessarily ordinals. In view of (5), the extension to values $r \geqq 2$ is false.

Let $k>0$, and consider any types β_ν $(\nu<k)$. Let $\overline{B}_\nu = \beta_\nu$, and denote by P the cartesian product of the sets B_ν, i.e. the set of all mappings $\nu \to x_\nu \in B_\nu$ defined for $\nu<k$. We order P alphabetically and call the order type π of P the *alphabetical product* of the types β_ν and write

$$\pi = \prod \times [\nu < k]\beta_\nu.$$

This multiplication has been considered by Hausdorff [10].

THEOREM 45. *If* $k>0$, *then* $\prod\times[\nu<k]\beta_\nu \to (\beta_0, \beta_1, \cdots)^1_k$.

PROOF. In spite of its somewhat complicated appearance the proof is, in fact, very simple, as can be seen by following it in the case $k=2$ or $k=3$. Let $P = \sum[\nu<k]K_\nu$. We want to find $X\subset P$ and $\nu_0<k$ such that $\overline{X} = \beta_{\nu_0}$; $X \subset K_{\nu_0}$. We use the notation $(z_0, z_1, \cdots, \hat{z}_n)$ for the system of all z_ν such that $\nu<n$, ordered according to increasing ν. Thus

$$P = \{\{x_0, x_1, \cdots, \hat{x}_k\} : x_\nu \in B_\nu \text{ for } \nu < k\}.$$

Case 1. There is $\nu<k$ such that the following condition is satisfied. There is a system of elements $x_\mu \in B_\mu$ $(\mu<\nu)$ such that, given any $x_\nu \in B_\nu$, there is some system $z(x_\nu) = (x_0, x_1, \cdots, \hat{x}_k)$ which belongs to K_ν. Then we may put $X = \{z(x) : x \in B_\nu\}$.

Case 2. There is no ν such that the condition of Case 1 holds. Then, given any $\nu<k$ and any elements $x_\mu \in B_\mu$ $(\mu<\nu)$, there is a function $f_\nu(x_0, x_1, \cdots, \hat{x}_\nu) \in B_\nu$ such that, for any choice of $x_\lambda \in B_\lambda$ $(\nu < \lambda < k)$,

$$(x_0, x_1, \cdots, \hat{x}_\nu, f_\nu(x_0, \cdots, \hat{x}_\nu), x_{\nu+1}, \cdots, \hat{x}_k) \notin K_\nu.$$

In particular, the function $f_0(\hat{x}_0)$ is constant. Then we define, inductively, elements $y_\nu (\nu < k)$ as follows. We put, for $\nu < k$, $y_\nu = f_\nu(y_0, \cdots, \hat{y}_\nu)$. Then $y_\mu \in B_\mu (\mu < k)$, and hence there is $\nu < k$ such that $(y_0, \cdots, \hat{y}_k) \in K_\nu$. But then

$$(y_0, y_1, \cdots, \hat{y}_\nu, f_\nu(y_0, \cdots, \hat{y}_\nu), y_{\nu+1}, \cdots, \hat{y}_k) \in K_\nu$$

which contradicts the definition of f_ν. The theorem is proved.

8. **Canonical partition relations.** Let S be an ordered set, and consider a partition

$$[S]^r = \sum{}' \, [\nu < k] K_\nu. \tag{110}$$

To every such *disjoint* partition there belongs an equivalence relation Δ on $[S]^r$ defined by the rule that elements X, Y of $[S]^r$ are equivalent for Δ, in symbols:

$$X \equiv Y(\cdot\Delta)$$

if, and only if, there is $\nu < k$ such that $X, Y \in K_\nu$. This equivalence relation Δ is unaltered if the classes K_ν are renumbered in any way. The partition (110) and the corresponding equivalence relation Δ is called *canonical*[13] if there is a system $(\epsilon_0, \epsilon_1, \cdots, \epsilon_{r-1})$ of numbers $\epsilon_\rho < 2$ such that, for $X = \{x_0, \cdots, x_{r-1}\}_< \subset S$; $Y = \{y_0, \cdots, y_{r-1}\}_< \subset S$, we have $X \equiv Y(\cdot\Delta)$ if, and only if, $x_\rho = y_\rho$ for every $\rho < r$, such that $\epsilon_\rho = 1$.

The canonical equivalence relation defined by means of the numbers ϵ_ρ is denoted by $\Delta^r_{\epsilon_0\epsilon_1 \cdots \epsilon_{r-1}}$. The *canonical partition relation*

$$\alpha \rightarrow * (\beta)^r \tag{111}$$

has, by definition, the following meaning. Whenever $\bar{S} = \alpha$, and (110) is any disjoint partition, with any arbitrary k, then there is $B \subset S$ such that $\bar{B} = \beta$, and such that the equivalence relation Δ belonging to (110), if restricted to $[B]^r$, coincides with some canonical equivalence relation $\Delta^r_{\epsilon_0 \cdots \epsilon_{r-1}}$. The main result of [4] is expressible in the form $\omega_0 \rightarrow * (\omega_0)^r$. The problem arises of finding canonical partition relations between types other than ω_0. The main difference between canonical and noncanonical relations derives from the fact that if the canonical relation (111) holds then a certain choice of a subset of S can be made irrespective of the number $|k|$ of classes of (110). The relation $\omega_0 \rightarrow * (\omega_0)^1$ is equivalent to the statement that, *if a denumerable set S is arbitrarily split into nonoverlapping subsets S_ν, then there are either infinitely many nonempty subsets S_ν, or else at least one of the subsets S_ν is infinite.*

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[13] [4; 5]. The notation used in the present note differs slightly from that used in the earlier papers.

The following theorem establishes a connection between canonical and noncanonical partition relations.

THEOREM 46. (i) *Let q_s denote the number of distinct equivalence relations which can be defined on the set $[0, s]$. Let*

$$s = \binom{2r}{r}; \qquad |\beta| > 2r; \qquad \alpha \rightarrow (\beta)^{2r}_{q_s}. \tag{112}$$

Then $\alpha \rightarrow * (\beta)^r$. If $|\beta| > 4$; $\alpha \rightarrow (\beta)^4_{203}$, then $\alpha \rightarrow * (\beta)^2$.

(ii) *If m, $r \geq 0$, and $2^{\aleph_\nu} \leq \aleph_n$ for $m \leq n < m+2r+1$; $\nu < n$, then* $\omega_{m+2r+1} \rightarrow * (\omega_m + 2r + 1)^{r+1}$.

REMARK. The first few values of q_s are:

$$q_0 = 1; \quad q_1 = 1; \quad q_2 = 2; \quad q_3 = 5; \quad q_4 = 15; \quad q_5 = 52; \quad q_6 = 203.$$

A rough estimate for all s is

$$q_s \leq 2^{\binom{s}{2}},$$

obtained by observing that an equivalence relation is fixed if for $\mu < \nu < s$ it is decided whether $\mu \equiv \nu$ or $\mu \not\equiv \nu$. It is easy to prove that, for $s > 0$,

$$q_{s+1} = \binom{s}{0} q_s + \binom{s}{1} q_{s-1} + \cdots + \binom{s}{s} q_0$$

and hence $q_s \leq s!$.

Deduction of (ii) *from* (i). By Theorem 39 (iii), we have

$$\omega_{m+2r+1} \rightarrow (\omega_m + 2r + 1)_k^{2r+2} \qquad (k < \omega_0),$$

and the conclusion follows from Theorem 46 (i).

PROOF OF (i). Suppose that (112) holds. Let $\bar{S} = \alpha$, and consider any disjoint partition (110). Let Δ be the equivalence relation on $[S]^r$ which belongs to (110). Our first aim is to define a certain equivalence relation Δ^* on $[S]^{2r}$.

Let $[[0, 2r]]^r = \{P_0, P_1, \cdots, P_{s-1}\}_{\neq}$. Then

$$s = \binom{2r}{r}.$$

Let $X=\{x_0, \cdots, x_{2r-1}\}_< \subset S$. Define the equivalence relation $\Delta(X)$ on $[0, s]$ by putting, for $\mu<\nu<s$,

$$\mu \equiv \nu(\cdot\Delta(X))$$

if, and only if, $\{x_\lambda:\lambda\in P_\mu\} \equiv \{x_\lambda:\lambda\in P_\nu\}(\cdot\Delta)$. Put, for X, $Y\in[S]^{2r}$,

$$X \equiv Y(\cdot\Delta^*)$$

if, and only if, $\Delta(X)=\Delta(Y)$.

Now, by definition of q_s, Δ^* has at most q_s nonempty classes. By (112), there is $B\subset S$ such that $\overline{B}=\beta$, and any two elements of $[B]^{2r}$ are equivalent for Δ^*. This means that, in the terminology of [4], Δ is invariant in $[B]^r$ (cf. [4, p. 253]). Choose any $A\subset B$ such that $2r<|A|<\aleph_0$, which is possible since $|\beta|>2r$. Then Δ is invariant in $[A]^r$ and hence, by [4, Theorem 2], canonical in $[A]^r$. Thus there is a canonical equivalence relation $\Delta(A)$ on $[B]^r$ such that $\Delta=\Delta(A)$ on $[A]^r$. It only remains to show that $\Delta(A)$ is independent of A.

Let A_0, $A_1\subset B$; $2r<|A_0|$, $|A_1|<\aleph_0$, and let $\Delta=\Delta(A_0)=\Delta^r_{\epsilon_0\cdots\epsilon_{r-1}}$ on $[A_0]^r$, $\Delta=\Delta(A_1)=\Delta^r_{\eta_0\cdots\eta_{r-1}}$ on $[A_1]^r$. Then $2r<|A_0+A_1|<\aleph_0$, and hence, for some $\kappa_\rho<2$, $\Delta=\Delta(A_0+A_1)=\Delta^r_{\kappa_0\cdots\kappa_{r-1}}$ on $[A_0+A_1]^r$. Let $A_0=\{y_0, y_1, \cdots, y_{2r-1}, \cdots\}_<$. Consider the sets

$$P = \{y_{2\lambda}:\lambda < r\}; \qquad Q = \{y_{2\lambda+1-\epsilon_\lambda}:\lambda < r\}.$$

By definition of $\Delta^r_{\epsilon_0\cdots\epsilon_{r-1}}$, we have $P\equiv Q(\cdot\Delta^r_{\epsilon_0\cdots\epsilon_{r-1}})$. Hence

$$P \equiv Q(\cdot\Delta); \qquad P \equiv Q(\cdot\Delta^r_{\kappa_0\cdots\kappa_{r-1}}); \qquad \kappa_\rho\leqq\epsilon_\rho\ (\rho<r).$$

Similarly, by considering the sets P and $Q'=\{y_{2\lambda+1-\kappa_\lambda}:\lambda<r\}$, we find that $P\equiv Q'(\cdot\Delta^r_{\kappa_0\cdots\kappa_{r-1}})$; $P\equiv Q'(\ \Delta)$;

$$P \equiv Q'(\cdot\Delta^r_{\epsilon_0\cdots\epsilon_{r-1}}); \qquad \epsilon_\rho \leqq \kappa_\rho(\rho < r).$$

Hence $\epsilon_\rho=\kappa_\rho$ for all ρ. For reasons of symmetry, $\eta_\rho=\kappa_\rho$, and so, finally $\epsilon_\rho=\eta_\rho$ $(\rho<r)$. Therefore $\Delta(A)$ is independent of A, which means that Δ is canonical in the whole set $[B]^r$. This proves Theorem 46.

9. **Polarised partition relations.** The relation $a\rightarrow(b_0, b_1)^2$ refers to a partition of the set of all pairs of elements of a set S of cardinal a. Instead of pairs of elements of one and the same set we shall now consider pairs of elements, one from each of two sets S_0, S_1. The relation $a\rightarrow(b_0, b_1)^2$ can be thought of as referring, in the terminology of combinatorial graph theory (linear combinatorial topology), to decompositions of the complete graph of cardinal a into two subgraphs. The new kind of relation to be defined now refers to decom-

positions of the "complete" even graph of cardinal-pair a_0, a_1, i.e. the graph obtained by joining every "point" of a set of cardinal a_0 to every point of a disjoint set of cardinal a_1.

More generally, we introduce the notation

$$[S_0, S_1, \cdots, S_{t-1}]^{r_0, r_1, \cdots, r_{t-1}}$$

for the cartesian product of the t sets $[S_\lambda|^{r_\lambda}$, i.e. we put

$$[S_0, \cdots, S_{t-1}]^{r_0, \cdots, r_{t-1}} = \{(X_0, \cdots, X_{t-1}) : X_\lambda \in [S_\lambda]^{r_\lambda} \text{ for } \lambda < t\}.$$

We shall always have $0 < t < \omega_0$. The introduction of this set leads naturally to the following definition of a corresponding partition relation. The relation

$$(113)\qquad \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{t-1} \end{pmatrix} \rightarrow \begin{pmatrix} b_{00} & b_{01} & \cdots \\ b_{10} & b_{11} & \cdots \\ \cdot & \cdot & \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \\ b_{t-1,0} & b_{t-1,1} & \cdots \end{pmatrix}_k^{r_0, r_1, \cdots, r_{t-1}}$$

has, by definition, the following meaning. Whenever $|S_\lambda| = a_\lambda$ for $\lambda < t$, and

$$[S_0, \cdots, S_{t-1}]^{r_0, \cdots, r_{t-1}} = \sum [\nu < k] K_\nu,$$

then there are sets $B_\lambda \subset S_\lambda$ and an ordinal $\nu < k$ such that $|B_\lambda| = b_{\lambda\nu}$ for $\lambda < t$, and $[B_0, \cdots, B_{t-1}]^{r_0, \cdots, r_{t-1}} \subset K_\nu$. There is a similar kind of relation involving order types which will, however, not be considered here. If the number of classes is finite we write in the rows on the right hand side of (113) a last element. The negation of (113) is obtained by replacing $\rightarrow$ by $\nrightarrow$. If, for $\lambda < t$, $b_{\lambda 0} = b_{\lambda 1} = \cdots$, we use the obvious abbreviation for (113). Cf. Theorem 49.

The passage from our former type of partition relation i.e. the case $t = 1$, to the more general kind (113) bears a certain resemblance to the process of polarisation used in the theory of algebraic forms, which accounts for the name *polarised partition relation* suggested for relations of the form (113).

We shall deduce some results for polarised relations but will not develop the theory to the same extent as was done in the case of the nonpolarised relation.

In [11] one of us has considered polarised canonical parition relations between finite numbers, and also involving ω_0.

THEOREM 47. *If $a' = b'$, then*

$$\begin{pmatrix} a \\ b \end{pmatrix} \nrightarrow \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}^{1,1}. \tag{114}$$

In particular, (114) *holds if* $1 < a$, $b < \aleph_0$.

THEOREM 48. *The relation*

$$\begin{pmatrix} \aleph_0 \\ b \end{pmatrix} \rightarrow \begin{pmatrix} \aleph_0 & \aleph_0 \\ \aleph_0 & b \end{pmatrix}^{1,1} \tag{115}$$

holds if, and only if, either $b=0$ *or* $b' > \aleph_0$. *In particular,*

$$\begin{pmatrix} \aleph_0 \\ 2^{\aleph_0} \end{pmatrix} \rightarrow \begin{pmatrix} \aleph_0 & \aleph_0 \\ \aleph_0 & 2^{\aleph_0} \end{pmatrix}^{1,1}. \tag{116}$$

PROOF OF THEOREM 47. If $1<a<\aleph_0$, then $b'=a'=2$; $1<b<\aleph_0$. Let, in this case, $A=[0, a]$; $B=[0, b]$. Then $[A, B]^{1,1}=K_0+'K_1$, where

$$K_0 = \{(\kappa, \lambda) : \kappa < a; \lambda < b; \kappa+\lambda \text{ even}\}.$$

Then, for any $\kappa<a$, the pairs $(\kappa, 0)$ and $(\kappa, 1)$ lie in different classes K_ν, and the same holds for $(0, \lambda)$ and $(1, \lambda)$. This proves (114). Now let $a \geqq \aleph_0$. Then $a'=b'=\aleph_n$, say, and we can write

$$a = \sum [\nu < \omega_n] a_\nu; \qquad b = \sum [\nu < \omega_n] b_\nu,$$

where $a_\nu<a$; $b_\nu<b$. Let $A=\sum' [\nu<\omega_n]A_\nu$; $B=\sum' [\nu<\omega_n]B_\nu$; $|A_\nu|=a_\nu$; $|B_\nu|=b_\nu$. Then $[A, B]^{1,1}=K_0+K_1$, where

$$K_0 = \{(x, y) : x \in A_\mu; \quad y \in B_\nu; \quad \mu < \nu < \omega_n\},$$
$$K_1 = \{(x, y) : x \in A_\mu; \quad y \in B_\nu; \quad \nu \leqq \mu < \omega_n\}.$$

Let $X \in [A]^a$; $y_0 \in B$. Then $y_0 \in B_\nu$ for some $\nu<\omega_n$. We have $|A_\mu X| \leqq |A_\mu| < a$ and hence, by definition of n, $\sum[\mu \leqq \nu] |A_\mu X| < a = |X|$. Therefore we can choose $x_0 \in X - \sum[\mu \leqq \nu]A_\mu$. Then $(x_0, y_0) \notin K_0$, and and hence $[X, \{y_0\}]^{1,1} \not\subset K_0$. By symmetry, it follows that, if $Y \in [B]^b$; $x_1 \in A$, then $[\{x_1\}, Y]^{1,1} \not\subset K_1$. This proves (114) and completes the proof of Theorem 47.

PROOF OF THEOREM 48. If $b=0$, then (115) obviously holds. If $0<b<\aleph_0$, then (115) is false, as is seen by considering the partition of the set in question in which $K_1=0$. Next, if $b'=\aleph_0$, then, by Theorem 47,

$$\begin{pmatrix} \aleph_0 \\ b \end{pmatrix} \nrightarrow \begin{pmatrix} \aleph_0 & 1 \\ 1 & b \end{pmatrix}^{1,1}.$$

Hence, a fortiori, (115) is false. It remains to prove (115) under the assumption $b' > \aleph_0$.

Let $|A| = \aleph_0$; $|B| = b$; $[A, B]^{1,1} = K_0 + K_1$. We may suppose that

$$(117) \qquad (X, Y) \in [A, B]^{\aleph_0, b} \quad \text{implies} \quad [X, Y]^{1,1} \not\subset K_1.$$

Let $(X, Y) \in [A, B]^{\aleph_0, b}$.

1. Put $Y_0 = \sum [x \in X] \{y : (x, y) \in K_0\}$. Then $[X, Y - Y_0]^{1,1} \subset K_1$, and hence, by (117), $|Y - Y_0| < b$,

$$\sum [x \in X] \left| \{y : y \in Y;\ (x, y) \in K_0\} \right| \geqq |Y Y_0| = b.$$

Since $b' > \aleph_0 = |X|$, this implies the existence of $x_0 \in X$ such that

$$\left| \{y : y \in Y;\ (x_0, y) \in K_0\} \right| = b.$$

Put

$$\phi(X, Y) = x_0; \qquad \psi(X, Y) = \{y : y \in Y;\ (x_0, y) \in K_0\}.$$

Then

$$\phi(X, Y) \in X; \qquad \psi(X, Y) \in [Y]^b,$$
$$[\{\phi(X, Y)\}, \psi(X, Y)]^{1,1} \subset K_0.$$

2. Put $f(y) = \{x : x \in X;\ (x, y) \in K_0\}$ $(y \in Y)$. If

$$(118) \qquad y \in Y \quad \text{implies} \quad |f(y)| < \aleph_0,$$

then $b = |Y| = \sum [P \subset X;\ |P| < \aleph_0] \left| \{y : y \in Y; f(y) = P\} \right|$. But, since there are only $\aleph_0$ distinct sets P, and $b' > \aleph_0$, there is $P_1 \subset X$ such that $|P_1| < \aleph_0$; $|Y_1| = b$, where $Y_1 = \{y : y \in Y;\ f(y) = P_1\}$. Then $[X - P_1, Y_1]^{1,1} \subset K_1$, which contradicts (117). Therefore (118) is false, and there is $y_1 \in Y$ such that $|f(y_1)| = \aleph_0$. Put $\phi_1(X, Y) = y_1$; $\psi_1(X, Y) = f(y_1)$. Then $\phi_1(X, Y) \in Y$; $\psi_1(X, Y) \in [X]^{\aleph_0}$;

$$[\psi_1(X, Y), \{\phi_1(X, Y)\}]^{1,1} \subset K_0.$$

3. We define sequences x_ν, y_ν, X_ν, Y_ν $(\nu < \omega_0)$ as follows.

$$x_0 = \phi(A, B);\ Y_0 = \psi(A, B);\ y_0 = \phi_1(A - \{x_0\}, Y_0);\ X_0 = \psi_1(A - \{x_0\}, Y_0).$$

For $0 < \nu < \omega_0$, we put

$$x_\nu = \phi(X_{\nu-1}, Y_{\nu-1} - \{y_{\nu-1}\}); \qquad Y_\nu = \psi(X_{\nu-1}, Y_{\nu-1} - \{y_{\nu-1}\});$$
$$y_\nu = \phi_1(X_{\nu-1} - \{x_\nu\}, Y_\nu); \qquad X_\nu = \psi_1(X_{\nu-1} - \{x_\nu\}, Y_\nu).$$

Then

$$x_\nu \in X_{\nu-1} \subset X_{\nu-2} - \{x_{\nu-1}\} \subset X_{\nu-3} - \{x_{\nu-2}, x_{\nu-1}\} \subset \cdots$$
$$\subset X_0 - \{x_1, \cdots, x_{\nu-1}\} \subset A - \{x_0, \cdots, x_{\nu-1}\};$$

$$\begin{aligned} y_\nu \in Y_\nu \subset Y_{\nu-1} - \{y_{\nu-1}\} \subset \cdots \subset Y_0 - \{y_0, \cdots, y_{\nu-1}\} \\ \subset B - \{y_0, \cdots, y_{\nu-1}\}; \\ [\{x_\nu\}, Y_\nu]^{1,1} \subset K_0; \ [\{x_\nu\}, \{y_\nu, y_{\nu+1}, \cdots\}]^{1,1} \subset K_0; \\ [X_\nu, \{y_\nu\}]^{1,1} \subset K_0; \ [\{x_{\nu+1}, x_{\nu+2}, \cdots\}, \{y_\nu\}]^{1,1} \subset K_0; \\ (x_\mu, y_\nu) \in K_0 \ (\mu, \nu < \omega_0). \end{aligned}$$

This proves (115). Finally, as is well known [**13**, p. 135], $(2^{\aleph_0})' > \aleph_0$, so that (116) is a special case of (115). This proves Theorem 48.

We introduce the notation

$$\left\{ \begin{matrix} a \\ b \end{matrix} \right\} = |[A]^b|,$$

where A is a set such that $|A| = a$. If $a, b < \aleph_0$, then

$$\left\{ \begin{matrix} a \\ b \end{matrix} \right\}$$

is the ordinary binomial coefficient

$$\binom{a}{b}.$$

The following lemma is probably well known.

LEMMA 6. *If* $a \geqq \aleph_0$, *then*

$$\left\{ \begin{matrix} a \\ b \end{matrix} \right\} = a^b \text{ for } b \leqq a \quad \text{and} \quad \left\{ \begin{matrix} a \\ b \end{matrix} \right\} = 0 \text{ for } b > a.$$

PROOF. The result is obvious for $b=0$ and for $b>a$. Now let $0<b<a$. Choose n such that $|n| = b$, and A_ν, A such that $|A_\nu| = a$ for $\nu < n$, and $A = \sum' [\nu < n] A_\nu$. Then $|A| = ab = a$. If $X \in [A]^b$, then $X = \{x_\nu : \nu < n\}$. Every x_ν has a possible values. Hence

$$\left\{ \begin{matrix} a \\ b \end{matrix} \right\} \leqq a^b.$$

On the other hand, if $y_\nu \in A_\nu$ for $\nu < n$, then $Y = \{y_\nu : \nu < n\} \in [A]^b$, and the set of all such Y has a cardinal a^b. Hence

$$\left\{ \begin{matrix} a \\ b \end{matrix} \right\} \geqq a^b,$$

and the lemma follows.

THEOREM 49. *Suppose that* $0<s<t<\aleph_0$; $k>0$;

$$|l| = |k|^{\left\{\begin{matrix}a_0\\ r_0\end{matrix}\right\}\left\{\begin{matrix}a_1\\ r_1\end{matrix}\right\}\cdots\left\{\begin{matrix}a_{s-1}\\ r_{s-1}\end{matrix}\right\}},$$

$$\left[\begin{matrix}a_0\\ \cdot\\ \cdot\\ \cdot\\ a_{s-1}\end{matrix}\right] \to \left[\begin{matrix}b_0\\ \cdot\\ \cdot\\ \cdot\\ b_{s-1}\end{matrix}\right]_k^{r_0,\cdots,r_{s-1}}, \tag{119}$$

$$\left[\begin{matrix}a_s\\ \cdot\\ \cdot\\ \cdot\\ a_{t-1}\end{matrix}\right] \to \left[\begin{matrix}b_s\\ \cdot\\ \cdot\\ \cdot\\ b_{t-1}\end{matrix}\right]_l^{r_s,\cdots,r_{t-1}}. \tag{120}$$

Then

$$\left[\begin{matrix}a_0\\ \cdot\\ \cdot\\ \cdot\\ a_{t-1}\end{matrix}\right] \to \left[\begin{matrix}b_0\\ \cdot\\ \cdot\\ \cdot\\ b_{t-1}\end{matrix}\right]_k^{r_0,\cdots,r_{t-1}}. \tag{121}$$

PROOF. We use the notation of the partition calculus explained in the proof of Theorem 25. In addition, if Δ is a partition of M, and $M'\subset M$, then the relation

$$|\Delta| \leq a \quad \text{in} \quad M'$$

expresses the fact that the number of classes of Δ containing at least one element of M' is at most a. (119) and (120) imply that $b_\lambda \leq a_\lambda$ for $\lambda<t$. If $b_\lambda<r_\lambda$ for some $\lambda<t$, then (121) holds trivially. Hence we may assume that

$$r_\lambda \leq b_\lambda \leq a_\lambda \qquad \text{for } \lambda < t. \tag{122}$$

Let $|A_\lambda| = a_\lambda$ for $\lambda<t$, and consider any equivalence relation Δ on $[A_0, \cdots, A_{t-1}]^{r_0,\cdots,r_{t-1}}$ such that $|\Delta| \leq |k|$. Our aim is to find $B_\lambda \in [A_\lambda]^{b_\lambda}(\lambda<t)$ such that

$$|\Delta| \leq 1 \quad \text{in} \quad [B_0, \cdots, B_{t-1}]^{r_0,\cdots,r_{t-1}}. \tag{123}$$

Put, for $X_\lambda \in [A_\lambda]^{r_\lambda}$ $(s\leq\lambda<t)$,

$$\Delta_1(X_s, \cdots, X_{t-1}) = \prod \Delta(X_0, \cdots, X_{t-1}),$$

where the last product is extended over all systems $(X_0, \cdots, X_{s-1}) \in [A_0, \cdots, A_{s-1}]^{r_0,\cdots,r_{s-1}}$. By (122), this product has at least one factor. It follows that $|\Delta_1| \leq |l|$. Hence, by (120), there is $B_\lambda \in [A_\lambda]^{b_\lambda}$ $(s\leq\lambda<t)$ such that

$$|\Delta_1| \leq 1 \quad \text{in} \quad [B_s, \cdots, B_{t-1}]^{r_s,\cdots,r_{t-1}}.$$

By (122), we can choose

$$Y_\lambda \in [B_\lambda]^{r_\lambda} \qquad (s \leqq \lambda < t).$$

Put, for $X_\lambda \in [A_\lambda]^{r_\lambda}$ $(\lambda < s)$,

$$\Delta_s(X_0, \cdots, X_{s-1}) = \Delta(X_0, \cdots, X_{s-1}, Y_s, \cdots, Y_{t-1}).$$

Then $|\Delta_s| \leqq |k|$, and therefore, by (119), there is $B_\lambda \in [A_\lambda]^{b_\lambda}$ $(\lambda < s)$ such that $|\Delta_s| \leqq 1$ in $[B_0, \cdots, B_{s-1}]^{r_0, \cdots, r_{s-1}}$. By (122), we can choose $Y_\lambda \in [B_\lambda]^{r_\lambda}$ $(\lambda < s)$. Then, for any $X_\lambda \in [B_\lambda]^{r_\lambda}$ $(\lambda < t)$,

$$\begin{aligned}(X_0, \cdots, X_{t-1}) &\equiv (X_0, \cdots, X_{s-1}, Y_s, \cdots, Y_{t-1}) \\ &\equiv (Y_0, \cdots, Y_{s-1}, Y_s, \cdots, Y_{t-1})(\cdot \Delta).\end{aligned}$$

This proves (123) and so establishes Theorem 49.

We note the following special case of Theorem 49.

Corollary. *If*

$$a_0 \to (b_0)^1_k; \qquad a_1 \to (b_1)^1_{|k|^{a_0}}$$

then

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \to \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}^{1,1}_k.$$

We give some applications of this last result.

(a) If $0 < d < \aleph_0 \leqq a_1$, then

$$\begin{pmatrix} 2d-1 \\ a_1 \end{pmatrix} \to \begin{pmatrix} d \\ a_1 \end{pmatrix}^{1,1}_2.$$

This is best possible in the sense that, if $2d-1$ is replaced by $2d-2$, the last relation becomes false. We even have, as is easily seen,

$$\begin{pmatrix} 2d-2 \\ a_1 \end{pmatrix} \nrightarrow \begin{pmatrix} d \\ 1 \end{pmatrix}^{1,1}_2 \qquad (0 < d < \aleph_0 \leqq a_1).$$

(b) If $a_0' > |k| > 0$; $a_1' > |k|^{a_0}$, then

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \to \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}^{1,1}_k.$$

In particular, if we assume that $2^{\aleph_0} = \aleph_1$, then

$$\begin{pmatrix} \aleph_0 \\ \aleph_2 \end{pmatrix} \to \begin{pmatrix} \aleph_0 \\ \aleph_2 \end{pmatrix}^{1,1}_2.$$

More generally, if $2^{\aleph_n}=\aleph_{n+1}$, then

$$\binom{\aleph_n}{\aleph_{n+2}} \rightarrow \binom{\aleph_n}{\aleph_{n+2}}_2^{1,1}.$$

This is best possible in the following strong sense.

(c) If $a' \leqq |k|$; $b>0$, then

$$\binom{a}{b} \nrightarrow \binom{a}{1}_k^{1,1}.$$

To prove (c), choose $n \leqq k$ such that $|n| = a'$. Then $a = \sum[\nu<n]a_\nu$, where $a_\nu<a$. Choose sets A_ν, A, B such that $|A_\nu| = a_\nu$ $(\nu<n)$; $A = \sum' [\nu<n]A_\nu$; $|B| = b$, and put $[A, B]^{1,1} = \sum' [\nu<k]K_\nu$, where $K_\nu = [A_\nu, B]^{1,1}$ $(\nu<n)$. If, now, $0 \neq X \subset A$; $y \in B$; $[X, \{y\}]^{1,1} \subset K_\nu$ for some $\nu<k$, then $\nu<n$; $X \subset A_\nu$; $|X|<a$. This proves (c).

The following theorem is a corollary of a result due to Sierpiński.

THEOREM 50. *If* $2^{\aleph_0}=\aleph_1$, *then*

$$\binom{\aleph_0}{\aleph_1} \nrightarrow \binom{\aleph_0}{\aleph_1}_2^{1,1}.$$

PROOF. Let $A = [0, \omega_0]$; $B = [0, \omega_1]$. According to Sierpiński[14] the assumption $2^{\aleph_0}=\aleph_1$ implies, and is, in fact, equivalent to, the existence of a sequence of functions $f_\lambda(y) \in B$ $(\lambda \in A)$, defined for $y \in B$, such that, given any $Y \in [B]^{\aleph_1}$, there is $\lambda_0 \in A$ such that $\{f_\lambda(y): y \in Y\} = B$ $(\lambda_0 \leqq \lambda < \omega_0)$. Then $[A, B]^{1,1} = K_0 +' K_1$, where $K_0 = \{(\lambda, y): \lambda \in A; y \in B; f_\lambda(y)=0\}$. If, now, $(X, Y) \in [A, B]^{\aleph_0, \aleph_1}$, then, by the property of the functions f_λ, there is $\lambda \in X$; $y_0, y_1 \in Y$ such that $f_\lambda(y_\nu) = \nu$ $(\nu<2)$. Then $(\lambda, y_\nu) \in [X, Y]^{1,1}K_\nu$ $(\nu<2)$. This proves the assertion.

THEOREM 51. *If* $a, b>1$;

$$\binom{a}{b} \rightarrow \binom{a}{b}_k^{1,1}.$$

Then

$$\binom{a'}{b'} \rightarrow \binom{a'}{b'}_k^{1,1}.$$

PROOF. Let l and m be such that $|l| = a'$; $|m| = b'$. Then $a = \sum[\lambda<l]a_\lambda$; $b = \sum[\mu<m]b_\mu$, where $a_\lambda<a$; $b_\mu<b$. Put $A' = [0, l]$; $B' = [0, m]$, and suppose that $[A', B']^{1,1} = \sum[\nu<k]K'_\nu$. Then we choose sets A_λ, A, B_μ, B such that $|A_\lambda| = a_\lambda$ $(\lambda<l)$; $|B_\mu| = b_\mu$ $(\mu<m)$;

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[14] [14], French translation in [16]. See also [1].

$A=\sum'[\lambda<l]A_\lambda$; $B=\sum'[\mu<m]B_\mu$. Then $|A|=a$; $|B|=b$; $[A,B]^{1,1}=\sum[\nu<k]K_\nu$, where $K_\nu=\{(x,y):x\in A_\lambda;\ y\in B_\mu;\ (\lambda,\mu)\in K'_\nu\}$ $(\nu<k)$. By hypothesis, there is $(X,Y)\in[A,B]^{a,b}$ and $\nu<k$ such that $[X,Y]^{1,1}\subset K_\nu$. Put

$$A''=\{\lambda:\lambda<l;\ A_\lambda X\neq 0\};\qquad B''=\{\mu:\mu<m;\ B_\mu Y\neq 0\}.$$

Then $a=|X|=\sum[\lambda\in A'']|A_\lambda X|$; $|A_\lambda X|\leq|A_\lambda|<a$. Hence, by definition of a', $|A''|\geq a'$. But, $|A''|\leq|l|=a'$. Hence $|A''|=a'$ and, by symmetry, $|B''|=b'$.

Let $\lambda\in A''$; $\mu\in B''$. Then we can choose $x\in A_\lambda X$; $y\in B_\mu Y$, and we then have $(x,y)\in[X,Y]^{1,1}\subset K_\nu$; $(\lambda,\mu)\in K'_\nu$. Hence $[A'',B'']^{1,1}\subset K'_\nu$; $(A'',B'')\in[A',B']^{a',b'}$, and Theorem 51 follows.

COROLLARY. *If* $a>1$, *then*

$$\begin{pmatrix}a'\\ a\end{pmatrix}\nrightarrow\begin{pmatrix}a'\\ a\end{pmatrix}^{1,1}_2.$$

For, if

$$\begin{pmatrix}a'\\ a\end{pmatrix}\rightarrow\begin{pmatrix}a'\\ a\end{pmatrix}^{1,1}_2,$$

then, by Theorem 51 and the known equation $a''=a'$, we conclude that

$$\begin{pmatrix}a'\\ a'\end{pmatrix}\rightarrow\begin{pmatrix}a'\\ a'\end{pmatrix}^{1,1}_2$$

which contradicts Theorem 47.

We may mention that there is an obvious extension of Theorem 51 to relations

$$\begin{pmatrix}a_0\\ \cdot\\ \cdot\\ \cdot\\ a_{t-1}\end{pmatrix}\rightarrow\begin{pmatrix}a_0\\ \cdot\\ \cdot\\ \cdot\\ a_{t-1}\end{pmatrix}\qquad\text{for any } t.$$

In conclusion, we collect some polarised partition relations involving the first three infinite cardinals. They follow from Theorems 47–50. We put $\aleph_0=a$; $\aleph_1=b$; $\aleph_2=d$.

$$\begin{pmatrix}a\\ a\end{pmatrix}\nrightarrow\begin{pmatrix}a&1\\ 1&a\end{pmatrix}^{1,1};\quad\begin{pmatrix}b\\ b\end{pmatrix}\nrightarrow\begin{pmatrix}b&1\\ 1&b\end{pmatrix}^{1,1};$$

$$\begin{pmatrix}d\\ d\end{pmatrix}\nrightarrow\begin{pmatrix}d&1\\ 1&d\end{pmatrix}^{1,1}\qquad\text{(Theorem 47)};$$

$$\begin{pmatrix}a\\ b\end{pmatrix}\rightarrow\begin{pmatrix}a&a\\ a&b\end{pmatrix}^{1,1};\quad\begin{pmatrix}a\\ d\end{pmatrix}\rightarrow\begin{pmatrix}a&a\\ a&d\end{pmatrix}^{1,1}\qquad\text{(Theorem 48)}.$$

If $2^a = b$, then

$$\begin{pmatrix} a \\ d \end{pmatrix} \rightarrow \begin{pmatrix} a & a \\ d & d \end{pmatrix}^{1,1} \qquad \text{(Theorem 49)}$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix} \nrightarrow \begin{pmatrix} a & a \\ b & b \end{pmatrix}^{1,1} \qquad \text{(Theorem 50).}$$

It seems curious that the continuum hypothesis should enable us both to strengthen

$$\begin{pmatrix} a \\ d \end{pmatrix} \rightarrow \begin{pmatrix} a & a \\ a & d \end{pmatrix}^{1,1}$$

to

$$\begin{pmatrix} a \\ d \end{pmatrix} \rightarrow \begin{pmatrix} a & a \\ d & d \end{pmatrix}^{1,1}$$

and to show that

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a & a \\ a & b \end{pmatrix}^{1,1}$$

cannot be strengthened to

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a & a \\ b & b \end{pmatrix}^{1,1}.$$

REFERENCES

1. F. Bagemihl and H. D. Sprinkle, *On a proposition of Sierpiński*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 726–728.

2. B. Dushnik and E. W. Miller, *Partially ordered sets*, Amer. J. Math. vol. 63 (1941) p. 605.

3. P. Erdös, *Some set-theoretical properties of graphs*, Revista Universidad Nacional de Tucuman, Serie A vol. 3 (1942) pp. 363–367.

4. P. Erdös and R. Rado, *A combinatorial theorem*, J. London Math. Soc. vol. 25 (1950) pp. 249–255.

5. ———, *Combinatorial theorems on classifications of subsets of a given set*, Proc. London Math. Soc. (3) vol. 2 (1952) pp. 417–439.

6. ———, *A problem on ordered sets*, J. London Math. Soc. vol 28 (1953) pp. 426–438.

7. P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. vol. 2 (1935) pp. 463–470.

8. P. Hall, *On representations of sub-sets*, J. London Math. Soc. vol. 10 (1934) pp. 26–30.

9. F. Hausdorff, *Grundzüge einer theorie der geordneten Mengen*, Math. Ann. vol. 65 (1908) pp. 435–506.

10. ———, *Mengenlehre*, 3d ed., 1944, §16.

11. R. Rado, *Direct decompositions of partitions*, J. London Math. Soc. vol. 29 (1954), pp. 71–83.

12. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. (2) vol. 30 (1930) pp. 264–286.

13. W. Sierpiński, *Leçons sur les nombres transfinis*, Paris, 1928.

14. ———, *O jednom problemu G Ruzjevića koji se odnosi na hipotezu kontinuuma*, Glas Srpske Kraljevske Akademije vol. 152 (1932) pp. 163–169.

15. ———, *Sur un problème de la théorie des relations*, Annali R. Scuola Normale Superiore de Pisa Ser. 2 vol. 2 (1933) pp. 285–287.

16. ———, *Concernant l'hypothèse du continu*, Académie Royale Serbe. Bulletin de l'Académie des Sciences Mathématiques et Naturelles. A. Sciences Mathématiques et Physiques vol. 1 (1933) pp. 67–73.

17. A. Tarski, *Quelques théorèmes sur les alephs*, Fund. Math. vol. 7 (1925) p. 2.

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AXIOMATIC APPROACH TO THE HOMOTOPY GROUPS

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1. **Introduction.** The history of homotopy groups traces back to the fundamental groups of Poincaré [9; 10; 14].[1] For a given topological space X and a given point x_0 in X, the funnamental group $\pi_1(X, x_0)$ is defined by considering the loops in X with x_0 as basic point, i.e., the continuous maps $\lambda: S^1 \to X$ of the circle S^1 into X with a given point s_0 of S^1 mapped into x_0.

Replacing the circle S^1 by a higher dimensional sphere S^n, Hurewicz [7] introduced the homotopy groups $\pi_n(X, x_0)$ in 1935 which turned out to be very useful and prolific. In 1941, relative homotopy groups $\pi_n(X, A, x_0)$, $n \geqq 2$, of a topological space X modulo a subspace A at a given point x_0 were introduced by a joint paper of Hurewicz and Steenrod [4] and also independently by J. H. C. Whitehead [16]. These groups are defined by considering the continuous maps of an n-dimensional cell E^n into X with the boundary sphere S^{n-1} mapped into A and a given point s_0 of S^{n-1} mapped into x_0.

For each $n \geqq 2$, a boundary homomorphism

$$\partial: \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$$

is defined by taking restrictions of the maps on the boundary sphere S^{n-1} of the cell E^n.

For any continuous map $f: (X, A, x_0) \to (Y, B, y_0)$, an induced homomorphism

$$f_*: \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$$

is defined by means of composition.

These are the entities of the so-called homotopy theory [5]. One observes that the homotopy theory looks quite like a homology theory.

Since Eilenberg and Steenrod [2] established their celebrated axiomatic approach to homology theory in 1945, it has been a natural problem to ask whether a similar approach is possible for homotopy theory.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Because of the obvious similarity between homotopy theory and homology theory, one would naturally first ask whether the axioms of Eilenberg and Steenrod hold in homotopy theory. A check of the axioms shows that all but one of the axioms are satisfied. The exception is the excision axiom. Simple examples can be constructed where the excision axiom is violated. See [13, p. 83].

Steenrod, in his book [13], suggested that a candidate to replace the excision axiom might be the fibering theorem in the theory of fiber bundles [13, p. 90, Theorem 17.1]. His conjecture is almost completely correct except that the notion of fiber bundles is too strong to include the spaces needed in the various constructions contained in the uniqueness proof unless the category of spaces is restricted to consist of all CW-complexes only.

If, in Steenrod's conjecture, one replaces fiber bundles by fiber spaces in the generalized sense of Serre [11, p. 443], then a complete axiomatic approach may be established. In the present address, such an axiomatic approach will be sketched.[3] In fact, only a very special kind of fiber space will be used, namely, the space of paths.

2. **Preliminaries.** By a *triplet* (X, A, x_0), we mean a topological space X, a nonvacuous subspace A of X, and a point x_0 in A. If $A = x_0$, then the triplet (X, A, x_0) will be simply denoted by (X, x_0) and may be considered as a pair consisting of a topological space X and a point x_0 in X.

Let (X, A, x_0) be a given triplet. Consider the set $P(X)$ of all path-components of X. The path-components of X containing points of A form a subset O of $P(X)$. If we identify O to be a single point, we obtain a quotient set $P(X, A)$ which will be called *the set of path-components of X modulo A*. If $A = x_0$, then $P(X, A) = P(X)$.

Next, let U denote the space of all paths in X issuing from x_0; that is to say, U consists of the continuous maps $\sigma\colon I \rightarrow X$ with $\sigma(0) = x_0$, where $I = [0, 1]$ denotes the closed unit interval, and is topologized by the compact-open topology. There is a natural map $p\colon U \rightarrow X$ defined by $p(\sigma) = \sigma(1)$. Let $C = p^{-1}(A)$ and u_0 denote the degenerate path $u_0(I) = x_0$. Thus we obtain a triplet (U, C, u_0) called the *associated triplet* of (X, A, x_0). The map p defines a continuous map

$$p\colon (U, C, u_0) \rightarrow (X, A, x_0)$$

called the *associated projection*.

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[3] The essential idea of this axiomatic approach was given independently by J.-P. Serre and J. Milnor. See [17].

Let (Y, B, y_0) be a second triplet and let $f: (X, A, x_0) \rightarrow (Y, B, y_0)$ be a given continuous map. Since f maps the path-components of X into those of Y, it induces a ***natural induced transformation***

$$f_*: P(X, A) \rightarrow P(Y, B)$$

in an obvious way. f_* maps the neutral element of $P(X, A)$ into that of $P(Y, B)$.

Let (V, D, v_0) denote the associated triplet of (Y, B, y_0) with the associated projection

$$r: (V, D, v_0) \rightarrow (Y, B, y_0).$$

Then f induces a continuous map $\phi: (U, C, u_0) \rightarrow (V, D, v_0)$ defined by $\phi(\sigma) = f\sigma$ for each σ in U. Obviously, $fp = \phi r$ holds in the following rectangle

$$\begin{tikzcd}
(U, C, u_0) \arrow[r, "\phi"] \arrow[d, "p"] & (V, D, v_0) \arrow[d, "r"] \\
(X, A, x_0) \arrow[r, "f"] & (Y, B, y_0).
\end{tikzcd}$$

3. Homotopy theories. A *homotopy theory*

$$H = \{\pi, *, \partial\}$$

consists of three functions π, $*$ and ∂ described as follows.

The first function π associates with each triplet (X, A, x_0) and each integer $m \geqq 0$ an abstrat set $\pi_m(X, A, x_0)$; we require that $\pi_0(X, A, x_0)$ be the set $P(X, A)$ of path-components of X modulo A and that $\pi_m(X, A, x_0)$ be a group in each of the following cases:

(HG1) $m \geqq 2$.

(HG2) $m = 1$ and $A = x_0$.

(HG3) $m = 1$ and (X, A, x_0) is the associated triplet of some triplet of the form (R, r_0, r_0).

Note. In the case (HG3), $\pi_0(A, x_0)$ is a group with usual multiplication among classes of loops.

The second function $*$ associates with each continuous map $f: (X, A, x_0) \rightarrow (Y, B, y_0)$ and each integer $m \geqq 0$ an ***induced transformation***

$$f_*: \pi_m(X, A, x_0) \rightarrow \pi_m(Y, B, y_0).$$

We require that f_* should be the natural induced transformation defined in §2 in case $m = 0$ and that f_* should be a homomorphism in each of the following cases:

(IT1) $m \geqq 2$.

(IT2) $m=1$, $A=x_0$ and $B=y_0$.

(IT3) $m=1$, (X, A, x_0) is the associated triplet of some triplet (R, r_0, r_0), (Y, B, y_0) is the associated triplet of some triplet (S, s_0, s_0), and f is induced by a continuous map $\phi: (R, r_0)\to(S, s_0)$.

(IT4) $m=1, B=y_0$, (X, A, x_0) is the associated triplet of (Y, B, y_0), and f is the associated projection.

Note. In the case (IT3), $f_*: \pi_0(A, x_0)\to\pi_0(B, y_0)$ is also a homomorphism.

The third function ∂ associates with each triplet (X, A, x_0) and each integer $m>0$ a *boundary operation*

$$\partial: \pi_m (X, A, x_0) \to \pi_{m-1} (A, x_0);$$

we require ∂ to be a homomorphism in each of the following cases:

(BO1) $m\geqq 2$.

(BO2) $m=1$ and (X, A, x_0) is the associated triplet of some triplet (R, r_0, r_0).

Furthermore, the collection $H=\{\pi, *, \partial\}$ must satisfy the following seven axioms.

Axiom I. *If f is the identity map on a triplet (X, A, x_0), then f_* is the identity on $\pi_m(X, A, x_0)$.*

Axiom II. *If $f: (X, A, x_0)\to(Y, B, y_0)$ and $g: (Y, B, y_0)\to(Z, C, z_0)$ are continuous maps, then $(gf)_*=g_*f_*$.*

Axiom III. *If the continuous maps $f: (X, A, x_0)\to(Y, B, y_0)$ and $g: (A, x_0)\to(B, y_0)$ are related by $g=f|(A, x_0)$, then $\partial f_*=g_*\partial$.*

The foregoing three axioms are called the *algebraic axioms*.

Axiom IV (Homotopy Axiom). *If the continuous maps f, g: $(X, A, x_0)\to(Y, B, y_0)$ are homotopic, then $f_*=g_*$.*

Let (X, A, x_0) be any given triplet. Then the inclusion maps $i: (A, x_0)\to(X, x_0)$ and $j: (X, x_0)\to(X, A, x_0)$ together with the boundary operations ∂ give rise to a beginningless sequence

$$\cdots \xrightarrow{j_*} \pi_{m+1}(X, A, x_0) \xrightarrow{\partial} \pi_m(A, x_0) \xrightarrow{i_*} \pi_m(X, x_0) \xrightarrow{j_*} \pi_m(X, A, x_0) \xrightarrow{\partial} \cdots$$

$$\cdots \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0) \xrightarrow{j_*} \pi_0(X, A, x_0)$$

which will be called the *homotopy sequence* of the triplet (X, A, x_0) in the homotopy theory H.

Axiom V (Exactness Axiom). *The homotopy sequence of any triplet (X, A, x_0) is weakly exact.*

This means that, if $\pi_m(X, x_0)=0$ for all $m\geqq 0$, then $\partial: \pi_m(X, A, x_0) \approx \pi_{m-1}(A, x_0)$ for every $m>0$. Here, $S=0$ denotes that the set S consists of a single element and $\tau: M\approx M$ denotes that the transformation τ maps M onto N in a one-to-one fashion.

AXIOM VI (Fibering Axiom). *If* (U, C, u_0) *is the associated triplet of* (X, A, x_0) *and if* $p: (U, C, u_0)\to(X, A, x_0)$ *is the associated projection, then*

$$p_*:\pi_m(U, C, u_0) \approx \pi_m(X, A, x_0), \qquad m > 0.$$

AXIOM VII (Dimension Axiom). *If* $X=x_0$, *then* $\pi_m(X, x_0)=0$ *for every* $m\geqq 0$.

Let $H=\{\pi, *, \partial\}$ be a homotopy theory. The group $\pi_m(X, A, x_0)$, $m\geqq 2$, is called the mth *relative homotopy group* of X modulo A at the basic point x_0 (in the homotopy theory H); and the group $\pi_m(X, x_0)$, $m\geqq 1$, is called the mth (absolute) *homotopy group* of X at the basic point x_0. In particular, $\pi_1(X, x_0)$ is usually called the *fundamental group* or the *Poincaré group* of X at the basic point x_0. For completeness, we shall also call $\pi_m(X, A, x_0)$ the mth *relative homotopy set* of X modulo A at x_0 and $\pi_m(X, x_0)$ the mth *homotopy set* of X at x_0 for every $m\geqq 0$.

4. **Properties.** Most of the elementary results in homotopy theory can be deduced right from the axioms. Precisely, for any given homotopy theory

$$H = \{\pi, *, \partial\},$$

we shall give in the present section a few properties of H as consequences of its definition given in §3.

(4.1) *For every triplet* (X, A, x_0) *and every* $m\geqq 0$, $\pi_m(X, A, x_0)$ *is nonempty and contains a special element* 0 *called the neutral element of* $\pi_m(X, A, x_0)$.

In fact, 0 is the neutral element of $P(X, A)$ defined in §2 if $m=0$. For $m>0$, 0 is the image under j_* of the group-theoretic neutral element of $\pi_m(X, x_0)$.

(4.2) *If* $f: (X, A, x_0)\to(Y, B, y_0)$ *is a homotopy equivalence, then* $f_*: \pi_m(X, A, x_0)\approx\pi_m(Y, B, y_0)$ *for every* $m\geqq 0$.

As immediate consequences of (4.2), we obtain the following three assertions:

(a) For every $m\geqq 0$, $\pi_m(X, A, x_0)$ is an invariant of the homotopy type of the triplet (X, A, x_0).

(b) If A is a deformation retract of X, then $i_*: \pi_m(A, x_0)\approx\pi_m(X, x_0)$ for every $m\geqq 0$.

(c) If X is contractible to the point x_0, then $\pi_m(X, x_0)=0$ for every $m\geqq 0$.

Now let (X, A, x_0) be a given triplet. Consider the associated triplet (U, C, u_0) of (X, A, x_0) and the associated projection p: $(U, C, u_0)\to(X, A, x_0)$. We have a diagram

$$\pi_m(X, A, x_0) \xleftarrow{p_*} \pi_m(U, C, u_0) \xrightarrow{\partial} \pi_{m-1}(C, u_0)$$

for every $m>0$. According to the fibering axiom, we have p_*: $\pi_m(U, C, u_0)\approx\pi_m(X, A, x_0)$. Hence we obtain a *natural transformation*

$$\nu = \partial p_*^{-1} : \pi_m(X, A, x_0) \to \pi_{m-1}(C, u_0)$$

for every $m>0$. In the cases (HG1) through (HG3), ν is obviously a homomorphism.

(4.3) $\nu:\pi_m(X, A, x_0) \approx \pi_{m-1}(C, u_0)$ *for every* $m > 0$.

The special case $A=x_0$ is of importance. In this case, C becomes the space of loops in X with basic point x_0 and ν is an isomorphism of $\pi_m(X, x_0)$ onto $\pi_{m-1}(C, u_0)$ for every $m>0$.

Next, let (X, A, x_0) be a given triplet and Y be any fiber space (in the sense of Serre) over a subspace X_0 of X relative to a projection f: $Y\to X_0$. Let $B=f^{-1}(A)$ and pick a point $y_0\in f^{-1}(x_0)$. Thus we obtain a triplet (Y, B, y_0) and a continuous map f: $(Y, B, y_0)\to(X, A, x_0)$. If X_0 contains the path-component of X containing x_0, f will be called a *fibering map*.

(4.4) THE FIBERING THEOREM. *If* f: $(Y, B, y_0)\to(X, A, x_0)$ *is a fibering map, then* f_*: $\pi_m(Y, B, y_0)\approx\pi_m(X, A, x_0)$ *for every* $m>0$.

If (U, C, u_0) denotes the associated triplet of (X, A, x_0), then it is well-known that U is a fiber space over the path-component of X containing x_0 and hence the associated projection p: $(U, C, u_0)\to(X, A, x_0)$ is a fibering map. Therefore, (4.4) is a generalization of the Fibering Axiom.

Let X_0 denote the path-component of X containing x_0 and $A_0=A\cap X_0$, then a direct consequence of (4.4) is that the inclusion map i induces i_*: $\pi_m(X_0, A_0, x_0)\approx\pi_m(X, A, x_0)$ for every $m>0$.

By a *triple* (X, A, B), we mean a topological space X together with two nonvoid subspaces A and B such that $A\supset B$. Pick a basic point x_0 in B. The inclusion maps

$$i:(A, B, x_0) \subset (X, B, x_0), \qquad j:(X, B, x_0) \subset (X, A, x_0)$$

give rise to the induced transformations i_* and j_*. On the other hand, the inclusion map $k\colon (A, x_0)\subset(A, B, x_0)$ has the induced transformations k_*. Define a *boundary operation*

$$\bar\partial\colon \pi_m(X, A, x_0) \to \pi_{m-1}(A, B, x_0)$$

for every $m>0$ by taking the composition $\bar\partial = k_*\partial$ of

$$\pi_m(X, A, x_0) \xrightarrow{\partial} \pi_{m-1}(A, x_0) \xrightarrow{k_*} \pi_{m-1}(A, B, x_0).$$

Further, let 0 denote a trivial set consisting of a single element called its neutral element and let $\bar\partial\colon \pi_0(X, A, x_0)\to 0$ denote the trivial transformation. Thus we obtain a beginningless sequence

$$\cdots \xrightarrow{j_*} \pi_{m+1}(X, A, x_0) \xrightarrow{\bar\partial} \pi_m(A, B, x_0) \xrightarrow{i_*} \pi_m(X, B, x_0) \xrightarrow{j_*} \pi_m(X, A, x_0) \xrightarrow{\bar\partial} \cdots$$
$$\cdots \xrightarrow{\bar\partial} \pi_0(A, B, x_0) \xrightarrow{i_*} \pi_0(X, B, x_0) \xrightarrow{j_*} \pi_0(X, A, x_0) \xrightarrow{\bar\partial} 0$$

called the *homotopy sequence of the triple* (X, A, B) at the basic point x_0. If $B=x_0$, then this reduces to the homotopy sequence of the triplet (X, A, x_0) augmented on its right end by a trivial set 0. In this sequence, every set has a specific neutral element. By the *kernel* of a transformation in this sequence, we mean the inverse image of the neutral element.

(4.5) THE EXACTNESS THEOREM. *The homotopy sequence of any triple (X, A, B) at any basic point $x_0\in B$ is exact; that is to say, the kernel of every transformation in the sequence coincides with the image of the preceding transformation.*

Note. (4.5) strengthens as well as generalizes the Exactness Axiom.

(4.6) THE COMMUTATIVITY THEOREM. *$\pi_m(X, A, x_0)$ is an abelian group in each of the following cases:*

(C1) *$m=1$, X is an H-space* [11, p. 474], *with x_0 as a two-sided homotopy unit, and $A=x_0$.*

(C2) *$m=2$ and (X, A, x_0) is the associated triplet of a triplet of the form (R, r_0, r_0).*

(C3) *$m=2$ and $A=x_0$.*

(C4) *$m>2$.*

REMARK. If $n\geqq 0$ is a given integer and if $H=\{\pi, {}^*, \partial\}$ is defined only for the dimensions $m\leqq n$ such that the axioms of §3 are satisfied for these dimensions, then the assertions in this section are also true for these dimensions. In fact, only these dimensions are used in the proofs.

5. **Existence.** The purpose of this section is to prove the existence of a homotopy theory. In fact, we shall construct a homotopy theory $H=\{\pi, *, \partial\}$ by induction as follows.

According to the definition, $\pi_0(X, A, x_0)$ and $f_*: \pi_0(X, A, x_0) \rightarrow \pi_0(Y, B, y_0)$ are well-defined for every triplet (X, A, x_0) and every continuous map $f: (X, A, x_0) \rightarrow (Y, B, y_0)$.

Let $n \geqq 1$ be a given integer and assume that we have already constructed the homotopy sets $\pi_m(X, A, x_0)$, for each $m<n$ and triplet (X, A, x_0), together with the induced transformations f_* and the boundary operations ∂ on these homotopy sets, such that all conditions and axioms are satisfied.

First, let us construct the homotopy set $\pi_n(X, A, x_0)$ of a given triplet (X, A, x_0). Consider the associated triplet (U, C, u_0) of (X, A, x_0) and the associated projection $p: (U, C, u_0) \rightarrow (X, A, x_0)$. We define

$$(5.1) \qquad \pi_n(X, A, x_0) = \pi_{n-1}(C, u_0).$$

Next, let us define the boundary operation $\partial: \pi_n(X, A, x_0) \rightarrow \pi_{n-1}(A, x_0)$. Let $q=p \mid (C, u_0)$. Then we define

$$(5.2) \qquad \partial = q_*: \pi_{n-1}(C, u_0) \rightarrow \pi_{n-1}(A, x_0).$$

Finally, let $f: (X, A, x_0) \rightarrow (Y, B, y_0)$ be a given continuous map. We shall construct the induced transformation $f_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0)$ as follows. Let (V, D, v_0) denote the associated triplet of (Y, B, y_0) with associated projection $r: (V, D, v_0) \rightarrow (Y, B, y_0)$. According to §2, f induces a continuous map $\phi: (U, C, u_0) \rightarrow (V, D, v_0)$. Let $\psi=\phi \mid (C, u_0)$. Then we define

$$(5.3) \qquad f_* = \psi_*: \pi_{n-1}(C, u_0) \rightarrow \pi_{n-1}(D, v_0).$$

It is verified that all the conditions and axioms in §3 are satisfied. This completes the inductive construction of a homotopy theory $H = \{\pi, *, \partial\}$. This theory H will be called the *natural homotopy theory.*

6. **Uniqueness.** Two homotopy theories $H=\{\pi, *, \partial\}$ and $H' = \{\pi', \#, \partial'\}$ are said to be *equivalent* if there exists, for each triplet (X, A, x_0) and each $m \geqq 0$, a transformation

$$h_m: \pi_m(X, A, x_0) \rightarrow \pi'_m(X, A, x_0)$$

satisfying the conditions:

(E1) h_0 is the identity on $\pi_0(X, A, x_0)=\pi'_0(X, A, x_0)$.

(E2) h_m is a homomorphism if $m \geqq 2$ or if $m=1$ and $A=x_0$.

(E3) $h_m: \pi_m(X, A, x_0) \approx \pi'_m(X, A, x_0)$.

(E4) $h_m f_* = f_\# h_m$.

(E5) $h_{n-1}\partial = \partial' h_n$.

A collection of transformations $h = \{h_n\}$ satisfying the conditions (E1) through (E5) is called an *equivalence* between H and H' and is denoted by $h: H \approx H'$.

We are going to prove the *uniqueness theorem* that any two homotopy theories H and H' are equivalent. For this purpose, we shall construct an equivalence $h: H \approx H'$ as follows.

The transformation h_0 is defined by (E1). Let $n \geqq 1$ be a given integer and assume that we have already constructed the transformations h_m for each $m < n$ and each triplet (X, A, x_0) such that (E1)–(E5) are satisfied. Let us construct h_n as follows.

Let (X, A, x_0) be any triplet. Consider the associated triplet (U, C, u_0) of (X, A, x_0) and the associated projection $p: (U, C, u_0) \rightarrow (X, A, x_0)$. According to the Fibering Axiom, we have

$$p_*: \pi_n(U, C, u_0) \approx \pi_n(X, A, x_0), \quad p_\#: \pi_n'(U, C, u_0) \rightarrow \pi_n'(X, A, x_0).$$

It is well-known that U is contractible to the point u_0; hence $\pi_m(U, u_0) = 0$ for every $m \geqq 0$ by the Axioms I, II, IV and VII. According to the Exactness Axiom, this implies

$$\partial: \pi_n(U, C, u_0) \approx \pi_{n-1}(C, u_0), \quad \partial': \pi_n'(U, C, u_0) \approx \pi_{n-1}'(C, u_0).$$

By our assumption of induction, we have

$$h_{n-1}: \pi_{n-1}(C, u_0) \approx \pi_{n-1}'(C, u_0).$$

Hence we may define a transformation

$$h_n: \pi_n(X, A, x_0) \rightarrow \pi_n'(X, A, x_0)$$

by taking the composition

$$h_n = p_\# \partial'^{-1} h_{n-1} \partial \overset{-1}{p_*}.$$

It can be verified that h_n satisfies the conditions (E1) through (E5). This completes the inductive construction of $h = \{h_m\}$ and proves the uniqueness theorem.

The uniqueness theorem shows that the natural homotopy theory constructed in §5 is essentially the only homotopy theory.

The equivalence $h = \{h_m\}$ constructed above will be called the *natural equivalence* between the homotopy theories H and H'.

7. The uniqueness of equivalence. The purpose of this section is to show that the natural equivalence constructed in §6 is the only equivalence[3] between any two given homotopy theories $H = \{\pi, *, \partial\}$

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[3] As a remark given by A. D. Wallace at the meeting, this means that there is a unique way to prove the uniqueness theorem.

and $H' = \{\pi', \#, \partial'\}$.

By an *admissible transformation*

$$k = \{k_m\} : H \to H'$$

of H into H', we mean for each triplet (X, A, x_0) and each integer $m \geqq 0$ a transformation

$$k_m : \pi_m(X, A, x_0) \to \pi'_m(X, A, x_0)$$

satisfying the conditions:

(AT1) k_0 is the identity.

(AT2) $k_m f_* = f_\# k_m$.

(AT3) $k_{m-1}\partial = \partial' k_m$.

By (E1), (E4) and (E5), every equivalence between H and H' is an admissible transformation. Conversely, we shall prove that *every admissible transformation k coincides with the natural equivalence h*, i.e., $k_m = h_m$ for every $m \geqq 0$.

By definition, we have $k_0 = h_0$. We shall prove the assertion by induction. Let $n > 0$ be a given integer and assume that $k_m = h_m$ for every $m < n$. We are going to show that $k_n = h_n$.

Let (X, A, x_0) be any triplet. Consider the associated triplet (U, C, u_0) of (X, A, x_0) and the associated projection $p: (U, C, u_0) \to (X, A, x_0)$. In the following diagram

$$\begin{tikzcd}
\pi_n(X, A, x_0) \arrow[d, "k_n"] & \pi_n(U, C, u_0) \arrow[l, "p_*"'] \arrow[r, "\partial"] \arrow[d, "k_n"] & \pi_{n-1}(C, u_0) \arrow[d, "k_{n-1} = h_{n-1}"] \\
\pi'_n(X, A, x_0) & \pi'_n(U, C, u_0) \arrow[l, "p_\#"'] \arrow[r] & \pi'_{n-1}(C, u_0)
\end{tikzcd}$$

we have commutativity in both rectangles as required by (AT2) and (AT3). Hence we obtain

$$k_n = p_\# k_n p_*^{-1} = p_\# \partial'^{-1} h_{n-1} \partial p_*^{-1} = h_n.$$

This result implies that the natural equivalence h is the only equivalence between H and H'. Furthermore, in order to construct geometrically the natural equivalence between two homotopy theories given by geometric definitions, it suffices to establish an admissible transformation by means of some natural geometric method.

8. The role of the basic point. In this section, we shall continue our axiomatic approach and study the role played by the basic point in a homotopy theory $H = \{\pi, *, \partial\}$.

Let us consider a given space X and two given points x_0, x_1 connected by a path

$$\sigma: I \to X, \qquad \sigma(0) = x_0, \qquad \sigma(1) = x_1.$$

By definition, we have $\pi_0(X, x_0) = P(X) = \pi_0(X, x_1)$. Further, since x_0 and x_1 are contained in the same path-component of X, the neutral element of $\pi_0(X, x_0)$ is the same as that of $\pi_0(X, x_1)$. For the relations between the homotopy groups $\pi_m(X, x_0)$ and $\pi_m(X, x_1)$, $m \geqq 1$, let us consider the spaces of loops W_0, W_1 in X with basic points at x_0, x_1 respectively. Let $w_0 \in W_0$ and $w_1 \in W_1$ denote the degenerate loops.

The path σ induces a continuous map ξ: $W_1 \to W_0$ defined as follows: for each $w \in W_1$, $\xi(w) \in W_0$ is the loop defined by

$$[\xi(w)](t) = \begin{cases} \sigma(3t), & \text{if } 0 \leqq t \leqq 1/3, \\ w(3t-1), & \text{if } 1/3 \leqq t \leqq 2/3, \\ \sigma(3-3t) & \text{if } 2/3 \leqq t \leqq 1. \end{cases}$$

On the other hand, σ also induces a path η: $I \to W_0$ defined as follows: for each real number $s \in I$, $\eta(s) \in W_0$ is the loop defined by

$$[\eta(s)](t) = \begin{cases} \sigma(3st), & \text{if } 0 \leqq t \leqq 1/3, \\ \sigma(s), & \text{if } 1/3 \leqq t \leqq 2/3, \\ \sigma(3s-3st), & \text{if } 2/3 \leqq t \leqq 1. \end{cases}$$

Then, obviously we have $\eta(0) = w_0$ and $\eta(1) = \xi(w_1)$.

By a *system of operations* in a homotopy theory $H = \{\pi, *, \partial\}$, we mean for each path σ: $I \to X$ in any topological space X and each integer $m \geqq 0$ a transformation

$$\sigma_m: \pi_m(X, x_1) \to \pi_m(X, x_0), \qquad x_0 = \sigma(0), \qquad x_1 = \sigma(1)$$

satisfying the conditions:

(SO1) σ_0 is the identity on $\pi_0(X, x_1) = \pi_0(X, x_0)$.

(SO2) If $m > 0$ and if ξ: $(W_1, w_1) \to (W_0, \xi(w_1))$ and η: $I \to W_0$ are induced by σ, then commutativity holds in the following diagram

$$\begin{tikzcd}
\pi_m(X, x_1) \arrow[rr, "\sigma_m"] \arrow[d, "\nu"] & & \pi_m(X, x_0) \arrow[d, "\nu"] \\
\pi_{m-1}(W_1, w_1) \arrow[r, "\xi_*"] & \pi_{m-1}(W_0, \xi(w_1)) \arrow[r, "\eta_{m-1}"] & \pi_{m-1}(W_0, w_0)
\end{tikzcd}$$

where ν denotes the natural transformation.

According to (4.3), ν^{-1} is well-defined. Hence we obtain

$$\sigma_m = \nu^{-1}\eta_{m-1}\xi_*\nu. \tag{8.1}$$

It is easy to see that σ_m is a homomorphism if $m > 0$. By (SO1) and (8.1), the inductive proof of the following theorem is obvious.

(8.2) *In any given homotopy theory* $H=\{\pi, *, \partial\}$, *there exists one and only one system of operations. Furthermore, for any two given homotopy theories* H *and* H', *the natural equivalence* h: $H\approx H'$ *commutes with the operations in* H *and* H'.

The usual general properties of the system of operations in H can be deduced from (SO1) and (SO2) without using the traditional geometric meaning of these operations. Hence one may prove the following assertion.

(8.3) *For any given topological space* X *and any given integer* $m\geqq 1$, *the collection of the homotopy groups* $\{\pi_m(X, x_0)\mid x_0\in X\}$ *in a homotopy theory* $H=\{\pi, *, \partial\}$ *forms a local system of groups in* X *in the sense of Steenrod.*

Consequently, if $\sigma: I\rightarrow X$ is a path joining x_0 to x_1, then $\sigma_m: \pi_m(X, x_1)\approx\pi_m(X, x_0)$. If X is pathwise connected, then all the groups $\pi_m(X, x_0)$, $x_0\in X$, are isomorphic and the abstract group $\pi_m(X)$ which is isomorphic to $\pi_m(X, x_0)$ for each $x_0\in X$ will be called *the* mth (abstract) *homotopy group of* X.

Another consequence of (8.3) is that for each $m\geqq 1$, the fundamental group $\pi_1(X, x_0)$ acts on the left of $\pi_m(X, x_0)$ as a group of operators. In the special case $m=1$, one can easily see that, for any two elements g and h in $\pi_1(X, x_0)$, h acts on g as follows:

$$h(g) = hgh^{-1}.$$

Next, let us construct the operations in the relative homotopy sets. Consider a given topological space X, a given subspace A of X, and two given points x_0, x_1 connected by a path $\sigma: I\rightarrow A$ with $\sigma(0)=x_0$ and $\sigma(1)=x_1$.

Let (U_0, C_0, u_0) and (U_1, C_1, u_1) denote the associated triplets of (X, A, x_0) and (X, A, x_1) respectively. The path σ induces a continuous map ξ: $C_1\rightarrow C_0$ defined as follows: for each $u\in C_1$, $\xi(u)\in C_0$ is the path defined by

$$[\xi(u)](t) = \begin{cases}\sigma(2t), & \text{if } 0\leqq t\leqq 1/2,\\ u(2t-1), & \text{if } 1/2\leqq t\leqq 1.\end{cases}$$

On the other hand, σ also induces a path η: $I\rightarrow C_0$ defined as follows: for each $s\in I$, $\eta(s)\in C_0$ is the path defined by

$$[\eta(s)](t) = \begin{cases}\sigma(2st), & \text{if } 0\leqq t\leqq 1/2,\\ \sigma(s), & \text{if } 1/2\leqq t\leqq 1.\end{cases}$$

Obviously we have $\eta(0)=u_0$ and $\eta(1)=\xi(u_1)$.

For each $m\geqq 0$, we define an operation

$$\sigma_m: \pi_m(X, A, x_1) \to \pi_m(X, A, x_0)$$

as follows: σ_0 is the identity on $\pi_0(X, A, x_1) = \pi_0(X, A, x_0)$; if $m > 0$, then σ_m is uniquely defined by the commutativity of the following diagram

$$\begin{tikzcd}
\pi_m(X, A, x_1) \arrow[rr, "\sigma_m"] \arrow[d, "\nu"] & & \pi_m(X, A, x_0) \arrow[d, "\nu"] \\
\pi_{m-1}(C_1, u_1) \arrow[r, "\xi_*"] & \pi_{m-1}(C_0, \xi(u_1)) \arrow[r, "\eta_{m-1}"] & \pi_{m-1}(C_0, u_0)
\end{tikzcd}$$

where ν denotes the natural transformations. Hence,

$$\sigma_m = \nu^{-1}\eta_{m-1}\xi_*\nu \qquad (m > 0).$$

One can easily establish the usual general properties of these operations. In particular, $\sigma_m: \pi_m(X, A, x_1) \approx \pi_m(X, A, x_0)$ and, if $m \geqq 2$, *the relative homotopy groups* $\{\pi_m(X, A, x_0) \mid x_0 \in A\}$ *form a local system of groups in* A. Consequently, $\pi_1(A, x_0)$ acts on the left of $\pi_m(X, A, x_0)$ as a group of operators for every $m \geqq 2$.

If A is pathwise connected, then, for a given $m \geqq 2$, all the groups $\pi_m(X, A, x_0)$, $x_0 \in A$, are isomorphic. The abstract group $\pi_m(X, A)$ which is isomorphic to $\pi_m(X, A, x_0)$ for each $x_0 \in A$ will be called *the* mth (abstract) *relative homotopy group of* X *modulo* A.

9. **Remarks and problems.** Although a homotopy theory looks quite like a homology theory, they differ in the following aspects:

(1) The excision axiom holds in homology theory but not in homotopy theory; on the other hand, the fibering axiom holds in homotopy theory but not in homology theory.

(2) The uniqueness of homotopy theory is proved for all possible triplets, while that of homology theory is proved only for triangulable pairs. For general topological spaces, there are essentially different homology theories.

(3) In the homotopy sets $\pi_m(X, A, x_0)$, the basic point x_0 plays an important role, while the homology groups $H_m(X, A)$ do not depend on any basic point.

(4) All the homology groups $H_m(X, A)$, $m \geqq 0$, are abelian groups. On the other hand, $\pi_2(X, A, x_0)$ is in general a nonabelian group, and $\pi_0(X, A, x_0)$, $\pi_1(X, A, x_0)$ are in general not groups.

Hence, it is natural to expect that the homotopy groups are in general different from the corresponding homology groups. This was known to Poincaré for the fundamental groups. The first higher dimensional example was given by Hopf [4], namely, $\pi_3(S^2) \approx Z$ while

$H_3(S^2)=0$. Since then, numerous examples have been given in the literature.

Since the excision axiom does not hold in the homotopy theory, it is desirable to invent something which would measure the extent by which the excision axiom fails. This was solved by Blakers and Massey [1] by introducing their homotopy groups of triads. These groups as well as Freudenthal's suspension [3], can be conveniently defined in terms of spaces of paths [6], and hence can be fixed nicely into the present axiomatic scheme.

However, there are important operations in homotopy theory, such as the Whitehead products [15], which are defined by means of some specific geometrical constructions. So far, the author fails to present new definitions of these operations so that they might be fixed into the axiomatic scheme without appeal to the geometric representation of the homotopy groups.

Originally the question of axiomatizing the homotopy groups was considered important because it was hoped that such an axiomatization would lead to important new results or would simplify proofs of existing theorems. The proofs of the elementary properties of §4 and many others are improved by this axiomatization. However, it is not likely that the axiomatization will help to simplify the proofs of such highly geometric theorems as the homotopy addition theorem, etc., because these theorems are essentially of the geometric representations of the homotopy groups.

BIBLIOGRAPHY

1. A. L. Blakers and W. S. Massey, *The homotopy groups of a triad.* I, Ann. of Math. vol. 53 (1951) pp. 161–205.

2. S. Eilenberg and N. E. Steenrod, *Axiomatic approach to homology theory*, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 117–120.

3. H. Freudenthal, *Über die Klassen der Sphärenabbildungen.* I. *Grosse Dimension*, Compositio Math. vol. 5 (1938) pp. 299–314.

4. H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. vol. 104 (1931) pp. 637–665.

5. S. T. Hu, *An exposition of the relative homotopy theory*, Duke Math. J. vol. 14 (1947) pp. 991–1033.

6. ———, *Homotopy properties of the space of continuous paths.* II. *The general case with arbitrary boundary sets*, Portugaliae Mathematica vol. 11 (1952) pp. 41–50.

7. W. Hurewicz, *Beiträge zur Topologie der Deformationen.* I–IV, Koninklijke Nederlandsche Akademie van Wetenschappen, Proceedings vol. 38 (1935) pp. 112–119, 521–528, vol. 39 (1936) pp. 117–126, 215–224.

8. W. Hurewicz and N. E. Steenrod, *Homotopy relations in fiber spaces*, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 60–64.

9. H. Poincaré, *Analysis situs*, J. École Polytech. (2) vol. 1 (1892) pp. 1–121.

10. ———, *5^e Complement à l'analysis situs*, Rend. Circ. Mat. Palermo vol. 18 (1904) pp. 45–110.

11. J.-P. Serre, *Homologie singulière des espaces fibrés*, Ann. of Math. (2) vol. 54 (1951) pp. 425–505.

12. E. H. Spanier, *Borsuk's cohomotopy groups*, Ann. of Math. (2) vol. 50 (1949) pp. 203–245.

13. N. E. Steenrod, *Topology of fibre bundles*, Princeton Math. Series, no. 14, 1951.

14. H. Tietze, *Über die topologischen Invarianten mehrdimensionalen Mannigfaltigkeiten*, Monatshefte für Mathematik und Physik vol. 19 (1908) pp. 1–118.

15. J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2) vol. 42 (1941) pp. 409–428.

16. ———, *On the groups $\pi_r(V_{n,m})$ and sphere bundles*, Proc. London Math. Soc. (2) vol. 48 (1944) pp. 243–291.

17. J. Milnor, *Construction of universal bundles*, I, Ann. of Math. vol. 63 (1956) pp. 272–284 (*added in proof*).

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BOOK REVIEWS

Vector and tensor analysis. By Nathaniel Coburn. New York, Macmillan, 1955. 12+341 pp.

While unified under a single title and forming a coherent sequence, the material covered in this book could well be taught in three separate courses: an undergraduate course in vector analysis, and two graduate courses, one on tensor analysis and the other on elasticity and fluid dynamics.

The first four chapters of the book are devoted to vector analysis. The treatment is distinguished by a strong tensor flavor and by the fact that it is not confined to a single coordinate system. Throughout the book ample problems are provided at the end of nearly every section. As with most treatments of vector analysis, the sequence is from algebra to differentiation, integration and applications. A number of excellent applications to differential geometry are incorporated into the original exposition of vectors. The author's background in physics is evidently focused on mechanical problems and all other fields of physics are almost entirely ignored. When thermal problems are analyzed, they are viewed from a mechanical point of view, as on p. 79 where the author reasons, "If we assume that *heat behaves like a fluid mass and satisfies the continuity principle* . . . ." The modern physicist would reason simply, "If we assume that thermal energy is conserved. . . ."

Many topics are marked by their absence. No physical picture of the meaning of the divergence and curl of a vector is developed. Integral definitions of these concepts are not employed, resulting in a loss of physical clarity and in an unnecessarily lengthy development of the theorems of Stokes and Gauss. Other topics which are untouched are the scalar quasi-potential, the vector potential, classification of vector fields, and the entire range of fruitful applications to electromagnetic theory. The study of vectors concludes with a rather sketchy treatment of rigid-body motion and a rather beautiful and fairly detailed exposition of the flow of perfect fluids.

Tensors, on the other hand, are presented with a strong vector flavor. The approach is cautious. Vector notions are retained as long as possible and unnecessarily frequent use is made of unit vectors. First, tensors in orthogonal cartesian coordinates are treated. Thus, index notation is introduced without regard for the distinction between contravariance and covariance, and the coordinates of tensors are spoken of as components. Some may regard this backwards approach to tensors as pedagogically desirable; others will feel that the treatment of the trivial special case of tensors in orthogonal cartesian coordinates is an unfortunate method of introduction.

The subject is then extended to general cartesian coordinates. The transformations are centro-affine, but the geometry of affine space and its interesting physical applications are ignored. The useful distinction between space transformations and coordinate transformations is not made. The affine nature of the discussion that might have been expected here is quickly obviated by the introduction of the metric tensor. Even the metric tensor is not introduced in a form having general validity but as a method of calculating distances in the large by taking the scalar product of the radius vector with itself. Here, at long last, the student learns that "contravariant and covariant components" of a univalent tensor are not always equal.

The third chapter of the section on tensor analysis deals with tensors in general curvilinear coordinates. The conditions which an allowable coordinate system must satisfy are presented clearly. As the contravariant and covariant *coordinates* of univalent tensors have hitherto been referred to as *components* and as they obviously are not the same as the physically measurable vector components, a third kind of component is here introduced: the physical component. Though the metric tensor is not defined for general curvilinear coordinates, and the definition given in the previous chapter is no longer valid, the linear affine connection Γ^i_{jk} is defined in terms of unit vectors and the metric tensor. Subsequent topics are covariant derivatives, integrability conditions and nonholonomic coordinates.

By this time the reader may have gleaned a general idea of what is meant by a tensor, but nowhere is a definition of a tensor or of a geometric object given. Nor does the presentation clarify the fundamental question of which of the results of tensor analysis depend merely on the existence of a coordinatization of the points of space, which depend on the definition of Levi-Civita parallelism, and which require the introduction of a metric tensor.

A final chapter develops a formulation of the theorems of Gauss and Stokes in n-dimensional euclidean space, which is needed for the study of compressible fluids.

The crowning achievement of the book lies in the chapters on the application of tensors. The first of these chapters deals with the differential geometry of surfaces in euclidean 3-space. The treatment of elasticity which follows is particularly interesting in that it deals with finite deformations. Subsequent topics which are presented in a convincing manner in terms of tensors are viscous fluids, compressible fluids including the general theory of discontinuities and shock waves, and the theory of homogeneous statistical turbulence.

In the application of tensors to modern problems of fluid dynamics, the book is noteworthy. But a really satisfactory book on tensors should perform two functions. It should present tensors and related geometric concepts with clarity and precision. It should also give a well-rounded picture of the many fertile fields of application of tensors. Such a book remains unwritten.

DOMINA EBERLE SPENCER

Espaces vectoriels topologiques. (Chapters I-V, plus Fascicule de résultats). By N. Bourbaki. (Actualités Scientifiques et Industrielles, Nos. 1189, 1229, 1230.) Paris, Hermann, 1953–1955. 2+123+2 pp.; 2+191+3 pp., 2000 fr.; 2+39+1 pp., 400 fr.

Confronted by the task of appraising a book by N. Bourbaki, this reviewer feels as if he were required to climb the Nordwand of the Eiger. The presentation is austere and monolithic. The route is beset by scores of definitions, many of them apparently unmotivated. Always there are hordes of exercises to be worked through painfully. One must be prepared to make constant cross-references to the author's many other works. When the way grows treacherous and a nasty fall seems imminent, one thinks of the enormous learning and prestige of the author. One feels that Bourbaki *must* be right, and that one can only press onward, clinging to whatever minute rugosities the author provides and hoping to avoid a plunge into the abyss. Nevertheless, even a quite ordinary one-headed mortal may have notions of his own, and candor requires that they be set forth. We proceed, then, to a description of the present book.

Chapter I is entitled *Topological vector spaces over a field with a valuation*. It consists mostly of definitions and elementary theorems. As the coefficient field in later chapters is always the real or complex numbers, the emphasis here on arbitrary fields with valuation is hard to understand.

Chapter II deals with convex sets and locally convex spaces. It provides an excellent introduction to the subject. The Hahn-Banach theorem is given in several useful forms; Kreĭn's theorem on the extension of positive linear functionals is given, as well as the Kreĭn-Mil'man theorem. A curious appendix contains the Markov-Kakutani fixed point theorem (why not Schauder-Leray's or Tihonov's?), with an application showing the existence of an invariant mean for the continuous bounded functions on a topological Abelian group (a fact provable in other and simpler ways).

Chapter III takes up spaces of continuous linear mappings. The main theorem proved is the Banach-Steinhaus theorem, which appears as a statement about filters on the space $\mathcal{L}(E, F)$, where E is a "tonneau" space, F is a locally convex space with Hausdorff separation, and $\mathcal{L}(E, F)$ is all continuous linear mappings of E into F. In this formulation, the classical Banach-Steinhaus theorem and its elegant applications seem far away (although one standard application is given as an example). A host of types of topological vector spaces appear, mostly in exercises. Their utility for the general mathematician seems small.

Chapter IV, entitled *Duality in topological vector spaces*, is in the reviewer's opinion the most useful of all the five chapters. Here is a complete and readable account of the various topologies for the space of continuous linear functionals on a topological vector space.

Chapter V contains a treatment of the elementary theory of Hilbert spaces. Aside from a liberal use of filters, there seems to be little novelty in this chapter. One notes a surprising concession to human weakness—the author has furnished a couple of diagrams to illustrate a well-known theorem on the existence of unique distance-minimizing elements in convex sets.

In an interesting Historical Note, the author traces the history of the subject, from the contributions of D. Bernoulli to those of L. Schwartz. This spirited and at the same time learned account is well worth reading.

The "Fascicule de résultats" is of doubtful value. It would seem difficult to appreciate or use this brief summary without first having studied the main text: and when this has been done, the summary is not needed. A similar comment applies to the folded inserts at the ends of the volume repeating the most important definitions and axioms. A dictionary giving various common terms in English, French, and German is provided, as well as brief lists of special symbols.

To summarize: Plus ça change, plus c'est le même Bourbaki.

EDWIN HEWITT

Die innere Geometrie der konvexen Flächen. By A. D. Alexandrow Berlin, Akademie Verlag, 1955, 38.50 DM.

This is a German translation of the Russian book with the analogous title which appeared in 1948. Except for corrections of misprints and of some minor errors and the translator's simplifications of one or two proofs, the book differs from the Russian edition only by the addition of two appendices which report without proofs on recent developments. The author regrets that he could not integrate these into the text owing to other duties—he is rector of Leningrad University.

For this reason, and because a detailed description of the Russian edition is available in the *Mathematical Reviews* (vol. 10 (1949) pp. 619, 620), the principal results of the theory in its *present* stage will be briefly outlined here, marking results found in the appendices by (A).

H. Weyl put and partially solved the problem of realizing a prescribed line element of positive curvature on a sphere as a closed convex surface in E^3.[1] The central problem in A. D. Alexandrow's work is the analogue to Weyl's problem for perfectly general convex surfaces.

The intrinsic metric of a polyhedron in E^3 is locally euclidean at all points except the vertices. A neighborhood of a vertex is intrinsically isometric to the neighborhood of the apex of a cone. If the polyhedron is convex then the total angle of this cone at the apex (i.e. the angle after unfolding the cone) is less than 2π. We can therefore describe the intrinsic metric of a closed convex polyhedron as an intrinsic metric (i.e. one where distance equals the length of the shortest connection) on a sphere such that every point has a neighborhood isometric to a neighborhood of the apex of a cone with total angle less than or equal to 2π. The author's first fundamental result is that these trivially necessary conditions are also sufficient, i.e., that any metric on the sphere with these properties can be realized by one, and up to motions only one, convex polyhedron in E^3.

For the corresponding problem on general convex surfaces the simple characterization by conical neighborhoods no longer works. Any two points a, b on a closed convex surface S can be connected by a shortest curve whose length we call the distance of a and b. If $x(t)$ and $y(t)$, $0 \leq t \leq 1$, are two shortest arcs issuing from $p = x(0) = y(0)$ and $\alpha(t)$ is the angle at p' in the euclidean triangle $p'x_t'y_t'$ for which the lengths of the sides equal the corresponding distances of p, $x(t)$, $y(t)$ on S, then $\alpha(t)$ is a nonincreasing function of t. This important fact is not easy to prove and is called the convexity condition. Thus the intrinsic metric of a closed convex surface may be described as an intrinsic (see above) metrization of the sphere which satisfies the convexity condition. Any metric on a sphere with these properties can be realized by one, and up to motions only one, closed convex surface in E^3. The author proves the existence by approximation with polyhedra, the uniqueness is, of course, equivalent to the rigidity of general convex surfaces and is due to Pogorelov (A). Because the convexity condition is not obvious and also in some respects too strong, it is shown to be equivalent to another condition which requires that, with a properly defined angle, the excess (i.e., the sum of the angles minus π) in a small geodesic triangle is non-negative.

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[1] *Über die Bestimmung einer geschlossenen Fläche durch ihr Linienelement*, Vierteljahrschrift der naturf. Ges. Zürich vol. 61 (1916) pp. 40–72. The problem was first solved completely in the analytic case by H. Lewy in: *On the existence of a closed convex surface realizing a given Riemannian metric*, Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 104–106.

This result is combined with another essential idea which the author calls the method of gluing. Although it applies to general convex surfaces, we formulate it here only for pieces of smooth surfaces, because this will bring out the idea more clearly. Assume we have a finite number of surfaces of non-negative curvature each homeomorphic to a disk and bounded by a finite number of smooth arcs. Assume further that boundary arcs are abstractly identified such that (1) the resulting surface is, topologically, a sphere, (2) any two identified (partial) arcs have the same length (3) at any point lying on exactly two boundary arcs the sum of the geodesic curvatures of these arcs (towards the domains they bound) is non-negative (4) the total angle at a point belonging to more than two domains is at most 2π. Then this abstract surface can be realized as a convex surface in E^3.

Let an intrinsic metrization of the plane as a complete space be given which satisfies the convexity or the excess condition. By the method of gluing the author constructs a sequence of closed surfaces containing larger and larger pieces of the given metric and thus proves that the given metric can be realized by a complete unbounded convex surface in E^3. The realization is unique if the integral curvature of the given metric equals 2π (Pogorelov (A) under certain additional hypotheses, and recently for the general case in Doklady Akad. Nauk U.S.S.R. vol. 106 (1956) pp. 19–20). It is not unique when the integral curvature is less than 2π (Oloyanišnikov (A)). The author also uses the method of gluing to realize partial surfaces, i.e., intrinsic metrics satisfying the excess condition and defined on connected open sets of the sphere. The realization is, of course, in general not unique.

These results changed Weyl's original problem entirely: the restriction to closed surfaces became unnatural, the existence of a realization was no longer a problem, but there emerged the novel problem of proving the smoothness of *all* realizations provided the given metric is smooth. This was accomplished by Pogorelov (A) in all cases under the assumption that the coefficients E, F, G have locally bounded fifth derivatives and that the curvature is positive. The, in some respects, most interesting cases, where E, F, G are of class C^2 or C^3 (the importance of this case derives from the Gauss-Codazzi Equations) remain open.

A simple example will exhibit the strength of these results. Consider a cap K_0 of the unit sphere with integral curvature less than 2π and bounded by a circle C_0. Imbed C_0 in any continuous family C_t of plane convex curves with the same length bounding domains D_t. By the gluing theorem the abstract surface resulting through identification of the boundaries C_t of D_t and C_0 of K_0 can be realized as a convex surface K_t' in E^3. Pogorelov's results guarantee that K_t' is unique (up to motions) and that the interior K_t of the part K_t' corresponding to K_0 is analytic. Thus we obtain a continuous deformation of K_0 through analytic surfaces.

The author uses his methods to prove analogous realization theorems in spaces of constant positive and negative curvature. The latter case is by far the more interesting because a convex surface in hyperbolic space may have negative Gauss curvature and is, topologically, any open connected subset of the sphere. An abstract surface of this general topological type whose curvature is bounded from below can be realized as a convex surface in a hyperbolic space.

This review of results does not give any indication about the methods. Most concepts, in particular that of angle and total geodesic curvature of an arc, require a detailed study with laborious proofs before they become applicable to the general situations treated by the author. In collaboration with some of his students the author showed (A) that these concepts are applicable to a wide class of surfaces, namely the surfaces of bounded curvature (A). They include all surfaces which can be uniformly approximated by sequences of smooth surfaces with uniformly bounded integral curvatures; they comprise, in particular, all polyhedra with a finite number of vertices. This fact alone should suffice to make the theory appealing to any true geometer.

In spite of some very long proofs the book is very easy to read, because the author spares no trouble in outlining his aims and procedure beforehand, and has a flair for setting the reader at ease. The translation, which is excellent throughout, is especially successful in reproducing this effect.

HERBERT BUSEMANN

E. Vidal Abascal. *Introduccion a la geometria diferencial.* Madrid, Editorial Dossat, 1956, 16+329 pp. 220 ptas.

This book, written by the professor of geometry at the University of Santiago de Compostela, is a textbook of classical differential geometry with the usual subjects. In its use and notation of vectors, its general treatment of material and problems it acknowledges the influence of reviewers text *Lectures on classical differential geometry* (Cambridge, Mass., 1950), now also available in a Spanish translation by Bravo Gala: *Geometría diferencial clásica* (Madrid, Aguilar, 1955, 11+256 pp.). It contains, moreover, an introduction on vectors, determinants, matrices, and a special chapter on curves in the euclidean plane with some problems on isometry and integral geometry. In the discussion of these problems, as well as in the exposition of Cartan's methods in the theory of Pfaffians, the author acknowledges the influence of Blaschke's books on Differential and on Integral geometry; and in his preface pays tribute to Bieberbach's *Differentialgeometrie* (1932). We can also, in the text, make the acquaintance of some of Professor Vidal Abascal's own contributions to geometry. Among the many illustrations we find some pictures of the creators of differential geometry.

D. J. STRUIK

Champs de vecteurs et de tenseurs. By E. Bauer. Paris, Masson, 1955. 204 pp. 2200 fr.

This text treats three main topics: the elementary theory of vectors and tensors and applications, the more advanced theory of vectors and tensors, an introduction to electromagnetic theory. In the first two topics, the theories of vectors and tensors are presented simultaneously by skillfully interweaving both subjects. First, the customary intuitive approach for three-dimensional Euclidean space is given and in the advanced theory a more detailed study in affine and metric n-dimensional space is presented. The introduction to electromagnetic theory furnishes the basic ideas of this subject.

After defining vectors as directed line segments which add by the parallelogram law and discussing the scalar product, the author examines rectangular and oblique Cartesian coordinates and transformations of coordinates. This leads to Gibbs dyads and the introduction of tensors in terms of their components. Other topics in this section are: vector algebra, differentiation theory, and the theorems of Gauss and Stokes. In treating the last topic, the author uses the physical approach to interpret the divergence and curl of a vector field and then defines these quantities as limits of integrals. The study of the del operator and Laplacian and Newtonian fields concludes this section of the text.

Vectors and tensors in affine and metric n-dimensional space are treated in Chapter IV. First, the author discusses the curl of a vector and the divergence of a vector density in affine n-space. This is followed by a discussion in metric space of the metric tensor, the absolute derivative, geodesics, and related topics.

Finally, the author considers the following topics in electromagnetic theory: electrostatic fields (force, energy, and polarization); electric currents (steady and non-steady) and the laws of Kirchoff, Joule, and Ampère; magnetic and electromagnetic fields. The last topic is presented with considerable skill and ranges from the Maxwell laws of classical three-dimensional Euclidean space (presented by classical vector methods) to the Lorentz-Einstein transformation in Minkowski space and the Maxwell tensor of relativity.

The level of the text is such that a mathematically mature student with a background in classical physics can follow the developments. The book should be of particular interest to physicists and engineers.

N. Coburn

Surface area. By L. Cesari. (Annals of Mathematics Studies, no. 35.) Princeton University Press, 1956. 10+595 pp. $8.50.

The length $l(C)$ of a curve C is the limit of the lengths $l(p_n)$ of inscribed polygons $p_1, \cdots, p_n, \cdots$ such that the maximum side-length of p_n converges to zero as $n \to \infty$. Almost eighty years ago, Schwarz and Peano noted that the analogous statement for surface area (in terms of the elementary areas of inscribed polyhedra) is false even in the simple case when the surface under consideration is the lateral surface of a circular cylinder. Subsequently, many other phenomena were noted which revealed further fundamental discrepancies between arc length and surface area. The immediate issue raised by the initial observations of Schwarz and Peano was, however, the formulation of a logically consistent definition of surface area. During the past eighty years, many such definitions have been proposed, and an enormous amount of effort has been expended in the study of these various concepts of surface area. As far as mathematical fields of a classical type are concerned, the reassuring inference from these studies is the fact that in reasonably decent cases the classical integral formula taught in calculus does indeed yield the correct value of surface area. On the other hand, the need for a comprehensive general theory of surface area became apparent several decades ago in various modern fields of Analysis and Geometry.

Thus surface area theory developed partly in response to needs arising in various mathematical fields, and partly in response to the challenge presented by apparent discrepancies between arc length and surface area. Furthermore, the confusion created by the extreme diversity of the definitions proposed for surface area constituted by itself a strong challenge to develop a theory of compelling unity, beauty, utility, and generality. The concept of surface area proposed by Lebesgue around the turn of the century (to be referred to as the Lebesgue area) has been adopted for this purpose by many workers in this field. In its present highly developed form, the theory of the Lebesgue area possesses several gratifying features. It reveals that the discrepancies between arc length and surface area (referred to above) are compensated for by a number of far-reaching analogies. Also, the theory shows that many of the conflicting definitions proposed for surface area become equivalent with the Lebesgue area provided that the underlying intuitive geometrical ideas are phrased in a reasonably relaxed manner. Furthermore, the theory yields important applications in relation to various classical and modern problems in other fields. Professor Cesari, the author of the book under review, made many contributions of the highest importance in surface area theory and in related fields, and therefore the publication of his present book is a significant event. For the general mathematical community, the book brings a beautifully organized presentation of many of the fundamental accomplishments in the theory of the Lebesgue area, as well as a wealth of informative and penetrating comments on historical background and motivation. For the specialist in surface area theory and in related fields, the book of Cesari is an indispensable library item, with many features of compelling interest. One finds here the first exposition in book form of the famous *Cesari inequality* which states that the Lebesgue area of a surface is less than or equal to the sum of the Lebesgue areas of its orthogonal projections (considered as *flat surfaces*) upon the three coordinate planes. From the intuitive point of view, this inequality is merely an obvious property of surface area, and yet the actual proof of the inequality by Cesari has been justly hailed as one of the most striking achievements in surface area theory. Indeed, the Cesari inequality furnished the missing link in establishing in full generality several fundamental theorems, and furthermore, its proof contained auxiliary results of a striking and novel character. One of these, the so-called *four-line theorem*, is concerned with a remarkable homotopy property of Euclidean 3-space. Subsequently, Cesari showed how to avoid the use of this theorem by ingenious analytic devices. Both of these alternative approaches are treated fully in the book. One finds here also the first exposition in book form of the fundamental *Cesari representation theorem*, which states that every surface (this term being taken in the appropriate precise meaning explained in the book) of finite Lebesgue area admits of a conformal representation (in a certain generalized sense) such that in terms of this representation the Lebesgue area is given by the classical integral formula. An important feature of the book is the generality of the mappings which occur as representations of surfaces. Indeed, Cesari operates with mappings from plane sets of a general type which include open sets and finitely connected Jordan regions. From the point of view of general methodology, the following feature of the book is of special interest. Cesari notes that several of the fundamental theorems of the theory are concerned with a given individual representation of the surface, and he suggests that in such cases the proofs should be made in terms of the given representation, without using the existence of more favorable representations for the same surface. In carrying out this program throughout the book, he devises novel and most instructive proofs for several fundamental theorems and develops remarkable new methods. This feature of the book should give much food for thought even to the most seasoned specialist in the field. To the reader interested in applications, the Appendix on the Weierstrass-type integral developed by Cesari should be especially valuable.

The presentation is careful, precise, and concise throughout. Both for reference and for study, the book is an invaluable addition to the literature on surface area theory.

T. RADO

Additive Zahlentheorie. Part I. *Allgemeine Untersuchungen*. By Hans-Heinrich Ostmann. (Ergebnisse der Mathematik und ihrer Grenzgebiete, New series, no. 7.) Berlin, Springer, 1956. 7+233 pp. 29.80 DM.

The author has chosen to present additive number theory from the point of view of the theory of density of sets of integers. The largest part of this volume is devoted to the development of this theory and closely related topics, but certain other parts of additive number theory are included.

The basic concepts are the system Σ of sets $\mathfrak{A}$ of non-negative integers and the operation $\mathfrak{A}+\mathfrak{B}=\{a+b; a\in\mathfrak{A}, b\in\mathfrak{B}\}$. Their properties are developed in a natural way and some connections with other parts of number theory are pointed out.

Various types of density are defined and compared, and an extensive theory is developed. In the course of this study, other concepts and methods are introduced, and thus the contacts with other parts of number theory are increased.

One chapter, concerned with the partition function, is practically independent of the rest. It is essentially a survey of this subject. A few theorems are proved and a large number of results are stated without proof but with adequate references.

Throughout the volume, proofs are given with a minimum of details and, in some cases, are omitted.

H. S. ZUCKERMAN

Algebraic threefolds. By L. Roth. (Ergebnisse der Mathematik und ihrer Grenzgebiete, New series, no. 6.) Berlin, Springer, 1956. 8+142 pp. 19.80 DM.

The contents of this book can be divided into three parts and an appendix (plus a good bibliography); the first part, consisting of the first three chapters, contains a very compressed outline of such topics as the genera of algebraic varieties, Severi's systems of (rational) equivalence, results of the Riemann-Roch type for 3-dimensional varieties, and the theory of the base. All this is given without proofs, or with sketches of proofs of the classical type (that is, over the field of complex numbers, and without refraining from the use of imperfectly defined concepts).

The second part, consisting of Chapters 4 and 5, is devoted to the main topic of the book, namely several criteria of rationality or unirationality for surfaces and 3-dimensional varieties; proofs are usually supplied here. The third part (Chapter 6) contains a classification of varieties, especially of dimension three, which admit continuous groups of transformations; the sections dealing with pseudo-abelian and para-abelian varieties, as well as some portions of the second part of the book, had previously appeared only in original papers by the same author. A 15-page appendix gives a condensed list of results on surfaces, with which the reader is assumed to be more or less familiar.

Actually, it seems to the reviewer that the reader should be quite familiar with the whole body of algebraic geometry, before he can attempt reading this work with understanding and profit; for a reader of this type, Chapters 1 to 3 could then have been omitted, or relegated to the appendix, thus making room for a more detailed and more critical exposition of the content of the remaining chapters, and in particular of the criteria of rationality; these will undoubtedly be the main attraction of the book, both for the "classical" and for the "abstract" algebraic geometer. The former will find here the only systematic collection of criteria of rationality, and a large number of special examples; the latter will, on the other hand, find a vast and challenging field which, by modern standards of rigour and generality, is virtually unexplored (it is enough to mention the fact that such a basic result as Castelnuovo's theorem on the rationality of plane involutions still awaits a convincing proof).

The abstract algebraic geometer has learned by experience that the results of classical algebraic geometry, although often based on debatable proofs, are nonetheless correct, and have been tested, by persons endowed with a keen geometric intuition, on a large number of concrete cases; thus, a readily accessible collection of theorems is an invaluable help for anybody desirous to test the power of the abstract methods against problems of rationality.

IACOPO BARSOTTI

RESEARCH PROBLEMS

13. A. D. Wallace: *A problem on modular lattices.*

Denote by L a connected compactum and suppose that L is supplied with a pair $\vee$, $\wedge$ of continuous lattice operations. It is known (L. W. Anderson, unpublished) that if L can be imbedded in R^3 then L is distributive. It is also known (D. E. Edmondson, to appear in Proc. Amer. Math. Soc.) that L may be topologically a 3-cell and be nonmodular. If L is modular and can be impedded in R^n it seems unlikely that L has to be distributive. If L is modular, if L can be imbedded in R^n, and if the boundary of L (relative to R^n) is a distributive sublattice of L does L have to be distributive? It may be helpful to use the fact (L. W. Anderson, to appear in Proc. Amer. Math. Soc.) that if $\dim L = 1$ then L is a chain. (Received April 27, 1956.)

14. A. D. Wallace: *Topological algebraic structures.*

A space has PRF if it is compact and if each proper retract has the fixed point property. It is clear that an absolute retract or any n-sphere has PRF. Many pathological spaces have this property, for example certain indecomposable continua. If S is a topological semigroup with PRF then either S is a group or else K, the minimal ideal of S, consists of idempotents. If $\dim S = n \geqq 1$ then $H^n(S) = 0$ (any coefficients) if S also has a unit and if S is not a group. If $n \geqq 2$ is $H^{n-1}(S) = 0$ under these hypotheses? Do either of these conclusions hold if the stipulation "S has a unit" is replaced by "$S = S \cdot S$"? For references see Bull. Amer. Math. Soc. vol. 61 (1955) pp. 95–112. (Received May 28, 1956.)

ALGEBRAIC GROUP-VARIETIES

IACOPO BARSOTTI

1. **Definitions and preliminaries.** Let k be an algebraically closed field, and let S be an n-dimensional projective space over k; the *points* of S are the ordered sets $\{\xi_0, \cdots, \xi_n\}$ of elements of k, with the exception of the set $\{0, \cdots, 0\}$, and where two sets $\{\xi\}$, $\{\xi'\}$ are identified if $\xi_i' = \rho\xi_i$ for each i, and for some $\rho \in k$. An (algebraic) *variety over k* is the set V of all the points $\{\xi\}$ whose coordinates $\xi_0, \cdots, \xi_n$ satisfy a given finite set of homogeneous algebraic equations with coefficients in k; V is *irreducible* if it is not the join of two nonempty varieties, neither of which contains the other. The restriction to algebraically closed fields is not strictly necessary, but is very convenient for expository purposes, and will be kept in force throughout this address. The case in which k is the complex field will be referred to as *the classical case.*

The (*cartesian*) *product* $V \times F$ of two varieties is defined in the usual way; it is a (pseudo-) variety embedded in a biprojective space, but it is also birationally equivalent (see below), in a one-to-one manner, to a variety (the *Segre product*) in the previous sense. A *cycle on* V is an element of the free abelian group generated by the subvarieties of V; a cycle is *effective* if all its (irreducible) *components* appear in it with a positive coefficient (*multiplicity*). If V, F are irreducible, an effective cycle D on $V \times F$ is also called an *algebraic correspondence between F and V*; to a point $P \in F$ corresponds the variety $D[P]$ consisting of all the $Q \in V$ such that $P \times Q \in D$; if P is *generic* (that is, if it does not belong to a certain proper subvariety of F), and if all the components of D have the same dimension and operate on the whole F, a certain multiplicity $e(U)$ can be attached to each component U of $D[P]$; the cycle $\sum_U e(U)U$ on V is then denoted by $D\{P\}$ (see [1] and [2] in the bibliography at the end of the paper).

The algebraic correspondence D between F and V is called a *rational mapping of F into V* if $D\{P\}$ is a point, with multiplicity 1, of V for a generic $P \in F$; if D is a rational mapping of F into V and also of V into F, it is called a *birational transformation*, and F, V are then said to be *birationally equivalent* [31]. These definitions, and this language, are used in [1; 2; 3; 4; 5; 6; 7; 8; 31; 32]; quite different,

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An address delivered before the Vancouver meeting of the Society on June 18, 1955, by invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings; received by the editors June 25, 1955.

but substantially equivalent, language is used by Weil and his school in [12; 13; 18; 19; 20; 26; 29; 30].

The idea of algebraic correspondences is not limited to correspondences between varieties: it can be extended, with definite advantages, to algebraic correspondences between a field and a variety; this is done in [1; 2], and systematically used in al lmy subsequent work.

In order to introduce the notion of group-variety, we shall first make the following convention: if V is a variety, we shall denote by V_1, V_2, $\cdots$ a set of "copies" of V; if x is any entity related to V, x_i shall mean the copy of x which is similarly related to V_i. Let then V be an irreducible variety over k, and let V_1, V_2, V_3 be copies of V; a rational mapping of $V_1 \times V_2$ into V_3 is called a *law of composition on* V; if $\{P, Q\}$ is a generic pair of points of V, and if $R_3 = D[P_1 \times Q_2]$, we shall write $R = PQ$; the law D is called a ***normal law*** [30] if it is also a rational mapping of $V_2 \times V_3$ into V_1, and of $V_3 \times V_1$ into V_2, and if in addition it is associative, that is, if $(PQ)R = P(QR)$ for any generic set $\{P, Q, R\}$ of points of V. We say that V is a ***group-variety*** with the law of composition D, if D is a normal law, and if V contains a proper subvariety W such that $V - W$ (set-theoretical difference) is a group with the law of multiplication prescribed by D; W is called the ***degeneration locus of*** V. It is easily verified that the group $V - W$ can be interpreted, in more than one way, as a group of birational transformations of V, or of automorphisms of $k(V)$ (= field of the rational functions on V) over k. When this is done, in a given way, σ_P shall denote the automorphism related to $P \in V - W$; it satisfies the relation $\sigma_P Q = PQ$ for $Q \in V - W$.

I shall not insist on the obvious definitions of ***direct product*** $V \times F$ of two group-varieties, of ***group-subvariety***, and of ***invariant*** group-subvariety; it must be noted, however, that a group-subvariety of a group-variety need not be irreducible. An immediate example of group-variety is given by an n^2-dimensional projective space G: if the nonhomogeneous coordinates of the points of G are arrayed in a matrix of order n, the matrix-product establishes a law of composition on G, which turns G into a group-variety whose degeneration locus consists of the hyperplane at infinity, and of the subvariety of the points represented by matrices with vanishing determinant. This group-variety G, and its irreducible group-subvarieties, form an important class of group-varieties, which I have called ***Vessiot varieties***, for the role they play in the algebraic Picard-Vessiot theory of differential equations [16; 17]. Vessiot varieties are usually noncommutative, but the following special commutative cases are noteworthy:

(1) *vector varieties*; these are isomorphic (that is, birationally equivalent as varieties, and isomorphic as groups) to direct products of finitely many straight lines, the law of composition on each straight line being given by the addition of the abscissae; (2) *logarithmic varieties*; these are isomorphic to direct products of finitely many straight lines, the law of composition on each straight line being given by the multiplication of the abscissae; (3) *periodic varieties*; these are commutative group-varieties G having the property that $P^e = E$ (identity) for each nondegenerate $P \in G$, and for a suitable positive integer e, independent of P; it is proved in [5] that periodic varieties exist only over fields of characteristic $p > 0$, in which case the smallest possible e is a power of p; the vector varieties over a field of characteristic $p > 0$ are all and only the periodic varieties of period p [5]. It is rather easy to prove that vector varieties and logarithmic varieties are Vessiot varieties; in [16; 17] it is also proved that any commutative Vessiot variety is the direct product of a logarithmic variety and a vector, or a periodic, variety, depending on whether k has zero or positive characteristic. The proof of the fact that every periodic variety is a Vessiot variety is given in [6]; an announcement of the same fact is contained in [26].

Vector varieties, logarithmic varieties, and periodic varieties share the property of being *rational*, that is, birationally equivalent to a projective space; any Vessiot variety over a field of characteristic zero has the same property (result unpublished);[1] most probably, the same is true for any characteristic, but no proof of this fact is available yet.

All Vessiot varieties have a nonempty degeneration locus; the last, and most important, example of group-varieties is given by the *abelian varieties*, which are, by definition, the group-varieties with empty degeneration locus; they are necessarily commutative [30].

2. Historical note and statement of the problems. The study of problems related to, or stemming from properties of group-varieties, mainly in the classical case, has formed, and still forms, the historical center at which several branches of mathematics meet. Abelian varieties originate in the celebrated problem of the inversion of what we now call the abelian integrals of the first kind on an algebraic curve of genus $g \geqq 1$; it now appears [27] that other types of commutative group-varieties originate in the problem of the inversion of integrals of the second and third kind (see §§3 and 4). For $g=1$, the solution of the first problem is expressed in the theory of elliptic functions; for $g>1$, the beautiful theory of abelian functions (or meromorphic functions of g complex variables with $2g$ periods) gives a transcendental tool of primary importance. The deeper results in this theory were obtained by transcendental and topological means, of which Conforto has given, in [14], a modern and rigorous account. The translation and generalization of the theory of abelian varieties to the abstract case (arbitrary k) has been made possible by the pioneering work of Zariski [31; 32], and by the revolutionary new tools which he forged in the arithmetical method in algebraic geometry; the actual reconstruction, within this framework, of the theory of abelian varieties, has been beautifully accomplished by Weil in a true masterpiece of modern mathematics [30].

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[1] *Added February*, 1956. This result was published at about the same time this address was delivered; see C. Chevalley, *On algebraic group-varieties*, J. Math. Soc. Japan vol. 6 (1954) p. 303.

Already in the classical works a basic problem manifests itself, although at times it is blurred by the general lack of rigor: it is not difficult to prove that, by means of abelian functions, an analytic and generally one-to-one correspondence can be established between a toroid (or the parallelotope of the periods) and an algebraic variety V; this, of course, establishes a normal law on V, reflecting the addition of the arguments of the abelian functions. The problem is to find whether V can be selected in such a way that the correspondence is strictly one-to-one, in which case V will be a group-variety. No rigorous solution to this problem can be found in the classical literature, although [14] contains a clear statement of it, and some useful hints concerning its solution. When the theory of group-varieties is put on a purely algebraic basis, the necessity of finding a solution to the analogous problem becomes even more acute, so acute in fact that Weil has developed, in [29], a new tool (the "abstract varieties") which allows him to by-pass the problem itself, at the cost of abandoning the geometrical interpretation of varieties as subsets of a projective space. Shortly after, however, the problem was completely solved in [4]; actually, when only abelian varieties are concerned, the solution is surprisingly simple, and a particular case of it is also given in [13].[1]

Before closing the subject of abelian varieties, it should be noted that through the theory of Riemann matrices in the classical case, or the ring of endomorphisms of an abelian variety in the abstract case, the theory of abelian varieties provides a powerful tool for the solution of difficult questions of analytic number theory.

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[1] A proof for the case of abelian varieties appears also in: [20]. In the classical case, and for abelian varieties only, the result is also contained in: K. Kodaira, *On Kahler varieties of restricted type*, Ann. of Math. vol. 60 (1954) p. 28.

Both abelian varieties and Vessiot varieties fit naturally (in the classical case) in the theory of Lie groups, but Vessiot varieties seem to have received more attention by the specialists on Lie groups; of course, the much discussed "classical groups" are, or are related to, very particular cases of Vessiot varieties. However, not much attention seems to have been paid to those groups of matrices which are algebraic, and to their algebraic nature, in spite of the fact that the theory of invariants is an algebraic theory. The only recent and rigorous works on the subject seem to be [10], [16] and [17]; while the precise scope and purpose of [10] is still not too clear to me, [16] and [17] contain abundant results on Vessiot varieties, as well as on the more immediate purpose of their author, namely the algebraic and rigorous reconstruction of the Picard-Vessiot theory of differential equations.

The main problem on Vessiot varieties is their classification and construction; in the classical case, both are achieved rather neatly by shifting the emphasis from the varieties to their Lie algebras; this artifice does not work in characteristic $p>0$, and [10] offers convincing proof of this statement. The reason for this failure is to be sought, in my opinion, in the fact that the coefficients of the Taylor expansion of a function over a field of positive characteristic are not obtainable by iterated derivations. It is also my opinion that a method dealing with these coefficients directly, rather than with the set of the derivations, would completely reëstablish the analogy with the classical case;[3] a modest example of this is given in §6 of [5].

We now come to the main problem on general group-varieties, namely their classification and construction (or structure). As usual in mathematics, this problem is to be considered solved when its solution is made to depend on problems which, in the logical or historical development of the theory, are parts of a "previous theory". Thus, abelian varieties and Vessiot varieties must be considered as "known" when investigating the structure of general group-varieties. It is quite natural to investigate first the structure of commutative group-varieties, and later the structure of general group-varieties; I shall say immediately that the second, more general problem has been completely neglected until now, except in so far as noncommutative group-varieties fit into the general theory of Lie groups. The first, more restricted problem, has fared a little better, since commutative group-varieties lend themselves, in the classical case, to be studied by means of quasi-abelian functions, that is, meromorphic functions of g complex variables, which satisfy an addition theorem in the sense of Weierstrass (and which consequently have $\mu \leqq 2g$ periods). An extensive literature on this subject is to be found in [27]. [27] and [28] are the most recent contributions to the subject along classical lines, and apparently the only classical contributions with a distinctly geometrical outlook (although the author skilfully avails himself of transcendental tools). It is unfortunate that [27] partakes of those shortcomings which for a few decades stifled serious progress in algebraic geometry, namely the lack of clear definitions and of rigorous proofs; for instance, it seems to me that group-varieties are never even taken into consideration in [27], the subject always being varieties with a commutative normal law.

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[3] This program has been realized, for "formal Lie groups", in: J. Dieudonné, *Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristique* $p>0$, Comment. Math. Helv. vol. 28 (1954) p. 87.

The general method followed by Severi in [27] can certainly be made logically rigorous, translated into algebraic language, and generalized to arbitrary characteristic. This translation and generalization has been undertaken in [24; 25; 26], and it is my understanding that it has now born the expected fruit.

3. Preparatory and structural results. I will now briefly present the main results of my studies on the subject in the course of the last three years. First of all, the free interchange of "varieties with a normal law of composition" and "group-varieties" is made possible by the following result [4]:

(1) THEOREM. *Let V be an (absolutely) irreducible variety over the (not necessarily algebraically closed) field k, and let D be a normal law of composition on V. There exists a variety G over k, birationally equivalent to V, such that G, with the law of composition induced by D, is a group-variety. G is unique but for isomorphisms.*

The most immediate application of (1) is to the homomorphism theorems [5; 12]; a *homomorphism* α *of* a group-variety G *into* a group-variety G' is a rational mapping of G into G' which is also, as far as nondegenerate points are concerned, a group-homomorphism; the *kernel of* α is the smallest subvariety of G which contains all the nondegenerate points of G which map on the identity of G'. The homomorphism α is *separable* if $\alpha\{P\}$ has no component of multiplicity >1 for a generic $P \in G'$. I will not state here the three homomorphism theorems, which are analogous to those of group-theory; the only difference lies in the fact that there may exist homomorphisms (purely inseparable) whose kernel is the identity, but which nonetheless are not isomorphisms. This is not the case, however, if the characteristic of k is zero.

All the structural theorems for commutative group-varieties depend on two preparatory results [5], the first of which reads:

(2) LEMMA. *A nonabelian commutative group-variety of dimension* >1 *has some positive dimensional proper group-subvariety.*

The proof of this result is approximately as follows: if G is the given variety, and F its nonempty degeneration locus, and if $\dim G=n>1$, let X be an $(n-1)$-dimensional irreducible subvariety of G. If there exist infinitely many $P\in G-F$ such that σ_P leaves X invariant, they lie on the desired group-subvariety. If not, $\sigma_P X$ depends on P, and when P approaches a suitable point of F, $\sigma_P X$ approaches a cycle Y which does have the previous property. Of course, "approaching" is replaced by a suitable algebraic notion.

Before stating the second preparatory result, we must introduce, in analogy to group-theory, the concept of factor set. Let G, V be commutative group-varieties over k, and let G_1, G_2 be copies of G. Let γ be a rational mapping of $G_1\times G_2$ into V, such that the relations $(\gamma[P_1\times Q_2R_2])(\gamma[Q_1\times R_2])=(\gamma[P_1Q_1\times R_2])(\gamma[P_1\times Q_2])$ and $\gamma[P_1\times Q_2]=\gamma[Q_1\times P_2]$ are satisfied for a generic set $\{P, Q, R\}$ of points of G; assume, moreover, $\gamma[E_1\times E_2]$ to be a nondegenerate point of V, if E is the identity of G. Then γ is called a *factor set of G into V*. The set of all such γ's is denoted by $\Gamma(G, V)$, and becomes a commutative group when a "natural" law of multiplication is introduced. The factor set γ is *associate to* the identity, or $\gamma\sim 1$, if there exists a rational mapping μ of G into V such that $\gamma[P_1\times Q_2]=(\mu[P])(\mu[Q])(\mu[PQ])^{-1}$ for generic P, $Q\in G$. The set $\Gamma_0(G, V)$ of the $\gamma\sim 1$ is a subgroup of $\Gamma(G, V)$. If $\gamma\in\Gamma(G, V)$, we can establish a normal law of composition D on the cartesian product $G\times V$ by setting $(P\times Q)(R\times S)=PR\times QS\gamma[P_1\times R_2]$, where P, $R\in G$, and Q, $S\in V$. The group-variety determined by $G\times V$ and D according to (1) is called a *crossed product of G and V*, and denoted by $\{G, V, \gamma\}$; it is isomorphic to the direct product $G\times V$ if and only if $\gamma\in\Gamma_0(G, V)$. Now, the result which we have in mind (and whose proof rests essentially on Lüroth's theorem) is the following [5]:

(3) LEMMA. *Let G be a commutative group-variety over k, and let V be a rational 1-dimensional irreducible group-subvariety of G (that is, a straight line, either additive or multiplicative); then G is a crossed product of G/V and V.*

Here, G/V is the image of G in the homomorphism whose kernel is V. From (2), (3), and from a result of [30], stating that any rational mapping of a rational variety R into an abelian variety maps the whole R onto a point, the complete structural theorems for commutative group-varieties follow, namely [5]:

(4) THEOREM. *A commutative group-variety G over k contains a unique maximal irreducible rational group-subvariety V; $A = G/V$ is abelian, and G is a crossed product of A and V.*

(5) THEOREM. *A commutative rational group-variety G over k contains a unique maximal irreducible logarithmic subvariety L, and a unique maximal irreducible vector (if k has characteristic zero) or periodic (if k has positive characteristic) subvariety V; moreover, G is isomorphic to the direct product $L \times V$.*

(6) THEOREM. *A periodic group-variety of dimension $n+1$ over k is the crossed product of a periodic group-variety of dimension n, and of a 1-dimensional vector variety.*

The case of noncommutative group-varieties lends itself to an attack from two directions, which combine to give a rather strong result, although not as strong as (4). The first of these (described in [5]) is brought about by considering two isomorphic general points X, Y of a group-variety G over k (so that X, Y will have coordinates in a suitable extension of k), and by expanding the coördinates of $X^{-1}YX$ in series of powers of the coordinates of Y. The coefficients of these power series are rational functions of the coordinates of X, and can be taken as coordinates of a general point of a new variety V. It turns out that if the series are truncated at a suitable point, V is a Vessiot variety, image of G in a homomorphism whose kernel is the center C of G. If the characteristic of k is zero, two concomitant facts concur to make the picture particularly simple, namely: (1) the series can be truncated after the first powers, and (2) V is necessarily isomorphic to G/C, and can accordingly be called the *adjoint* variety. Both properties are, substantially, consequences of the applicability of the theory of Lie algebras. If the characteristic of k is positive, neither property is necessarily true, and examples of their failure are readily constructed. In this case, V is called the *stem of G*; it remains true, however, that G/C is a Vessiot variety, but this is a consequence of the second method of attack [6].

The other line of attack (described in [6]) involves the consideration of the set of the elements x of $k(G)$ (rational functions on G) which have no poles on G outside the degeneration locus; such elements form a ring, whose field of quotients is also the field of rational functions on a Vessiot variety F, homomorphic image of G in a separable homomorphism α. In the language of the theory of group representations, α is a maximal representation of G. Various consequences can be drawn from these two results, of which the most important are [6]:

(7) THEOREM.[4] *A group-variety G over k contains a unique maximal Vessiot subvariety H; H is invariant in G, and G/H is abelian.*

(8) COROLLARY. *Every rational group-variety is a Vessiot variety.*

Result (7) is not as strong as the analogous result for commutative group-varieties, namely (4). If (8) could be inverted, it would be possible to refine (7) in a manner similar to (4); in any case, the following generalization of two results of [30] is helpful [5]:

(9) THEOREM. *Let A be an abelian group-subvariety of a group-variety G; then A is a subvariety of the center of G, and G contains an invariant irreducible group-subvariety B such that the intersection $A \cap B$ is a finite set of points.*

4. **Factor sets.** Result (4) gives complete information on all the possible G if the structure of $\Gamma(A, V)/\Gamma_0(A, V)$ is known. The study of this group forms the object of §4 of [5], of [7], and, in part, of [8]. It can be easily shown that it is sufficient to study the structure of $\Gamma(A, V)/\Gamma_0(A, V)$ for the simple case in which V is 1-dimensional, and either logarithmic or a vector variety.

In the classical case, let us denote by $\mathfrak{D}$, $\mathfrak{D}_e$, $\mathfrak{D}_1$, $\mathfrak{D}_2$ the additive groups of, respectively, the closed differentials on A with integral residues,[5] the exact differentials on A, the closed differentials of the first kind on A, and the closed differentials of the second kind on A; let $\mathfrak{D}_l$ denote the additive group of the differentials of the type dx/x, for $0 \neq x \in k(A)$. Then it can be proved by transcendental means (and is implicitly stated in [27]) that if V is a 1-dimensional vector variety, $\Gamma(A, V)/\Gamma_0(A, V)$ is isomorphic to $\mathfrak{D}_2/\mathfrak{D}_1+\mathfrak{D}_e$, while $\Gamma(A, V)/\Gamma_0(A, V)$ is isomorphic to $\mathfrak{D}/\mathfrak{D}_2+\mathfrak{D}_l$ if V is a 1-dimensional logarithmic variety. In the first case, it is also known, by transcendental and topological means, that $\mathfrak{D}_2/\mathfrak{D}_1+\mathfrak{D}_e$ is a k-module of order $n = \dim A$, so that the same is true of $\Gamma(A, V)/\Gamma_0(A, V)$. Now, this has been proved in [7], by algebraic means only, for any k of characteristic zero, and leads to the following result:

(10) THEOREM. *Let A be an n-dimensional abelian variety over the field k of characteristic zero; let V be a 1-dimensional vector variety over k.*

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[4] *Added May*, 1956. I have been informed that a proof of this result, valid for the case of characteristic zero, was announced by C. Chevalley in the summer of 1953.

[5] This condition concerning the residues was erroneously left unmentioned in §7 of [5].

Then $\Gamma(A, V)/\Gamma_0(A, V)$ *is a* k*-module of dimension* n, *isomorphic to* $\mathfrak{D}_2/\mathfrak{D}_1+\mathfrak{D}_e$.

The core of the algebraic proof of (10) lies in the fact that $\mathfrak{D}_1$ is the set of all the ***invariant*** differentials on A (invariant under the transformations σ_P for $P\in A$), while $\mathfrak{D}_2$ is the set of all the ***semi-invariant*** closed differentials on A, namely of those closed differentials ω such that $\sigma_P\omega-\omega\in\mathfrak{D}_e$ for any $P\in A$.

The second case, or the case in which V is logarithmic, could probably also be treated by means of the differentials; but there is a more natural and geometrical manner, which I shall explain after introducing a few new concepts.

Quite in general, let V be a normal [31] n-dimensional irreducible variety over k, and let $\mathfrak{Z}(V)$ be the group of the $(n-1)$-dimensional cycles on V. An element $\mathfrak{z}\in\mathfrak{Z}(V)$ is *algebraically equivalent to zero*, or $\mathfrak{z}\approx 0$, if there exist an irreducible variety F over k, an algebraic correspondence D between F and V, and two points P, Q of F, such that $\mathfrak{z}=D\{P\}-D\{Q\}$; if F can be selected to be rational, and D can be selected so that it operates on the whole F, $\mathfrak{z}$ is said to be *linearly equivalent to zero*, in symbols $\mathfrak{z}\sim 0$. Finally, $\mathfrak{z}$ is *arithmetically equivalent to zero*, or $\mathfrak{z}\equiv 0$, if the intersection [3] $(\mathfrak{z}\cap C, V)$ of $\mathfrak{z}$ with any curve C of V has *order* zero whenever it exists. The sets of, respectively, the cycles $\mathfrak{z}\sim 0$, $\mathfrak{z}\approx 0$, $\mathfrak{z}\equiv 0$ will be denoted by $\mathfrak{L}(V)$, $\mathfrak{C}(V)$, $\mathfrak{A}(V)$ respectively; they are subgroups of $\mathfrak{Z}(V)$, and satisfy the inclusion $\mathfrak{L}(V)\subseteq\mathfrak{C}(V)\subseteq\mathfrak{A}(V)\subset\mathfrak{Z}(V)$. The following relations have been proved within the framework of classical algebraic geometry, and by the combined efforts of, mainly, Picard, Castelnuovo, Poincaré, Enriques, Severi and Lefschetz: (a) $\mathfrak{Z}(V)/\mathfrak{A}(V)$ is a finite free module over the ring of integers; (b) $\mathfrak{A}(V)/\mathfrak{C}(V)$ is a finite group, whose order is the *torsion of* V; (c) $\mathfrak{C}(V)/\mathfrak{L}(V)$ is isomorphic, as a group, and "birationally" so, in a well determined way, to an abelian variety, the *Picard variety of* V. A mixture of properties (a) and (b), namely the fact that $\mathfrak{Z}(V)/\mathfrak{C}(V)$ is a finite module over the ring of integers, has been rigorously reëstablished, by algebraic means and for any characteristic, by Néron in [22]; property (c) has also been reëstablished under the same conditions in [18] and [19] by Matsusaka, or in [23] (also in an unpublished work by the author). In the case in which $V=A=$ an abelian variety, property (a) can be proved independently [8]; property (b) follows from the fact that (by (19) of [8]) every element of $\mathfrak{A}(A)/\mathfrak{C}(A)$ has a finite, bounded, period, and that consequently (by Proposition 4 of [22]) is itself finite. As a matter of fact, the quoted results imply that $\mathfrak{A}(A)=\mathfrak{C}(A)$ if k has characteristic zero, and that $\mathfrak{A}(A)/\mathfrak{C}(A)$ has an order which is a power of the characteristic p of k if $p>0$. Finally, as far as property (c) is concerned, it is proved in [8] that the Picard variety of A is *isogenous to* A, namely, has the same dimension of A and is a homomorphic image of A. It must be noted also that if $\mathfrak{z} \in \mathcal{Z}(A)$, then $\mathfrak{z} \equiv 0$ if and only if $\mathfrak{z}$ is linearly equivalent to each $\sigma_P \mathfrak{z}$ for $P \in A$; this definition of the equivalence $\equiv$ is profitably used in [30], and its identity with arithmetic equivalence is proved in [8].[6]

This being established, we have [5; 8]:

(11) THEOREM. *Let A be an abelian variety over the field k of characteristic p; let V be a 1-dimensional logarithmic variety over k. Then $\Gamma(A, V)/\Gamma_0(A, V)$ is isomorphic to $\mathcal{A}(A)/\mathcal{L}(A)$; this, in turn, is isomorphic to the group of the points of the Picard variety B of A if $p=0$. If $p>0$, $\mathcal{A}(A)/\mathcal{L}(A)$ contains a subgroup isomorphic to B, and having in $\mathcal{A}(A)/\mathcal{L}(A)$ an index which is a (finite) power of p.*

The only remaining case is that in which k has positive characteristic, and V is a vector variety. Work on this case is not completed, but what there is would indicate that a result similar to (10) holds true. The method of proof, however, is completely different, and is based on the consideration of the k-module $\mathfrak{D}_1 \cap \mathfrak{D}_a$, which in this case is not necessarily zero-dimensional; the same method, through the consideration of the group $\mathfrak{D}_1 \cap \mathfrak{D}_l$, throws light on the value of the torsion of A, or of the index mentioned in (11).[7]

BIBLIOGRAPHY

1. I. Barsotti, *Algebraic correspondences between algebraic varieties*, Ann. of Math. vol. 52 (1950) p. 427; see also *Errata*, ibid. vol. 53 (1951) p. 587.

2. ———, *Local properties of algebraic correspondences*, Trans. Amer. Math. Soc. vol. 71 (1951) p. 349.

3. ———, *Intersection theory for cycles of an algebraic variety*, Pacific Journal of Mathematics vol. 2 (1952) p. 473.

4. ———, *A note on abelian varieties*, Rend. Circ. Mat. Palermo vol. 2 (1953) p. 236.

5. ———, *Structure theorems for group-varieties*, Annali di Matematica Pura ed Applicata vol. 38 (1955) p. 77.

6. ———, *Un teorema di struttura per le varietà gruppali*, Rend. Acc. Naz. Lincei vol. 18 (1955) p. 43.

7. ———, *Factor sets and differentials on abelian varieties*, to appear.

8. ———, *Il teorema di dualità per le varietà abeliane ed altri risultati*, Rendiconti di Matematica e delle sue Applicazioni vol. 13 (1954) p. 98.

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[6] See (16) of [8]; the result of n. 72 of [30] mentioned in the proof of (16) of [8] is valid only when A is simple. A standard manipulation shows, however, that if (16) of [8] is valid when A is simple, it is also valid for any A.

[7] *Added February*, 1956. It has now been proved that result (10), with the exception of its last three words, remains true when k has characteristic $\neq 0$, and that the index mentioned in (11) is equal to 1.

9. C. Chevalley, *Theory of Lie groups*, Princeton, 1946.

10. ———, *Théorie des groupes de Lie*, II, *Groupes algébriques*, Paris, 1951.

11. ———, *Introduction to the theory of algebraic functions of one variable*, New York, 1951.

12. W. L. Chow, *On the quotient variety of an abelian variety*, Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) p. 1039.

13. ———, *The jacobian variety of an algebraic curve*, Amer. J. Math. vol. 76 (1954) p. 453.

14. F. Conforto, *Funzioni abeliane e matrici di Riemann, parte prima*, Roma, 1942.

15. ———, *Problèmes résolus et non résolus de la théorie des fonctions abéliennes dans ses rapports avec la géométrie algébrique*, Coll. de Géom. Alg., Liège, 1952.

16. E. R. Kolchin, *Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations*, Ann. of Math. vol. 49 (1948) p. 1.

17. ———, *On certain concepts in the theory of algebraic matric groups*, Ann. of Math. vol. 49 (1948) p. 774.

18. T. Matsusaka, *On the algebraic construction of the Picard variety*, Jap. J. Math. vol. 21 (1951) p. 217.

19. ———, *On the algebraic construction of the Picard variety* (II), Jap. J. Math. vol. 22 (1952) p. 51.

20. ———, *Some theorems on abelian varieties*, Natural Science Reports of the Ochanomizu University vol. 4 (1953) p. 22; see also *A remark* . . . , ibid., p. 172.

21. H. Morikawa, *Cycles and endomorphisms of abelian varieties*, Nagoya Mathematical Journal vol. 7 (1954) p. 95.

22. A. Néron, *Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps*, Bull. Soc. Math. France vol. 80 (1952) p. 101.

23. A. Néron and P. Samuel, *La variété de Picard d'une variété normale*, Annales de l'Institut Fourier vol. 4 (1952) p. 1.

24. M. Rosenlicht, *Equivalence relations on algebraic curves*, Ann. of Math. vol. 56 (1952) p. 169.

25. ———, *Differentials of the second kind for algebraic function fields of one variable*, Ann. of Math. vol. 57 (1953) p. 517.

26. ———, *Generalized jacobian varieties*, Ann. of Math. vol. 59 (1954) p. 505.

27. F. Severi, *Funzioni quasi abeliane*, Pont. Acad. Sci. Scripta Varia, vol. 4, 1947.

28. ———, *Sulla caratterizzazione dei corpi di funzioni quasi abeliane*, Convegno Intern. Geom. Diff., 1953, p. 21.

29. A. Weil, *Foundations of algebraic geometry*, Amer. Math. Soc. Colloquium Publications, vol. 29, 1946.

30. ———, *Variétés abéliennes et courbes algébriques*, Paris, 1948.

31. O. Zariski, *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc. vol. 53 (1943) p. 490.

32. ———, *The concept of simple point of an abstract algebraic variety*, Trans. Amer. Math. Soc. vol. 62 (1947) p. 1.

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RIEMANN'S METHOD IN THE THEORY OF SPECIAL FUNCTIONS

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To E. L. Post

The theory of special functions is known as a "painful" subject. My talk today might as well be entitled, "special functions without pain." The painful nature of the subject springs from this: that at first sight, it seems to consist of an enormous number of uncorrelated individual cases, the particular significance, connection, or depth of any specific formula being hard to judge—an enormous mass of chaotic detail, the worst sort of material with which to deal. What is needed, then, is some way of ordering and correlating the material, which should, in order of importance, accomplish the following ends.

(1) To make evident in a general intuitive way what sorts of formulae are likely to hold "on standard grounds" and, conversely, which formulae depend on really special or on deeper properties of a given special function.

(2) To give in a mechanical way the exact form of as wide a class of formulae as possible.

(3) To furnish exact proofs of these formulae.

In attempting to develop such a scheme, it is clear that the question of notation will play a role of importance. In contrast to the conventionally meaningless ad hoc notations for the various kinds of special functions, one wants to develop a "natural" notation, which will bring out as much of the structure of a given function as possible. What I would like to do today is to indicate how one such system can be developed, to indicate the derivation of a few formulae on the basis of this system, and to describe in a general way the range of applicability of the methods involved.

The complete system would consist of three parts:

(a) the function theoretic method (Riemann's method);

(b) the Laplace transform method;

(c) the method of partial differential equations.

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Today I shall discuss only the first of these, since the latter two are more standard. However, I should remark that

(a) covers transformation, recursion, differentiation, multiplication, and a number of expansion formulae, and provides a general framework into which (b) and (c) fit.

(b) covers most integral formulae, if used in a sufficiently systematic way. For an excellent introduction to the application of the Laplace transform method, the recent book of van der Pol and Bremmer may be consulted.

(c) covers a class of addition and expansion formulae. For a most interesting treatment of this method, the papers of P. Henrici listed in the bibliography appended to the present paper should be consulted.

Together, (a), (b), and (c), if used systematically, can cover over 80% at least of the body of formulae constituting the standard corpus of "special function theory," leaving a residue of 20% or so which count from the point of view of (a), (b), and (c) as special formulae.

The basic idea of the method (a) is taken from Riemann's famous paper of 1857 on the hypergeometric function, and is simply this—to characterize a differential equation with rational coefficients not in terms of the parameters entering into its coefficients, but in terms of its singularities, and in terms of parameters describing the nature of its solutions in the neighborhood of these singularities. We shall consequently consider an ordinary homogeneous linear differential equation of order n

$$(1)\qquad (Lf)(z) = f^{(n)}(z) + r_{n-1}(z)f^{(n-1)}(z) + \cdots + r_0(z)f(z) = 0,$$

the coefficients r_j being rational functions. A "special function" will be a solution of such an equation. This definition includes all the "special functions of mathematical physics"—hypergeometric, Whittaker, Bessel, Lame, Mathieu, ellipsoidal wave, etc., functions, but excludes abelian integrals (though not strictly), θ-functions, the gamma function, etc.

The singularities of equation (1) are $z=\infty$ and the singularities of its coefficients. As is customary, we distinguish the singularities into regular and irregular singularities, and the irregular singularities into irregular singularities of various types. This may be done in the conventional way by examining the orders of the poles of the coefficients r_j at the various singularities. More significantly for the present purpose, it may be done by considering the behavior of the solutions of (1) at a singularity ζ. For simplicity, take $\zeta=0$. Then it is known that in general (1) possesses linearly independent solutions

$$\sigma_1, \cdots, \sigma_n$$

which are asymptotically of the form

$$\sigma_i(z) \sim \exp\left(q_{i,k_i} z^{-k_i/\nu_i} + \cdots + q_{i,1} z^{-1/\nu_i}\right) z^{\alpha_i} \left(1 + p_{i,1} z^{1/\nu_i} + p_{i,2} z^{2/\nu_i} + \cdots\right), \tag{2}$$

in the neighborhood of $z=0$, ν_i being a certain integer. The series here are not convergent but only asymptotic, and that only in certain angular sectors of the plane. Moreover, these series may in particular cases contain logarithmic terms also. However, for simplicity in the present presentation, I will ignore these possibilities.

As a convenient symbolic representation of the asymptotic series (2), let us write

$$\left[\nu_i \mid q_{i,k_i}, \cdots, q_{i,1}, \alpha_i;\ p_{i,1}, \cdots, p_{i,N_i}\right] \tag{3}$$

to as many terms as is profitable, calling N_i the number of times the symbol (3) is *augmented*. We normally assume that q_{i,k_i} is not zero; that is, if $q_{i,l}=0$ for $l \geqq p+1$, we normally begin the symbol (3) with $[\nu_i \mid q_{i,p}, \cdots]$. If $\nu_i = 1$, it will normally be omitted from the symbol (3). Finally, we agree to write $[\alpha]$ as α. Then if the asymptotic forms of the solutions of equation (1), $Lf=0$, at the singularity $\zeta = 0$ are represented by the symbols (3), the *type* $\tau(\zeta)$ of the singularity ζ is given by the simple formula

$$\tau(\zeta) = \text{smallest integer not less than } \max_{i=1\ldots n} (k_i/\nu_i) + 1. \tag{4}$$

If $\tau(\zeta) = 1$, the singularity is regular; otherwise it is irregular.

The quantity R defined by the sum $R = \sum \tau(\zeta)$ extended over all the finite and infinite singularities of the equation (1) may be called the Riemann number of equation (1). The Riemann number R measures the total number of singularities of equation (1), counted according to type. If we count the total number $P = P(R, n)$ of parameters on which an equation of Riemann number R and order n depends, we easily find

$$P(R, n) = (R-2)\frac{n(n+1)}{2} + n. \tag{5}$$

The symbol (3) has been introduced as a description of the asymptotic form of a given solution of equation (1) at one of its singularities ζ. The singularity ζ itself may now be described by the collection of symbols (3) describing the n asymptotic forms of n linearly independent solutions a ζ, that is, by the collection of symbols

$$\begin{array}{c} \zeta \\ \left[\nu_1 \mid q_{1,k_1}, \cdots, \alpha_1; p_{1,1}, \cdots, p_{1,N_1}\right], \\ \left[\nu_2 \mid q_{2,k_2}, \cdots, \alpha_2; p_{2,1}, \cdots, p_{2,N_2}\right], \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;, \\ \left[\nu_n \mid q_{n,k_n}, \cdots, \alpha_n; p_{n,1}, \cdots, p_{n,N_n}\right]. \end{array} \tag{6}$$

The quantities $\alpha_1, \cdots, \alpha_n$ are called the exponents of the singularity ζ. If (6) describes a single singularity of equation (1), and if $\zeta_1, \cdots, \zeta_p$ is the collection of all singularities of equation (1), it is natural to describe equation (1) itself by the conglomerate symbol

$$\Phi \left(\begin{array}{lll} \qquad\qquad\quad \zeta_1 & \quad\;\; \zeta_2 \;\cdots & \zeta_p \\ \left[\nu_1 \mid q_{1,k_1}, \cdots, \alpha_1; p_{1,1}, \cdots p_{1,N_1}\right] & \left[\mu_1 \mid q_{1,l_1}, \cdots \right] & \left[\lambda_1 \mid \cdots \right] \\ \left[\nu_2 \mid q_{2,k_2}, \cdots \qquad\qquad\quad p_{2,N_2}\right] & \left[\mu_2 \mid \cdots \qquad\;\; \right] & \left[\lambda_2 \mid \cdots \right] \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;; z \\ \left[\nu_n \mid q_{n,k_n}, \cdots \qquad\qquad\quad p_{n,N_n}\right] & \left[\mu_n \mid \cdots \qquad\;\; \right] & \left[\lambda_n \mid \cdots \right] \end{array} \right) \tag{7}$$

composed of the symbolic descriptions of all its singularities. The symbol (7) may be called the *Riemann symbol* of equation (1). I have followed Riemann in writing it as a symbolic function of z, which may, according to the requirements of convenience, denote either the equation (1) itself, or an arbitrary branch of the multi-valued function which is its solution. The parameters entering into the generalized Riemann symbol (7) are not all independent; the exponents are bound together by the generalized Fuchs relation, which may be stated in case equation (1) has regular singularities only in the form

$$\sum \alpha_\iota = \frac{n(n-1)}{2}(R-2), \tag{8}$$

the sum in (8) being extended over all singularities and all exponents. Relation (8) holds even in the presence of irregular singularities in suitable "nondegenerate" cases. In the presence of "degeneracies," formula (8) must be appropriately modified.

By augmenting the entries in the Riemann symbol (7) a sufficient number of times, we may ensure that the symbol (7) contains at least as many independent parameters as the number (5) of parameters entering into the coefficients of equation (1). In this case, it is reasonable to surmise that the symbol (7) determines the corresponding equation (1) uniquely. It is in fact easy to state general conditions under which the symbolic description (7) determines the corresponding equation uniquely. Without stating these conditions explicitly, I shall write only such symbols (7) as do determine a unique differential equation. To find the symbolic representation of a given equation (1), we have only to substitute trial solutions of the form (2) into equation (1), and to compare the coefficients of the lowest few powers of z on the left and right of (1). This will yield the symbolic description of (1) after a small amount of algebraic computation.

Thus, for example, the hypergeometric equation

$$\delta(\delta+\alpha-1)f(z)-z(\delta+\gamma_1)(\delta+\gamma_2)f(z)=0, \qquad \delta=zd/dz,$$

is associated with Riemann's symbol

$$\Phi\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \gamma_1 & ;\ z \\ 1-\alpha & \alpha-\gamma_1-\gamma_2 & \gamma_2 & \end{Bmatrix};$$

the confluent hypergeometric equation

$$\delta(\delta+\alpha-1)f(z)-z(\delta+\gamma)f(z)=0, \qquad \delta=zd/dz,$$

with the symbol

$$\Phi\begin{Bmatrix} 0 & \infty & \\ 0 & \gamma & ;\ z \\ 1-\alpha & [1,\alpha-\gamma] & \end{Bmatrix}.$$

More generally, the generalized hypergeometric equation

$$\delta(\delta+\alpha_1-1)\cdots(\delta+\alpha_q-1)f(z)-z(\delta+\gamma_1)\cdots(\delta+\gamma_p)f(z)=0, \tag{9}$$
$$\delta=zd/dz,$$

is associated if $p\leq q$ with the symbol

$$\Phi\begin{Bmatrix} 0 & \infty & \\ 0 & \gamma_1 & \\ 1-\alpha_1 & \vdots & \\ \vdots & \gamma_p & ;\ z \\ \vdots & [e\,|\,e\omega_1, f] & \\ \cdots & \cdots & \cdots \\ 1-\alpha_q & [e\,|\,e\omega_e, f] & \end{Bmatrix},$$

where $e=q+1-p$; $\omega_1, \cdots, \omega_e$ are the eth roots of unity, and $f=e^{-1}((q+p)/2-\sum_{i=1}^{q}\alpha_i-\sum_{i=1}^{p}\gamma_i)$. In order to write a Riemann symbol characterizing the Fuchsian case $p=q+1$ of equation (9), it is convenient to introduce an auxiliary notational convention. It is known that if a group of several of the exponents at a regular singularity of equation (1) differ by an integer, the corresponding solutions may possibly contain logarithmic terms. If in a particular case such terms may apparently but do not in actuality appear, we indicate this by bracketing the corresponding entries in the associated Riemann symbol together on the left by a single curly bracket. In terms of this notational convention, the Fuchsian case $p = q + 1$ of equation (9) is determined by the Riemann symbol

$$\Phi\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ \begin{array}{c} 0 \\ 1-\alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \end{array} & \left\{\begin{array}{c} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ q-1 \end{array}\right. & \begin{array}{c} \gamma_1 \\ \cdot \\ \cdot \\ \cdot \\ \\ \end{array} & ; \quad z \\ 1-\alpha_q & \sum\alpha - \sum\gamma & \gamma_p & \end{array}\right\}.$$

As a final example, illustrating the employment of an augmented Riemann symbol, let me mention that the equation

$$(1 - z^2)f''(z) - 2zf'(z) + (\lambda + 4\theta(1 - z^2) - \mu^2(1 - z^2)^{-1})f(z) = 0$$

of spheroidal wave functions corresponds to the symbol

$$\Phi\left\{\begin{array}{cccc} +1 & -1 & \infty & \\ [\mu/2; k] & \mu/2 & [2\theta^{1/2}, 1] & ; \quad z \\ -\mu/2 & -\mu/2 & [-2\theta^{1/2}, 1] & \end{array}\right\},$$

where

$$k = k(\lambda, \theta, \mu)$$

is an easily computed algebraic function. This symbol might also be written as

$$\Phi\left\{\begin{array}{cccc} +1 & -1 & \infty & \\ [\mu/2; k] & [\mu/2; e^{-\pi i(1+\mu/2)}k] & [2\theta^{1/2}, 1] & ; \; z \\ -\mu/2 & -\mu/2 & [-2\theta^{1/2}, 1] & \end{array}\right\},$$

indicating explicitly how the fact that the spheroidal wave equation is even is reflected in the corresponding symbol.

As a first example illustrating the use of the Riemann symbols (7) in the derivation of identities among special functions, let me note that since $e^{(-z)} = e^{-z}$, since $(-z)^\beta$ is a constant multiple of z^β, and since e^{-z} is regular and nonvanishing in the finite plane, we have

$$
\begin{aligned}
\Phi\left\{\begin{matrix} 0 & \infty & & \\ 0 & \gamma & ; & -z \\ 1-\alpha & [1, \alpha-\gamma] & & \end{matrix}\right\} &= \Phi\left\{\begin{matrix} 0 & \infty & & \\ 0 & \gamma & ; & z \\ 1-\alpha & [-1, \alpha-\gamma] & & \end{matrix}\right\} \\
&= e^{-z}\Phi\left\{\begin{matrix} 0 & \infty & & \\ 0 & [1, \gamma] & ; & z \\ 1-\alpha & \alpha-\gamma & & \end{matrix}\right\} = e^{-z}\Phi\left\{\begin{matrix} 0 & \infty & & \\ 0 & \alpha-\gamma & ; & z \\ 1-\alpha & [1, \gamma] & & \end{matrix}\right\}.
\end{aligned} \tag{10}
$$

Note that the Riemann symbol at the end of equation (10) is of the same general form as that which begins equation (10). If we write equation (10) in the more conventional notation of Pochammer for confluent hypergeometric functions, we see that we have established Kummer's transformation

$$
{}_1F_1(\gamma; \alpha; -z) = e^{-z}{}_1F_1(\alpha-\gamma; \alpha; z).
$$

It is easy to see, making use of the Riemann symbol for the generalized hypergeometric equation (9) in case $p \leqq q$, that this is the only formula of its sort which applies to the solutions of equation (9) in case $p \leqq q$. A more fanciful change of variable, which applies to the solutions of equation (9) if $p=3$, $q=2$, and which may be justified in the same way as (10), is

$$
\begin{aligned}
&\Phi\left\{\begin{matrix} 0 & 1 & \infty & & \\ 0 & \left\{\begin{matrix} 0 \\ 1 \end{matrix}\right. & \begin{matrix} \gamma_1 \\ \gamma_2 \end{matrix} & ; & -4z(1-z)^{-2} \\ 1-\alpha & & & & \\ 1-\beta & 1/2 & \gamma_3 & & \end{matrix}\right\} \\
&\qquad = \Phi\left\{\begin{matrix} 0 & \infty & 1 & -1 & & \\ 0 & 0 & 2\gamma_1 & \left\{\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}\right. & ; & z \\ 1-\alpha & 1-\alpha & 2\gamma_2 & & & \\ 1-\beta & 1-\beta & 2\gamma_3 & & & \end{matrix}\right\} \\
&\qquad = \Phi\left\{\begin{matrix} 0 & 1 & \infty & & \\ 0 & 2\gamma_1 & 0 & & \\ 1-\alpha & 2\gamma_2 & 1-\alpha & ; & z \\ 1-\beta & 2\gamma_3 & 1-\beta & & \end{matrix}\right\} \\
&\qquad = (1-z)^{2\gamma_1}\Phi\left\{\begin{matrix} 0 & 1 & \infty & & \\ 0 & 0 & 2\gamma_1 & & \\ 1-\alpha & 2(\gamma_2-\gamma_1) & 1-\alpha+2\gamma_1 & ; & z \\ 1-\beta & 2(\gamma_3-\gamma_1) & 1-\beta+2\gamma_1 & & \end{matrix}\right\}.
\end{aligned}
$$

In case $\gamma_2 = \gamma_1 + 1/2$, the last Riemann symbol in this equation evidently denotes a solution of (9) with $p = 3$, $q = 2$, and we have evidently derived Whipple's transformation

$$
\begin{aligned}
&{}_3F_2(\gamma_1, \gamma_1 + 1/2, \gamma_3; \alpha, \beta; -4z(1 - z)^{-2}) \\
&\qquad = (1 - z)^{2\gamma_1}{}_3F_2(2\gamma_1, 2\gamma_1 + 1 - \alpha, 2\gamma_1 + 1 - \beta; \alpha, \beta; z).
\end{aligned}
$$

Let us now consider the process of multiplying two functions. For this purpose, first note that the differential equation with single-valued coefficients of lowest order satisfied by a given multi-valued function is the smallest number of its branches on which all its branches are linearly dependent. Suppose then that we take two multi-valued functions σ and ρ, each satisfying a second order differential equation with single-valued coefficients. Then there exist two branches σ_1, σ_2 of σ on which all the continuations of σ are linearly dependent, and two branches ρ_1, ρ_2 of ρ on which all the branches of ρ are linearly dependent. It is then apparent that all the branches of the product $\sigma\rho$ are dependent on the four branches

$$\sigma_1\rho_1, \quad \sigma_2\rho_2, \quad \sigma_1\rho_2, \quad \sigma_2\rho_1.$$

This observation justifies the following calculation.

$$
\begin{aligned}
&\Phi\left\{\begin{matrix} 0 & \infty & & \\ 0 & \gamma & ; & z \\ 1-\alpha & [1, \alpha-\gamma] & & \end{matrix}\right\}
\Phi\left\{\begin{matrix} 0 & \infty & & \\ 0 & \gamma & ; & z \\ 1-\alpha & [1, \alpha-\gamma] & & \end{matrix}\right\} \\
&\qquad = \Phi\left\{\begin{matrix} 0 & \infty & \\ [0; 0] & 2\gamma & \\ 2-2\alpha & 2(\alpha-\gamma); & z \\ 1-\alpha & [1, \alpha] & \\ 2-\alpha & [-1, \alpha] & \end{matrix}\right\} \\
&\qquad = \Phi\left\{\begin{matrix} 0 & \infty & & \\ 0 & \gamma & & \\ 1-\alpha & \alpha-\gamma & ; & z^2/4 \\ 1/2-\alpha/2 & [2 \mid 2, \alpha/2] & & \\ 1-\alpha/2 & [2 \mid -2, \alpha/2] & & \end{matrix}\right\}.
\end{aligned}
$$

In more conventional notation, this is the following formula of Ramanujan

$$(11)\quad {}_1F_1(\alpha; \gamma; z)\,{}_1F_1(\alpha; \gamma; -z) = {}_2F_3(\gamma, \alpha - \gamma; \alpha, \alpha/2, \alpha/2 + 1/2; z^2/4).$$

An amusing example of the sort of formula of this type sometimes to be met is the following formula of Bailey.

$$
\begin{aligned}
&{}_0F_2(\rho_1,\rho_2;z)\,{}_0F_2(\rho_1,\rho_2;-z) = {}_3F_8 \\
&\qquad = \Big[\frac{1}{3}(\rho_1+\rho_2-1),\frac{1}{3}(\rho_1+\rho_2),\frac{1}{3}(\rho_1+\rho_2+1); \\
&\qquad\quad -\rho_1,\rho_2,\frac{1}{2}\rho_1,\frac{1}{2}\rho_2,\frac{1}{2}(\rho_1+1),\frac{1}{2}(\rho_2+1), \\
&\qquad\qquad \frac{1}{2}(\rho_1+\rho_2-1),\frac{1}{2}(\rho_1+\rho_2);\left(\frac{3}{4}\right)^{3}z^{2}\Big].
\end{aligned}
\tag{12}
$$

The derivation of (12) by Riemann's method is just as trivial as the derivation of (11).

As a final example, we calculate the Laplace transform of the square of a Bessel function, so as to illustrate the way in which the function theoretic method links up with the method of the Laplace transform. Rather than giving the explicit statement of the various general rules governing the Laplace transform of a Riemann symbol, we will display the various steps of the calculation, which will make the principles involved reasonably clear. If $\mathfrak{L}f$ denotes the Laplace transform of f, we have

$$
\begin{aligned}
\int_0^\infty e^{-st}(J_\nu(t))^2dt &= \mathfrak{L}\left[\Phi\begin{pmatrix} 0 & \infty & & \\ \nu & [i,1/2] & ; & \cdot \\ -\nu & [-i,1/2] & & \end{pmatrix}^2\right](s) \\
&= \mathfrak{L}\left[\Phi\begin{pmatrix} 0 & \infty & & \\ [0;0] & 1 & & \\ 2\nu & [2i,1] & ; & \cdot \\ -2\nu & [-2i,2] & & \end{pmatrix}\right](s) \\
&= \Phi\begin{pmatrix} 2i & -2i & 0 & \infty & & \\ 0 & 0 & [0;0] & 1-2\nu & ; & s \\ 0 & 0 & 0 & 1+2\nu & & \end{pmatrix} \\
&= \Phi\begin{pmatrix} 0 & -4 & \infty & & \\ 0 & 0 & 1/2-\nu & ; & s^2 \\ 0 & 0 & 1/2+\nu & & \end{pmatrix} \\
&= \Phi\begin{pmatrix} -1 & +1 & \infty & \\ 0 & 0 & 1/2-\nu; & \dfrac{s^2+2}{2} \\ 0 & 0 & 1/2+\nu & \end{pmatrix}.
\end{aligned}
\tag{13}
$$

Here we have used most notably the fact that the Laplace transform of a function satisfying a differential equation with (even) rational coefficients satisfies a differential equation with even rational coefficients. By noting the asymptotic form of the Laplace integral in (13) as $s \to \infty$ we may determine the particular branch of

$$\Phi\left\{\begin{matrix} -1 & +1 & \infty & \\ 0 & 0 & 1/2-\nu\ ; & \dfrac{s^2+2}{2} \\ 0 & 0 & 1/2+\nu & \end{matrix}\right\}$$

represented by this integral. In this way we find that the integral in (13) represents the Legendre function of the second kind, and that (13) may be written in more conventional notation as

$$\int_0^\infty e^{-st}(J_\nu(t))^2 dt = \frac{1}{\pi} Q_{\nu-1/2}\left(\frac{s^2+2}{2}\right).$$

This is a special case of a formula of Hankel.

BIBLIOGRAPHY

A. Erdélyi et al.

1. *Higher transcendental functions*, vols. 1–3, New York, 1953.

P. Henrici

1. *On generating functions of the Jacobi polynomials*, Pacific J. Math. vol. 5, supplement 2 (1955) pp. 923–931.
2. *Addition theorems for general Legendre and Gegenbauer functions*, Journal of Rational Mechanics and Analysis, vol. 4, no. 6 (1955) pp. 983–1017.
3. *Zur Funktionentheorie der Wellergleichung*, Comment. Math. Helv. vol. 27 (1953) pp. 235–293.

B. van der Pol and H. Bremmer

1. *Operational calculus based on the two-sided Laplace integral*, Cambridge University Press, 1955.

C. Truesdell

1. *An essay toward a unified theory of special functions*, Princeton, 1948.

YALE UNIVERSITY

THE SUMMER MEETING IN SEATTLE

The sixty-first Summer Meeting of the American Mathematical Society was held from Tuesday, August 21, through Friday, August 24, 1956, at the University of Washington, Seattle, Washington. Other organizations meeting in Seattle at about the same time were the Mathematical Association of America, the Institute of Mathematical Statistics, the Biometric Society, and the Econometric Society.

Over 580 people registered for the meeting, including about 370 members of the Society.

Professor Salomon Bochner of Princeton University delivered the Colloquium Lectures on *Harmonic analysis and probability*, with Professors Richard Brauer, L. V. Ahlfors, C. B. Allendoerfer, and R. S. Phillips presiding.

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, the Society was addressed by Professors C. B. Allendoerfer, Kenkichi Iwasawa, and Sarvadaman Chowla, their topics being (respectively) *Some recent advances in differential geometry in the large*, *A theorem on Abelian groups and its application in algebraic number theory*, and *Difference sets and finite geometries*. They were introduced (respectively) by Professors C. O. Oakley, Deane Montgomery and R. H. Bruck.

There was a Session of Invited Papers in Applied Mathematics, presided over by Dr. R. E. Gaskell. Speakers were Professors Bernard Friedman, P. L. Chambré, J. W. Carr III, Dr. William Nachbar, and Dr. R. L. Brock. Abstracts of their papers appear in this report along with those of other papers in Applied Mathematics.

A session for contributed papers in Probability and Statistics was held jointly with the Institute of Mathematical Statistics, Dr. I. J. Good presiding. There were thirteen other sessions for contributed papers, presided over by Dr. F. J. Weyl and Professors R. V. Churchill, J. L. Kelley, R. P. Dilworth, M. F. Smiley, Casper Goffman, R. D. James, Edwin Hewitt, R. A. Beaumont, H. F. Bohnenblust, G. B. Price, F. A. Valentine, and M. M. Day.

From the smooth progress of the meeting and the fine recreation and entertainment provided, it was evident that the Committee on Arrangements had worked hard and well. In this connection, the Society owes special thanks to Professor A. E. Livingston, Chairman of the Committee.

The Council met on Tuesday evening, August 21, 1956.

The Secretary announced the election of the following ninety-eight persons to ordinary membership in the Society:

Professor Bernhard Banaschewski, McMaster University;
Dr. William D. Barcus, Jr., Brown University;
Mrs. Anne Breese Barnes, University of Texas;
Professor Clyde Raymond Barrow, Westminster College;
Mr. Kenneth Reynold Blake, Marquette University;
Mr. James Robert Boyd, Chance Vaught Aircraft, Dallas, Texas;
Mr. Arthur Eugene Bragg, Tuskegee Institute;
Mr. Donald George Brennan, Massachusetts Institute of Technology;
Miss Lena Celeste Brown, Southern University;
Dr. James Floyd Carpenter, Boeing Airplane Company, Seattle, Washington;
Mr. Louis Child, New Mexico College of Agriculture and Mechanic Arts;
Mr. Charles Cyprian Chouteau, University of Wichita;
Mr. Robert William Cowan, University of Georgia;
Mrs. Alice Roth Crilly, Mount St. Mary's College;
Professor Geoffrey Crofts, Occidental College;
Professor Jacob Cukierman, Acadia University;
Mr. William M. Cunnea, University of Chicago;
Professor Frances Elizabeth Davis, University of Hawaii;
Professor Mamie Myrtis Davis, Wesleyan College;
Mr. Richard Thomas Dillon, University of Oregon;
Mrs. Elsie Phelps Elder, Washington University;
Sister M. Barbara Ann Foos, SSJ, Nazareth College;
Mr. Donald Roy Fredkin, New York University;
Mr. Eugene John Gehrig, Northwestern University;
Mr. Charles Joseph Gentry, University of Florida;
Dr. Bruce Gilchrist, Syracuse University;
Dr. Allen A. Goldstein, Convair-Astronautics, San Diego, California;
Mr. George Ross Grainger, Convair-Astronautics, San Diego, California;
Professor Mauritz Leland Granberg, Sioux Falls College;
Mr. Robert Karl Gruenewald, Washington University;
Mr. Herschel Nixon Hadley, Naval Powder Factory, Indian Head, Maryland;
Mr. Walter Elliott Haggerty, Louisiana State University;
Dr. Dilla Hall, Southern Illinois University;
Mr. Robert L. Helmbold, Carnegie Institute of Technology;
Mr. Michael Herschorn, McGill University;
Mr. Robert George Heyneman, University of California, Berkeley;
Professor William Floyd Hill, Howard Payne College;
Mr. Lewis Eugene Hulbert, Battelle Memorial Institute, Columbus 1, Ohio;
Mr. Lambert S. Joel, National Bureau of Standards, Washington 25, D. C.;
Mr. Elgy Sibley Johnson, The Joel E. Spingarn High School, Washington, D. C.;
Miss Katherine Everts Kellner, University of Illinois;
Mr. Merrill Jay Kemp, University of Utah;
Mr. James Edward Kiefer, University of Minnesota;
Mr. Donald Ross King, Rutgers University;
Mr. Ralph Ellis Kleinman, University of Michigan;
Dr. Herbert Knothe, Holloman Air Development Center, Holloman, New Mexico;

Mr. George Richard Kuhn, St. Louis University;
Mr. Donald Irvine Kurth, University of Minnesota;
Mr. John Grieg Leghorn, University of Colorado;
Mr. Hugh Lloyd Lewis, Southwest Texas State Teachers College;
Mr. Arthur Libenson, General Electric Company, Ithaca, New York;
Mr. Irving J. Lieberman, Litton Industries, Beverly Hills, California;
Dr. Stuart P. Lloyd, Bell Telephone Laboratories, Inc.;
Dr. Wilhelmus Antonius J. Luxemburg, University of Toronto;
Mr. Rogers William Martin, School Board, Wilmington, Delaware;
Professor Joseph Courtney McCully, University of Rhode Island;
Mr. E. Gregory McNiel, DePaul University;
Professor Jack R. Meagher, Western Michigan College;
Mr. Richard Frederick Meyer, Arthur D. Little, Inc., Cambridge, Massachusetts;
Mr. R. A. Moreland, Jr., Texas Technological College;
Mr. Florio Joseph Moretti, Fairfield University;
Mr. Herbert Alfred Morrissey, Reynolds Metals Company, Sheffield, Alabama;
Professor Charles E. Moulton, Shurtleff College;
Sister Rose Marian Mulligan, O. S. F., Ladycliff College;
Mr. Patrick Lawrence O'Dea, White Sands Proving Ground, New Mexico;
Professor George Harry Pavlakos, Elmhurst College;
Dr. Rudolph William Preisendorfer, Scripps Institute of Oceanography, San Diego, California;
Mr. Robert Pruitt, University City High School, University City, Missouri;
Mr. Ronald Pyke, University of Washington;
Mr. Gerhard Rayna, Lehigh University;
Professor John Reckzeh, State Teachers College, Jersey City, New Jersey;
Professor Thelma Austin Rice, Alabama State College;
Mr. John Astwood Riley, Boston College;
Comdr. Horace Leland de Rivera, (U.S.N., retired), Northeastern University;
Mr. James Noel Rogers, University of Wisconsin;
Mr. Robert Rosen, Columbia University;
Mr. Adolph Andrew Rutoskey, Fort McPherson, Georgia;
Professor Hans J. Sagan, Montana State College;
Miss Lily H. Seshu, University of Illinois;
Dr. Joseph Richard Sligo, University of Nevada;
Miss Emma D. Breedlove Smith, South Carolina State College;
Mr. Lloyd Benton Smith, Jr., Louisiana State University;
Mr. William Frank Steele, Heidelberg College;
Rev. Thomas James Taylor, St. Ambrose College;
Mr. J. Richard Tessmer, Massachusetts Institute of Technology;
Professor Robert Harry Thompson, Sterling College;
Sister M. Denis Treacy, O.S.F., The Assisium, New York, New York;
Professor Donald Robert Truax, University of Kansas;
Mr. Edward J. Tully, Jr., Master Beach, New York;
Professor Mary K. Tulock, California State Polytechnic College;
Sister Mary Robert Von Wolff, O.P., St. Mary's Dominican College;
Mr. Max Leslie Weiss, Cornell University;
Mr. Frederick R. White, University of Buffalo;
Mr. Jay Charles Williams, University of Kansas;
Dr. Joseph Clement Wilson, Redstone Arsenal, Huntsville, Alabama;

Professor George Anderson Yates, Tennessee Wesleyan College;
Miss Sallyann Elizabeth Zimm, University of Illinois;
Professor Geraldyne Pierce Zimmerman, South Carolina State College.

It was reported that the following four persons had been elected to membership on nomination of institutional members as indicated:

Haverford College: Mr. Alexander Hamilton Frey, Jr.
University of Missouri: Mr. Billy J. Attebery
New York University: Mr. Martin Schechter
Stanford University: Mr. Richard Lee Van de Wetering.

The Secretary reported that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche-Mathematiker-Vereinigung: Dr. Artur Bergmann, Ricarda Huch Schule, Professor E. F. Peschl, Mathematisches Institut der Universität, and Dr. Hans-Egon Richert, Mathematisches Institut der Universität; Société Mathématique de France: Professor Henri Eyraud, University of Lyon, Professor István Fáry, University of Montreal, and Dr. František Vyčichlo, University of Prague.

The following appointments by the President were reported: as a Program Committee for a Summer Seminar in Applied Mathematics to be held in 1957: Paul Garabedian, A. S. Householder, Mark Kac, R. E. Langer, C. C. Lin, M. H. Martin, Chairman, William Prager, and J. J. Stoker; as a Committee on Arrangements for the 1956 Annual Meeting to be held at the University of Rochester: Edward Batho, Dorothy Bernstein, H. M. Gehman, Norman Gunderson, Hewitt Kenyon, Robert MacDowell, Walter Rudin, Chairman, and R. D. Schafer; as a Committee on Arrangements for a meeting to be held November 30–December 1 at the University of Kentucky: V. F. Cowling, Chairman, H. H. Downing, J. C. Eaves, J. G. Horne, Frank Levin, S. E. Pence, J. H. Roberts, and W. C. Royster; as a member of the Committee on Translations for a period of three years beginning July 1, 1956: J. V. Wehausen; as Chairman of the Committee on Translations for a period of one year beginning July 1, 1956: Hans Samelson. (Committee now consists of R. E. Bellman, R. P. Boas, Jr., Irving Kaplansky, Hans Samelson, Chairman, and J. V. Wehausen); as members of the Organizing Committee for Summer Institutes for three years beginning July 1, 1956: R. H. Bing and H. S. M. Coxeter; as Chairman of the Organizing Committee for Summer Institutes for a period of one year beginning July 1, 1956: Edwin Hewitt. (Committee now consists of R. H. Bing, H. S. M. Coxeter, Nelson Dunford, Edwin Hewitt, Chairman, E. R. Kolchin, and J. J. Stoker); as a delegate to the Fourth Congress of Roumanian

Mathematicians, May 27–June 4, 1956: D. G. Bourgin, as delegates to the International Symposium on Algebraic Topology held in Mexico City, August 6–September 1, 1956: Samuel Eilenberg, W. Hurewicz, and N. E. Steenrod.

The Secretary reported that Professors C. B. Allendoerfer, Kenkichi Iwasawa, and S. Chowla had accepted invitations to deliver hour addresses at the Summer Meeting of 1956 in Seattle, Washington; that Professor Bertram Yood will address the Society at the meeting in Pasadena, California, November 17, 1956; that Professor J.-P. Serre will address the Society at the meeting to be held at the Massachusetts Institute of Technology on October 27, 1956; that Professor George Piranian will address the Society at the meeting to be held at Northwestern University on November 23–24, 1956; that Professor O. G. Harrold, Jr. will address the Society at the meeting to be held at the University of Kentucky, November 30–December 1, 1956; that Professor H. J. Muller has accepted an invitation to deliver the Gibbs Lecture in January, 1958; that Professor J. L. Doob has accepted an invitation to deliver the Colloquium Lectures in 1959 (the lectures for 1958 have been omitted); that Professors P. R. Halmos, S. C. Kleene, W. V. Quine, J. B. Rosser, Chairman, and Alfred Tarski have been named as an Invitations Committee for the 1957 Summer Institute; that Professor W. S. Massey will replace Professor Samuel Eilenberg on the Transactions and Memoirs Editorial Committee for the period June 1, 1956 to July 15, 1957 and will serve as Chairman of this committee for the remainder of 1956, that Dr. S. H. Gould has been appointed Executive Editor of Mathematical Reviews for a term of one year beginning June, 1956.

The Council set the following dates and places of meetings: February 23, 1957 at Yale University; April 20, 1957 at the University of California at Berkeley; and January 28–30, 1958 at the University of Cincinnati and the Sheraton-Gibson Hotel, Cincinnati, Ohio.

The Council voted to approve a Symposium on Applied Mathematics on the topic "Orbit Theory" to be held in conjunction with the meeting at New York University in April, 1957.

A business meeting of the Society was held at 10:00 a.m., August 23, 1956, with President Elect Brauer presiding. The Secretary reported briefly on the affairs of the Society.

The following amendments to the by-laws were adopted: Sections 9 and 10 of Article IX were amended to read as follows:

Section 9. An ordinary or contributing member shall receive the *Bulletin* and *Proceedings* as privileges of membership during each year for which his dues have been discharged, except that a member may substitute for the *Proceedings*, the *Transactions* or *Mathematical Reviews* by paying a premium established by the Trustees for this privilege.

Section 10. Ten dollars of the annual dues of those who receive the *Bulletin* and the *Proceedings*, or the *Transactions* or *Mathematical Reviews* in substitution for the *Proceedings*, under the provisions of this Article shall be allocated in payment therefor. Additionally, in the case of substitution of the *Transactions* or *Mathematical Reviews* for the *Proceedings*, the entire amount of the premium referred to in Section 9 shall also be so allocated.

A resolution of thanks to the host institution was adopted.

The abstracts of the papers follow. Those having the letter "*t*" following the abstract number were read by title. Where a paper, presented in person, has more than one author, the symbol (p) follows the name of the author who presented it. Dr. E. H. Bareiss was introduced by Dr. Feodor Theilheimer, Professors Fryer and Helperin by Professor H. W. Ellis, Mr. R. P. Hunter by Professor P. M. Swingle, Professor P. W. M. John by Professor V. L. Klee, Jr., Professor Erwin Kreyszig by Professor Stefan Bergman, Dr. T. C. H. Li by Dr. H. E. Salzer, Mr. W. D. Maness by Dr. P. O. Bell, Mr. George Marsaglia by Professor V. L. Klee, Jr., Dr. Richard Montague by Dr. R. L. Vaught, Mr. R. M. Smullyan by Professor John McCarthy, and Dr. R. C. Thorne, by Professor H. F. Bohnenblust.

ALGEBRA AND THEORY OF NUMBERS

594. N. C. Ankeny: *Quadratic forms.*

A new proof of formula for the representation of a number as the sum of n squares, $n \leq 8$ is presented. Let $V_n(p)$ denote the n dimensional Euclidean vector space over the finite field with p elements, p a rational prime, with the usual dot product and metric. Then count the number of non-null lines, a line being all scalar multiples of a vector, a null line being all multiples of a vector of zero length. By the use of Geometry of Numbers it is then shown that each null line gives $2n$ solutions of our original problem. Conversely by Witt's theorem, almost every solution corresponds to a reflection with respect to a non-null line. The solutions left over are shown to correspond to certain isometrics in planes orthogonal to null-lines. This yields the formulae. (Received June 25, 1956.)

595*t*. H. W. Becker: *On the triangles of Bancroft Brown.*

These are pairs of Heronian triangles $(H\blacktriangle)a, b, c$ and a, b, c' with 2 sides and the radius r of the inscribed circle in common (Brownian triangles, $B\blacktriangle$). Brown gave 7 numerical solutions, (Math. Mag. vol. 29 (1956) pp. 275–276). #1, 2, 4 generalize to (I) $r, a, b, c, c' = 2mn(m^2-n^2)$, $m^4+6m^2n^2+n^4$, $2(m^2+n^2)^2$, $(3m^2+n^2)(m^2-n^2)$, $m^4+14m^2n^2+n^4$. #3 generalizes to (II) $r, a, b, c, c' = mn(m^2-n^2)(m^2-mn+n^2)$, m^6+n^6, $2mn(m^4+n^4)$, $(m^2-mn+n^2)(m^4+m^3n-2m^2n^2+mn^3+n^4)$, $(m^2+n^2)(m^4+m^2n^2$

$+n^4$). #4 with both $\blacktriangle$ obtuse (0-0) generalizes in (I) to any $m>4$ with $n=1$; (II) is type 0-0 when $m^4-3m^2n^2+n^4$ is negative. Conjecture: In all $B\blacktriangle$, $6|r$. It is evident from comparison of r in (I) and (II) that they can yield no common numerical solution. They were arrived at by the problem solving principles of Polya (guessing!) based on consideration of the generators e, f and g, h of the component Pythagorean $\blacktriangle$. The condition for $B\blacktriangle$ is that the quartic $e^2g^2(eh-fg)^2+4f^2h^2(e^2g^2+efgh-f^2h^2)=\square$. Very likely no complete solution, in polynomials of any finite degree or weight, can exist. For every $B\triangle$ pair, there is a third "algebraic triangle" with the same r, a, b, but third "side" c'' negative. What is the density of $B\blacktriangle$ among $H\blacktriangle$? What types of $H\blacktriangle$ cannot be $B\blacktriangle$? (Received July 6, 1956.)

596t. Gerald Berman and R. J. Silverman: *Algebraic systems with n-operations.*

A Γ_s-system is a system $\{S, \sigma_1, \sigma_2, \cdots, \sigma_s\}$ where S is a nonempty set and σ_i is a mapping from a nonempty subset S_i of $S\times S$ to S, $i=1, 2, \cdots, s$. A Γ_0-system is a system $\{S\}$. A $\Gamma_s(n)$-system is a Γ_{s+n}-system with the additional properties (i) $S_i=S\times S$, $i>s$, (ii) σ_i is associative, $i>s$, and (iii) σ_i is left-distributive with respect to σ_j for $s>i>j$. It is shown that (a) corresponding to any set S and any integers s and n there exists a $\Gamma_s(n)$-system, (b) given any Γ_s-system $\mathfrak{S}$ and any integer n, there exists a $\Gamma_s(n)$-system $\mathfrak{T}$ which contains a Γ_s-subsystem isomorphic to $\mathfrak{S}$, such that if $S_i=S\times S$ for any i, the corresponding $T_i=T\times T$, (c) given any $\Gamma_s(n)$-system $\mathfrak{S}$ and any integer m, there exists a $\Gamma_s(n+m)$-system containing a $\Gamma_s(n)$-subsystem isomorphic to $\mathfrak{S}$, and (d) given any $\Gamma_s(n)$ system $\mathfrak{S}$ with set S, there exists a $\Gamma_s(n)$-system $\mathfrak{T}$ with set T, having cardinality greater than that of S, such that $\mathfrak{T}$ contains a $\Gamma_s(n)$-subsystem isomorphic to $\mathfrak{S}$. (Received July 5, 1956.)

597. Gerald Berman and R. J. Silverman (p): *Embedding of general algebraic systems into systems of transformations.*

A $\Gamma_s(n)$-system is defined in the abstract *Algebraic systems with n-operations.* Several theorems are proved on the isomorphic embedding of a $\Gamma_s(n)$-system into a $\Gamma_s(n)$-system whose elements are transformations on a set, where the embedding satisfies certain restrictions. For example, it is shown that a $\Gamma_s(n)$-system $\{S, \sigma_1, \sigma_2, \cdots, \sigma_{s+n}\}$ can be embedded isomorphically into a $\Gamma_s(n)$-system $\{T^n, \tau_1^n, \tau_2^n, \cdots, \tau_{s+n}^n\}$ where (i) T^1 is the set of transformations on a set T (T may be taken as the set of transformations on S) and T^k is the set of transformations on T^{k-1}, (ii) τ_i^n corresponds to σ_i, $i=1, 2, \cdots, s+n$, (iii) σ_i is left-distributive with respect to σ_j for $i>j$, $i>s$. These embedding theorems generalize many of the well known embedding theorems in algebra. As a special case, a ring can be embedded isomorphically into the endomorphisms of a group. It is not necessary to first embed the ring into a ring with identity as in the usual proof. The embedding theorems also subsume such theorems as: Every semi-group, (group, near-ring, semi-ring, associative near-ring) can be embedded isomorphically into the right multiplications on a semi-group (group, group, semi-group, loop). (Received July 5, 1956.)

598. J. L. Brenner: *New proof of a theorem of Taussky and Geiringer.*

If any principal minor of a matrix $A=(a_{ij})$ is singular and if, for each i, $|a_{ii}| \geq \sum_j |a_{ij}|$, the principal minor of maximum dimension is singular. By renumbering the subscripts, the matrix can be reduced. (Received June 25, 1956.)

599*t*. V. Bohun-Chudyniv: *On methods of determining all types of Grave-Cayley algebras and some applications.*

An oktonion algebra [see V. Bohun-Chudyniv, *On a general method for constructing completely orthogonal* $2^k \times 2^k$ *squares by using orthogonal systems of K-nions*, Proceedings of the International Congress of Mathematicians, 1954, Asmterdam, Holland, vol. II, pp. 281–282] is a Grave-Cayley algebra if the norm of the product of two arbitrary oktonions P_1, P_2, satisfy the condition (1): $N(P_1P_2) = N(P_1)N(P_2)$. At the present time only a very few types of Grave-Cayley algebras are known, differing from one another by their multiplication tables. Among them are: (I) J. T. Graves' type, communicated to W. R. Hamilton, Dec. 26, 1843 and Jan. 18, 1884, Proc. Royal Irish Academy (3) 1845–47; (II) A. Cayley's type, Phil. Mag., London, vol. XXX (1847) pp. 57–58; (III) L. E. Dickson's type, *Linear algebras*, Cambridge, 1914, p. 14; and others. The aims of this paper are: (A) to determine all types of oktonion algebras, the multiplication tables of which imply the condition (1), and including abovementioned algebras as separate cases; (B) to apply the multiplication table of algebras (A) for determining orthogonalizable Latin 7×7 Squares, and pertaining to each of them completely orthogonal Latin Squares of the same order (and constructing same). (Received July 5, 1956.)

600*t*. R. H. Bruck: *The structure of commutative Moufang loops.*

Let G be a commutative Moufang loop with centre $Z(G)$, associator subloop G', inner mapping group $I(G)$, multiplication group $M(G)$. (1) If G is finitely generated, each of $G/Z(G)$, G', $I(G)$, $M(G)/Z(M)$ is finite. (2) G is locally centrally nilpotent. (3) $G/Z(G)$ and G' are locally finite loops of exponent 3. (4) $I(G)$ and $M(G)/Z(M)$ are locally finite 3-groups. (5) The following statements are equivalent: (i) G is centrally nilpotent of (finite positive) class c; (ii) $I(G)$ is nilpotent of class $c-1$; (iii) $M(G)$ is nilpotent of class $2c-1$. Statement (2) is proved in a form conjectured by Slaby and originally proved for $n \leqq 5$ by Slaby and the author (Bull. Amer. Math. Soc. Abstract 60-1-102): Every commutative Moufang loop which can be generared by n elements ($n>1$) is centrally nilpotent of class at most $n-1$. The class, $k(n)$, of the free commutative Moufang loop on n free generators, satisfies the inequalities $1+[n/2] \leqq k(n) \leqq n-1$ for $n \geqq 3$. It is still unknown whether every infinitely generated commutative Moufang loop is (transfinitely) centrally nilpotent. (Received July 5, 1956.)

601*t*. Eckford Cohen: *Generalizations of the Euler ϕ-function.*

New proofs of the characteristic properties of certain generalized totient functions are obtained. The Jordan function $J_k(n)$ and the author's generalized totient $\phi_k(n)$, (Bull. Amer. Math. Soc. Abstract 61-3-389), are considered. It is shown that both functions can be reformulated in terms of the ϕ-function alone. On the basis of these formulas, the characteristic properties of $J_k(n)$ and $\phi_k(n)$ result easily from the corresponding properties of $\phi(n)$. (Received June 4, 1956.)

602*t*. Eckford Cohen: *Cauchy products of certain types of arithmetical functions.*

This paper is concerned primarily with approximation to the Cauchy product $\sum_{a=1}^{n} f(a)g(n-a)$ of certain types of almost periodic arithmetical functions $f(n)$, $g(n)$. The method employed is based on Wintner's theory (*Eratosthenian averages*, 1943, Chapter 2) generalizing Ramanujan's trigonometric expansions, in connection with the author's orthogonality relation (Duke Math. J. vol. 19 (1952) p. 120) for the

Ramanujan sum $c(n, r)$. Estimates associated with the mean values of arithmetical functions are also obtained, and the general theory is illustrated by means of divisor and totient functions. (Received June 4, 1956.)

603*t*. Harvey Cohn: *Some algebraic number theory estimates based on the Dedekind eta-function.*

For the pure cubic field, the zeta-function residue reduces to a finite expression involving the Dedekind eta-function. (See Dedekind, Crelle 121, 1900, pp. 40–123.) From this expression, using the Farey sub-division, the author obtains the estimate $h \log \epsilon = O(|d|^{1/2} \log |d| \operatorname{loglog} |d|)$, where d is the field discriminant, h is the class number, and ϵ (>1) is the fundamental unit. This is a slight improvement over Landau's more general $O(|d|^{1/2} \log |d|^{\epsilon})$. From this follows the result $h = O(|d|^{1/2} \cdot \operatorname{loglog} |d|)$, in contrast to Landau's $O(|d|^{1/2} \log^2 |d|)$. Bounds on norms of certain modular invariants are included. (Received March 28, 1956.)

604*t*. D. W. Crowe: *A simple ordering for linear graphs.* Preliminary report.

For an arbitrary connected linear graph, G, a class of words $\{w(G)\}$ is described according to the following recipe: Beginning at some vertex of G follow an arbitrary path (traversing each edge exactly once) recording the degree of each vertex as it occurs until either (a) a vertex of degree 1 is reached or (b) a vertex u is reached which has already been recorded. In case (a) return to the first vertex at which there are edges not yet traversed and continue the process. In case (b) u is the nth distinct recorded vertex having degree $d(u)$. Attach the subscript n to this last $d(u)$ and proceed as in case (a). Continue the process until all edges have been traversed. If $G \neq G'$ then $\{w(G)\} \cap \{w(G')\} = \phi$. In the usual lexicographic order let $W(G)$ be the smallest element of $\{w(G)\}$. Then define $G < G'$ if and only if $W(G) < W(G')$. For an arbitrary string w necessary and sufficient algebraic conditions are given for the existence of a graph G such that $w \in \{w(G)\}$. (Received June 22, 1956.)

605*t*. R. L. Dąvis: *Torsion in Engel modules.*

One standard approach to the Restricted Burnside Problem has been through the Engel ring defined by $[fg^{p-1}] = 0$, over the field of p elements. It was thus that A. L. Kostrikhin (*Izvestia Akad. Nauk SSSR.* vol. 19 (1955) pp. 233–244) just recently solved the problem affirmatively for the case $p = 5$ and the number of generators $q = 2$. This paper studies rather the free Lie J-module (J the integers) on two generators. If in this case the Engel ideal generated by elements $[fg^4]$ is replaced by an ideal I generated by some of its "linear consequences," it can be proved that L/I has torsion, which first appears in L_6 (L_k is the submodule of Lie forms of degree k). It can then further be shown that actually $5L_7 \subset I$, and hence from this point on Kostrikhin's computations apply also to this J-module. (Received May 14, 1956.)

606*t*. M. M. Day: *Amenability of semigroups.* II.

The concepts of amenability and strong amenability for semigroups and groups were defined and discussed in Trans. Amer. Math. Soc. vol. 69 (1950) pp. 276–291 and Bull. Amer. Math. Soc. Abstracts 55-11-506 and 56-1-40. It was shown then that strong amenability of a semigroup Σ implied amenability of Σ; by finding new formulations of these properties of Σ in terms of the semigroup algebra $l_1(\Sigma)$, it is proved here that the converse is also valid. (Received June 29, 1956.)

607. M. M. Day: *The second conjugate of a semigroup algebra.*

Arens (Proc. Amer. Math. Soc. vol. 2 (1951) pp. 839–848) gave an example of a commutative Banach algebra whose second conjugate algebra is not commutative. This note shows that this is the natural situation for the semigroup algebra $l_1(\Sigma)$ of a semigroup Σ. In particular, if Σ is the set of positive integers, there exists more than one invariant mean on $m(\Sigma)$ and if two such means are multiplied together the product is always the second one, so the subsemigroup of invariant means in the algebra $l_1(\Sigma)^{**}$ is not commutative. (Received June 29, 1956.)

608. R. P. Dilworth: *Homomorphisms of distributive lattices.*

The following theorem is proved: Let M be a sublattice of a distributive lattice L. Let $\theta \geqq \phi$ be congruence relations on M and let ϕ^* be an extension of ϕ to L. Then there exists an extension ϕ^* of θ to L such that $\theta^* \geqq \phi^*$. This theorem is used to prove the following results: (1) A lattice is distributive if and only if every congruence relation on each sublattice can be extended to a congruence relation on the entire lattice. (2) If M is a sublattice of a distributive lattice L, then the lattice of congruence relations on M is a meet homomorphic image of the lattice of congruence relations on L. (3) If M is a sublattice of a distributive lattice L, then the lattice of congruence relations on M is a homomorphic image of the lattice of congruence relations on L if and only if M is projectively dense in L. (Received July 2, 1956.)

609. N. J. Divinsky. *D-regularity.*

An element x of a ring A is r.D.r. (right D-regular), if there exists an element y in A such that $x = xy$. An ideal is r.D.r. if every element in it is r.D.r. In every ring there exists a maximal r.D.r. ideal M_r and a maximal l.D.r. (left D-regular) ideal M_l which are in general different from one another. The right subradical is the Jacobson radical of M_r. When A has either DCC on right ideals or ACC on left ideals, $M_r = A$ if and only if A has a right unity. Also $M_r = M_l = A$ if and only if A has a unity when A has any one of the following: DCC on one-sided ideals; ACC on one-sided ideals; a unity in A/J where J is the Jacobson radical; A has a nondivisor of zero. If A has DCC on one-sided ideals then A is nilpotent if and only if $M_r = M_l = 0$ and there are no nonzero total divisors of zero in A^n for every positive integer n. For commutative rings with $M(= M_r = M_l) = 0$ it is shown that when A has DCC, A is nilpotent; when A has ACC, A is a subdirect sum of nilpotent rings (though it may be Jacobson semisimple!); and in general A is a subdirect sum of rings bound to their maximal nilideals. (Received June 19, 1956.)

610. D. O. Ellis: *Notes on the foundations of lattice theory*, III.

Various propositions are derived relating m-lattices (Garrett Birkhoff, *Lattice Theory*, American Math. Soc., New York, 1948), delta-lattices (Randall Conkling and David Ellis, *Metric delta-lattices*, Revista (Tucuman), vol. 10 (1954) pp. 75–82), c-groupoids (David Ellis, *Notes on the foundations of lattice theory* II, Archiv der Math. vol. 4 (1953) pp. 257–260), partially ordered and lattice ordered groupoids, and lattice properties in a given lattice. The methods are straightforward in general and are algebraic and lattice-theoretic in nature. (Received June 25, 1956.)

611. C. M. Fulton and D. A. Norton (p): *A fixed point theorem for affine transformations.*

Let the affine transformation σ on an n-dimensional hyperspace be defined by

$X' = \sigma X = AX + C$ (where A is an $n \times n$ square matrix and C is a column vector). Define inductively $X^{(i)} = \sigma X^{(i-1)}$. If the hypervolume $V_A(X) = |X' - X, X'' - X, \cdots, X^{(n)} - X|$ is a nonzero constant, σ can have no fixed point. This paper shows that a necessary and sufficient condition that $V_A(X)$ be a nonzero constant is that the minimal equation of A be $(X-1)^n = 0$ and that C not be the null space of A^{n-1}. (Received May 28, 1956.)

612*t*. Lawrence Goldman: *Algebraic results on homogeneous linear differential equations.*

Let K be any differential field and let $L(y) \in K\{y\}$ be a homogeneous linear differential polynomial of order n. Let $K\langle \omega_1, \cdots, \omega_n \rangle$ be a Picard-Vessiot extension of K (E. R. Kolchin, Ann. of Math., vol. 49, pp. 1–42) where $(\omega_1, \cdots, \omega_n)$ is a fundamental system of zeros of $L(y)$. Let $F(y) \in K\{Y\}$ be the lowest order irreducible differential polynomial such that $F(\omega_1) = 0$, let the order of $F(y)$ be r. The following results are proved: (1) Over the differential field $K\langle \omega_1 \rangle$, $L(y)$ factors (iteratively) into $L_1 L_2(y)$ where L_1, L_2 are of orders $n-r$, r respectively. (2) If F is neither linear nor homogeneous then dimension of $K\langle \omega_1, \cdots, \omega_n \rangle$ over K is $(n-1)r$, if F is not linear but is homogeneous then dimension of $K\langle \omega_1, \cdots, \omega_n \rangle$ over K is $\leqq (n-1)r+1$. (3) If $r = n-1$ and degree of F is m write $F = \sum_0^m F_i$, where F_i is homogeneous of degree i, then the general manifold (J. F. Ritt, *Differential algebra*) of $\sum_0^m c_i F_i$ (c_i any constants) belongs to the manifold of $L(y)$. (4) Let the $n \times n$ algebraic matric group of all automorphisms of $K\langle \omega_1, \cdots, \omega_n \rangle$ over K be the orthogonal group then F is of order $n-1$, $F = F_m + F_0$ and the lowest order irreducible differential equation over K which any zero of $L(y)$ satisfies is of the form $F_m + cF_0 = 0$, where c is some constant. (Received June 26, 1956.)

613. Franklin Haimo: *An order preserving map on the subgroup lattice of the automorphism group.*

Let H be a group of automorphisms of a group α, and let $\lambda(H)$ be the set of all automorphisms of α such that to each pair $a \in \alpha$ and $\bar{k} \in \lambda(H)$ there exists $k_{(a,\bar{k})} \in H$ with $k_{(a,\bar{k})}\bar{k}(a) = a$. Then $\lambda(H)$ is a subgroup of automorphisms, $\lambda(H) \supset H$, and λ is monotonic on the lattice of subgroups of the automorphism group, carrying normal subgroups onto normal subgroups. H and $\lambda(H)$ share their sets of universal fixed points in α. If the elements of H are precisely those automorphisms of α which induce the identity on some fixed quotient α/β, then $\lambda(H) = H$. In any case λ, $\lambda(H) = \lambda(H)$. If H is the group J of inner automorphisms of α, then $\lambda(H)$ consists of all the class preserving automorphisms of α. In 1913 Miller showed that $\lambda(J)$ may include J strictly, a result frequently neglected. [See G. E. Wall, J. London Math. Soc. vol. 22 (1947) pp. 315–320.] In any event, $\lambda(J)$ is part of the second centralizer of J. (Received July 5, 1956.)

614. William Hanf (p) and Alfred Tarski: *New results concerning isomorphism of Boolean algebras and commutative semigroups.* Preliminary report.

The results below supplement and partly improve those in Bull. Amer. Math. Soc. Abstracts 60-6-651 and 60-6-668. The following two theorems have been established by Hanf: (I) *There are denumerable Boolean algebras* $\mathfrak{A}$ *and* $\mathfrak{B}$ *such that* $\mathfrak{A}$ *and* $\mathfrak{A} \times \mathfrak{B} \times \mathfrak{B}$ *are isomorphic while* $\mathfrak{A}$ *and* $\mathfrak{A} \times \mathfrak{B}$ *are not.* This result was obtained, following Tarski's suggestion, by modifying the construction of Kinoshita in Fund. Math. vol. 40, pp.

39–41; it improves Kinoshita's result. (II) *There is a (nondenumerable) Boolean algebra* $\mathfrak{A}$ *such that* $\mathfrak{A}$ *and* $\mathfrak{A}\times\mathfrak{A}\times\mathfrak{A}$ *are isomorphic while* $\mathfrak{A}$ *and* $\mathfrak{A}\times\mathfrak{A}$ *are not.* Here for $\mathfrak{A}$ the algebra $\mathfrak{A}\times\mathfrak{B}_1$ of Abstract 60-6-651 can be taken. From (I) Tarski has derived the following corollary: (III) *There are denumerable Boolean algebras* $\mathfrak{A}$ *and* $\mathfrak{C}$ *such that* $\mathfrak{A}\times\mathfrak{A}$ *and* $\mathfrak{C}\times\mathfrak{C}$ *are isomorphic while* $\mathfrak{A}$ *and* $\mathfrak{C}$ *are not.* This solves Problem (1) in Abstract 60-6-668. Moreover, Tarski has pointed out that (I), (II), *and* (III) *can be carried over to commutative semigroups.* The problem of extending (II) to denumerable Boolean algebras and denumerable commutative semigroups remains open. (Received July 5, 1956.)

615*t*. I. Heller: *On sets of generators in a free Abelian group.*

If G is a free Abelian group of rank n, a subset S of G containing at least n linearly independent elements is called a set of generators when every set of n linearly independent elements of S is a basis of G. Using a result on matrices (the author, *On linear systems with integral valued solutions*, Bull. Amer. Math. Soc. Abstract 62-6-743 and 62-6-744) the note proves: If a set of generators contains $n(n+1)$ elements (not counting the null element), it is maximal and of the form $D=\{a_i, a_i-a_j\}$, where the a_i are a basis of G. For $n\geq 4$ there are maximal sets of generators with less than $n(n+1)$ elements (and hence not of form D). (Received June 4, 1956.)

616*t*. J. H. Hodges: *Weight partitions for skew matrices over a finite field.*

For $\alpha\in GF(q)$, $q=p^n$, p odd, let $e(\alpha)=e^{2\pi i t(\alpha)/p}$, where $t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}}$. If $A=(\alpha_{ij})$ is a square matrix with elements in $GF(q)$, let $\sigma(A)=\sum_i \alpha_{ii}$. Consider the sum $S=S(B, U, A)=\sum e\{\sigma(U'X+X'U)\}$, where A is a nonsingular skew matrix of order $2m$, B is a skew matrix of order t, U is an arbitrary $2m\times t$ matrix, and the summation is over all $2m\times t$ matrices X satisfying the equation $X'AX=B$, all matrices having elements in $GF(q)$. If $U=0$, S is the number $Z_t(A, B)$ of solutions X of $X'AX=B$, which is given in (L. Carlitz, *Representations by skew forms in a finite field*, Archiv der Mathematik, vol. 5 (1954) pp. 19–31). In the present paper it is shown that S can be expressed in terms of generalized Kloosterman sums defined for skew matrices over $GF(q)$. A number of properties of these Kloosterman sums are also given. The analogous problems for symmetric and for general matrices have been considered previously by the author and will appear presently in Mathematische Zeitschrift and the Duke Mathematical Journal, respectively. The symmetric case with $t=1$ has been considered with more explicit results by Carlitz in (L. Carlitz, *Weighted quadratic partitions over a finite field*, Canadian Journal of Mathematics vol. 5 (1953) pp. 317–323). (Received July 5, 1956.)

617*t*. D. R. Hughes: *Collineations and generalized incidence matrices*. Preliminary report.

Let π be a (v, k, λ) configuration, let G be a collineation group of π, where G has order m. Let w be the number of transitive classes of points of π, with respect to G; then w is also the number of transitive classes of lines. There exist integers r_i, s_i, $i=1, 2, \cdots, w$, where each r_i and s_i divides m, such that if $C_1=\operatorname{diag}(s_1^{-1}, s_2^{-1}, \cdots, s_w^{-1})$, $C_2=\operatorname{diag}(r_1^{-1}, r_2^{-1}, \cdots, r_w^{-1})$, and if S is the square matrix of order w with $+1$ in every position, then $AC_1A^T=(k-\lambda)C_2^{-1}+\lambda mS$, $A^TC_2A=(k-\lambda)C_1^{-1}+\lambda mS$, where A is a square matrix with non-negative integral entries. The Hasse-Minkowski theory of rational congruence can be applied to these matrix equations: for example, if m

is an odd prime and if the number N of points fixed by a nonidentity element of G is even, then $x^2=(k-\lambda)y^2+(-1)^e mz^2$, $e=(m-1)/2$, possesses a nontrivial solution in integers. Applications of these results to projective planes (i.e., $\lambda=1$) are numerous, yielding many restrictions on the structure of possible planes of non-prime-power order. (Received July 11, 1956.)

618. Burrowes Hunt: *Continued fractions and indefinite forms.*

There is given an explicit matric formulation of the connection between indefinite binary forms and regular continued fractions; this is extendable to n-ary forms and a continued-fraction-like algorithm. Let A and B be the matrices with top row (1, 1) and bottom rows (0, 1) and (1, 0). If a_0 is an integer and $a_1, a_2, \cdots$ are non-negative integers, then $F=\cdots BA^{a_2}BA^{a_1}BA^{a_0}B$ is called a Farey Matrix. Let the rows of F be (a, b) and (c, d): c/d is equal to the r.c.f. $[a_0+1; a_1+1, a_2+1, \cdots]$. To F is associated the form $bx^2+(a-d)xy-cy^2$ of discriminant trace^2 $F-4\det F$. Cyclic permutation of the letters spelling F gives rise to a chain of equivalent reduced forms. If F' has the same trace and determinant as F but is not a cyclic permutation of F, then F and F' are in different classes. If forms are associated to all 2×2 matrices in this way then M is equivalent to N iff $M=U^{-1}NU$ for a unimodular U. Every unimodular U has a unique representation $U=GF$, where F is a Farey Matrix and G is one of a group of 8 matrices whose effect on F is to permute the rows or multiply either or both rows by -1. To each class of indefinite forms is associated the equivalent (to each other) Farey Matrices which are automorphs of appropriate forms in the class. (Received July 5, 1956.)

619.* W. E. Jenner: *On representations of the full linear group and some of its subgroups over infinite fields of prime characteristic.*

It is shown that if k is an infinite field of characteristic $p>0$ then for $n>1$ the groups $GL(n, k)$, $SL(n, k)$ and $Sp(2n, k)$ have rational irreducible representations of arbitrarily high degree. For the first two groups this had been shown by M. Rosenlicht (unpublished). This result is in contrast to the fact that the degrees of the absolutely irreducible representations of any finite-dimensional Lie algebra over a field of characteristic $p>0$ are bounded (Jacobson, Zassenhaus). Preliminary results are obtained towards a complete representation theory for these groups. (Received July 12, 1956.)

620t. Bjarni Jónsson: *Results on direct decompositions of groups and other algebraic systems.*

Theorem. *Given an infinite Boolean algebra $\mathfrak{A}$ and a denumerable, centerless, indecomposable group $\mathfrak{C}$, there exists a centerless group $\mathfrak{G}$ with the same cardinal as $\mathfrak{A}$, and an isomorphism F of $\mathfrak{A}$ onto the (Boolean) algebra of factors of $\mathfrak{G}$, such that $F(a)\cong F(b)$ if and only if the ideals* (a) *and* (b) *of $\mathfrak{A}$ are isomorphic, and $F(p)\cong\mathfrak{C}$ for every atom p of $\mathfrak{A}$.* Combined with results by Hanf and Tarski (Bull. Amer. Math. Soc. Abstracts 60-6-651, 60-6-668 and 62-6-614), this yields (I) a (nondenumerable) group $\mathfrak{G}$ and a finite group $\mathfrak{H}$ such that $\mathfrak{G}\cong\mathfrak{G}\times\mathfrak{H}\times\mathfrak{H}$ but not $\mathfrak{G}\cong\mathfrak{G}\times\mathfrak{H}$, (II) denumerable groups $\mathfrak{G}$ and $\mathfrak{H}$ such that $\mathfrak{G}\cong\mathfrak{G}\times\mathfrak{H}\times\mathfrak{H}$ but not $\mathfrak{G}\cong\mathfrak{G}\times\mathfrak{H}$, (III) a (nondenumerable) group $\mathfrak{G}$ such that $\mathfrak{G}\cong\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}$ but not $\mathfrak{G}\cong\mathfrak{G}\times\mathfrak{G}$, (IV) denumerable groups $\mathfrak{G}$ and $\mathfrak{H}$ such that $\mathfrak{G}\times\mathfrak{G}\cong\mathfrak{H}\times\mathfrak{H}$ but not $\mathfrak{G}\cong\mathfrak{H}$. Actually, for "group" one may substitute "K-algebra," K being any class of algebras with zero and finitary operations, closed under the operation of taking subdirect powers and containing a denumerable, centerless, indecomposable algebra $\mathfrak{C}$, except that for (I) $\mathfrak{C}$ must be assumed finite. (For terminology see Jónsson and Tarski, *Direct decompositions of finite algebraic systems.*) Thus generalized, (I)–(IV) include the results by Hanf and Tarski. (Received July 5, 1956).

621*t*. Bjarni Jónsson and Alfred Tarski: *Two general theorems concerning free algebras.*

Given a class K of algebraic systems, an algebra $\mathfrak{A} \in K$ is called a K-algebra freely generated by a set G of its elements if $\mathfrak{A}$ is generated by G and every mapping of G into an algebra $\mathfrak{B} \in K$ extends to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Many familiar classes K (groups, rings, lattices) satisfy the conditions: (C_1) *a K-algebra $\mathfrak{A}$ freely generated by a finite set with n elements is never generated* (*freely or not*) *by a set with $p < n$ elements*; (C_2) *if, moreover, $\mathfrak{A}$ is generated by another set H with n elements, it is freely generated by H*. Theorems: (I) *Every class K containing a finite algebra with more than one element satisfies* (C_1). (II) *If every equation holding in all finite K-algebras holds identically in all K-algebras, then K satisfies* (C_2). In connection with (I) compare Fujiwara, Proc. Jap. Acad. 31, p. 135. (C_1) fails in the class K_1 of algebras (with operations $\circ$, $^+$, *) characterized by the axioms $x^+ \circ x^* = x$, $(x \circ y)^+ = x$, $(x \circ y)^* = y$; in fact, any two K_1-algebras freely generated by finite sets are isomorphic. The class K_2 characterized by the single axiom $x^{+*} = x$ satisfies the hypothesis of (I) and hence condition (C_1); but (C_2) fails in K_2 even for $n = 1$. (Received July 5, 1956.)

622*t*. C. W. Kohls: *Ideals in rings of continuous functions.* I.

Let $C(X)$ be the ring of all continuous real functions on a completely regular Hausdorff space X; $N^p = \{f \in C(X) : f = 0$ on some X-neighborhood of $p\}$; and $M'^p = \{f \in C(X)$: the zero-set of f meets every deleted nbhd. of $p\}$, if p is a nonisolated point of X, $= M^p$ otherwise. For any $p \in \beta X$, p is a βF-point (with respect to X) if whenever $f \in C(X)$ is extendable to p and vanishes there, then $f \geqq 0$ or $f \leqq 0$ on some X-nbhd. of p. (1) p is a βF-point iff N^p is prime, and iff the zero-sets of any pair of functions extendable to p and vanishing there are comparable in some X-nbhd. of p. (2) Let p be a βF-point. Then $C(X)/N^p$ and $C(X)/M'^p$ are ordered integral domains containing the reals R, which contain infinitely large elements iff $p \notin \upsilon X$ (largest subspace of βX over which *all* $f \in C(X)$ can be extended). If $p \notin \upsilon X$, then $C(X)/N^p$ is an η_1-set, but if $p \in \upsilon X$, $C(X)/N^p$ has a countable cofinal subset. The field of quotients of $C(X)/N^p$ is a real-closed field. $C(X)/M'^p$ is a valuation ring, and hence the prime ideals containing M'^p form a chain. (3) On the other hand, if p is not a βF-point, there are incomparable prime ideals contained in M^p, and hence $C(X)/N^p$ is not a valuation ring. (The author was supported by the National Science Foundation.) (Received April 10, 1956.)

623*t*. C. W. Kohls: *Ideals in rings of continuous functions.* II.

Let $C^*(X)$ be the ring of bounded continuous real functions on a locally compact Hausdorff space X; $C_\infty(X)$, the functions "vanishing at ∞"; $C_c(X)$, the functions with compact supports. An ideal A of an arbitrary ring B is a $\mathcal{B}$-ideal if for $b \in B$, whenever there exists $a \in A$ such that $a \in M$ iff $b \in M$, for all $M \in \mathfrak{M}_s(B)$ ($=$structure space of B), then $b \in A$. (1) If A is an ideal of $C^*(X)$, then (a)–(c) are equivalent: (a) $C_c(X) \subseteq A \subseteq C_\infty(X)$; (b) for all $M \in \mathfrak{M}_s(C^*(X))$, $M \supseteq A$ iff M is a free ideal; (c) the mapping $p \rightarrow M^{*p} \cap A$ ($p \in X$) is a homeomorphism from X to $\mathfrak{M}_s(A)$. (2) $C_c(X)$ is an ideal of $C_\infty(X)$ contained in no maximal ideal. (3) $C_\infty(X)$ is a regular ring iff X is finite. Using a result of L. J. Heider (Bull. Amer. Math. Soc. Abstract 62-1-158), the set of $\mathfrak{Z}$-ideals A of $C^*(X)$ satisfying $C_s(X) \subseteq A \subseteq C_\infty(X)$ is characterized algebraically. For $C_s(X)$, the conditions become: (i) A is a commutative semi-simple algebra over R such that $M \in \mathfrak{M}_s(A)$ implies $A/M \cong R$; (ii) each $a \in A$ has a relative identity, i.e., there exists an $e \in A$ such that $ae = a$; (iii) for each $a \in A$, there is an $r \in R$ such that $a \leqq re$, where e is a suitable relative identity for a and $\leqq$ is the induced partial order on A; (iv) if B is an extension of A satisfying (i)–(iii) and such that the mapping $M \rightarrow M \cap A$ is one-to-one from $\mathfrak{M}_s(B)$ onto $\mathfrak{M}_s(A)$, then $B = A$. (The author was supported by the National Science Foundation.) (Received April 10, 1956.)

624t. C. W. Kohls: *The space of prime ideals of a ring.*

Let $\mathfrak{S}(A)$ be any set of ideals of the ring A which admits the Stone topology; $\mathfrak{P}(A)$, the set of primitive ideals; and $\mathfrak{Q}(A)$, the set of prime ideals—endowing each set with the Stone topology. For commutative A, $e \in A$ is a *relative identity* for $a \in A$ if $ae = a$. (1) Let A be a ring in which every proper ideal is contained in an $\mathfrak{S}$-ideal. Then $\mathfrak{S}(A)$ is compact iff A is finitely generated. (2) The space $\mathfrak{S}(A) \subseteq \mathfrak{Q}(A)$ is Hausdorff iff for each distinct pair S, $T \in \mathfrak{S}(A)$, there exist $a, b \in A$ such that $a \notin S$, $b \notin T$ and $axb \in \Delta\mathfrak{S}(A)$ ($=$intersection of all $S \in \mathfrak{S}(A)$) for all $x \in A$. (3) Suppose A is commutative, $\mathfrak{Q}(A) \supseteq \mathfrak{S}(A) \supseteq \mathfrak{P}(A)$, and for each $S \in \mathfrak{S}(A)$, there is an $a \notin S$ with a relative identity. Then $\mathfrak{S}(A)$ is locally compact. And if $\Delta\mathfrak{S}(A) = (0)$, then $\mathfrak{S}(A)$ is compact iff A has an identity. (4) Let I be a proper ideal of A, $P \in \mathfrak{Q}(I)$, and $Q = \{a \in A : Ia \subseteq P\}$. Then $Q \in \mathfrak{Q}(A)$, and $P = Q \cap I$. (5) If I is a proper ideal of A, $\{Q \in \mathfrak{Q}(A) : Q \not\supseteq I\}$ is homeomorphic to $\mathfrak{Q}(I)$, under the mapping $Q \rightarrow Q \cap I$. (6) Suppose A is commutative with characteristic $n \neq 0$, $\mathfrak{Q}(A)(\mathfrak{P}(A))$ is Hausdorff, and for each $P \in \mathfrak{Q}(A)(\mathfrak{P}(A))$ there is an $a \notin P$ with a relative identity. Then $\mathfrak{Q}(A; I_n) \cdot (\mathfrak{P}(A; I_n))$ is Hausdorff, where $(A; I_n)$ is the ring consisting of pairs (a, i), $a \in A$, $i \in$ integers mod (n), with operations defined so that $(0, 1)$ is the identity. (The author was supported by the National Science Foundation.) (Received June 4, 1956.)

625. L. A. Kokoris: *On algebras of (γ, δ) type.*

An algebra of (γ, δ) type is a finite dimensional algebra A over a field F satisfying the identities $z(xy) = (zx)y + \gamma(xz)y - \gamma x(zy) + \delta(yz)x - \delta y(zx)$ and $(xy)z = x(yz) + \gamma(xz)y - \gamma x(zy) + (\delta - 1)(yz)x - (\delta - 1)y(zx)$ where γ and δ are elements in F such that $\gamma^2 - \delta^2 + \delta = 1$. The structure of algebras with $\delta \neq 0, 1$ has been studied previously. (See A. A. Albert, *Almost alternative algebras*, Portugaliae Mathematica, vol 8 (1949) pp. 23–36, and the author's, *On a class of almost alternative algebras*, Canadian Journal of Math. vol. 8 (1956) pp. 250–255.) Algebras of type $(\gamma, \delta) = (-1, 0)$ and $(1, 1)$ are studied here. The results for simple and semisimple algebras are the same as when $\delta \neq 0, 1$, but the proofs differ due to the fact that the decompositions relative to an idempotent have different properties in these two cases. (Received June 4, 1956.)

626t. Joseph Landin and Irving Reiner: *Automorphisms of the general linear group over a principal ideal domain.*

Let R be an integral domain, possibly noncommutative, in which all left and right ideals are principal. Let E be a free R-module of rank n, and E^* its dual. Let $\Gamma L_n(R)$ be the group of invertible semi-linear maps of E onto itself, relative to automorphisms of R, and let $GL_n(R)$ be its subgroup of linear maps. Define a subgroup U of $GL_n(R)$ by $U = \{\lambda_\alpha : \alpha = \text{unit in the center of } R\}$, where λ_α is the homothetic map $x \rightarrow x\alpha$. If $\phi : GL_n(R) \rightarrow U$ is any homomorphism such that $\lambda_\alpha \in U$, $\phi(\lambda_\alpha) = \lambda_\alpha^{-1}$ implies $\alpha = 1$, and if $g \in \Gamma L_n(R)$, then the mapping $u \to \phi(u) g u g^{-1}$ gives an automorphism of $GL_n(R)$. Likewise, so does $u \to \phi(u) h^{-1} u^* h$, where h is a one-to-one semi-linear map of E onto E^* relative to an anti-automorphism of R, and where u^* is the contragredient of u. It is shown here that, conversely, every automorphism of $GL_n(R)$ for $n \geqq 3$ is given by one of these formulas. The proof uses methods of Mackey (Ann. of Math. vol. 43 (1942) pp. 244–260), Rickart (Amer. J Math. vol. 72 (1950) pp. 451–464), Dieudonné (*La géométrie des groupes classiques*, 1955), and Reiner (Trans. Amer. Math. Soc. vol. 79 (1955) vol. 459–476). (Received April 30, 1956.)

627. W. J. LeVeque: *On the frequency of small fractional parts in certain real sequences.*

Let x be a number in $[0, 1]$, and let $X_n = \langle nx \rangle$, the distance between nx and the nearest integer to nx. Let $f(x)$ be a suitable positive decreasing function with $\sum_1^\infty f(k) = \infty$, and put $g(x) = f(\log x)/x$, $A(n) = \sum_1^n g(k)$. Let $T_n(x)$ be the number of positive integers $m \leqq n$ such that $X_m < g(m)$. It is shown that for fixed ω, $\lim_{n \to \infty} \operatorname{meas} \{x \mid T_n(x) < 12\pi^{-2} A(n) + \omega (12\pi^{-2} A(n))^{1/2}\} = \phi(\omega)$, where $\phi(\omega)$ is the normal distribution function $(2\pi)^{-1} \int_{-\infty}^{\omega} e^{-t^2/2} dt$. This refines a well-known theorem of Khinchin, which guarantees the existence of infinitely many solutions of the inequality $X_m < g(m)$. The result reflects the strong statistical dependence of the quantities $\langle mx \rangle$, since if the X_n were redefined as independent random variables uniformly distributed on $[-1/2, 1/2]$, the same theorem would hold with the factor $12\pi^{-2}$ replaced by 2. The same problem is considered for the case that $X_n = (r_1 r_2 \cdots r_n x)$, where $\{r_i\}$ is a suitable increasing sequence of positive integers, and here the theorem to be expected from the independent case again holds. Moreover, it is shown in this case that for almost all x, $T_n(x) \sim 2 \sum_1^n g(k)$. (Received July 6, 1956.)

628. D. J. Lewis: *Strongly normic polynomials.*

To each homogeneous polynomial $F(x_1, x_2, \cdots, x_n) = F(X)$ over the ring of integers Θ of a ρ-adic field such that $F(X)$ has only the trivial zero in Θ, there exists a smallest rational integer m such that $F(a_1, a_2, \cdots, a_n) \equiv 0 \pmod{\rho^m}$, a_i in Θ, only if every $a_i \equiv 0 \pmod{\rho}$. (Here ρ is the prime ideal in Θ.) If m is equal to the degree of $F(X)$, then $F(X)$ is said to be a strongly normic polynomial over Θ. It is shown that there exists polynomials $\phi_d(y)$ of degree $d-1$ over the rational integers such that if $F(X)$ is a strongly normic polynomial over Θ of degree d, and if d is less than the number q of elements in Θ/ρ; then $n < d^2 \phi_d(q-1)$. The proof makes use of certain results concerning ideals in rings of polynomials. (See Amer. J. Math. vol. 78 (1956) pp. 71–77.) (Received July 2, 1956.)

629t. Saunders MacLane: *The categorical bar construction and the homology of rings.*

For a ring Λ there is a natural complex $R(\Lambda)$ of Λ-bimodules giving the homology of Λ, with 2-dimensional cohomology the group of all ring extensions by Λ with a nilring as kernel. R is formed by a modification of the bar construction $B(Q, \psi)$, with Q a graded differential ring, $\psi : Q \to \Lambda$ a homomorphism of such. This construction, described by means of abelian categories, provides a universal pattern for all standard constructions of homological algebra. Take $\mathfrak{K}$ the category of graded differential right Λ-modules $\mathfrak{K}$, $\mathfrak{L}$ the subcategory of Q-Λ-bimodules, and $T : \mathfrak{K} \to \mathfrak{L}$ the universal functor $T(K) = Q \otimes K$ with the immersion $\tau(k) = 1 \otimes k$ of degree $+1$. Set $B_n = T^{n+1}(\Lambda)$. There is a given differential ∂_r in B_n, and in addition a differential $\partial_s : B_n$

$\to B_{n-1}$ with τ as its contracting homotopy. Then $\partial_s\partial_r = -\partial_r\partial_s$ and B is also acyclic for the differential $\partial_r+\partial_s$. In the quotient $\overline{B} = \Lambda \otimes_Q B$ the ∂_r and ∂_s are given by the formulas of Eilenberg-MacLane (Ann. of Math. vol. 58 (1953) p. 73) with G replaced by Q, J by Λ. The cubical complex $Q(\Lambda)$ of the additive group of Λ with augmentation η to Λ becomes a graded differential ring with a suitable product. One then defines $R(\Lambda) = \overline{B}(Q(\Lambda), \eta)$. (Received June 26, 1956.)

630t. Saunders MacLane: *Relative homology for abelian groups and modules.*

Relative homology theory, known to be intractable for groups, can be managed for abelian groups A in terms of the cubical complex $Q(A)$. (cf. Eilenberg-MacLane, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 294–330). In fact, for an exact sequence $0 \to A \to E \to C \to 0$ of abelian groups one proves that the induced map $Q(E)/Q(A) \to Q(C)$ is a chain equivalence. The requisite reverse map and the homotopy are constructed by a method of successive approximations due to Eilenberg-Mac Lane. Suppose next that Λ is a ring and A, E, and C, are Λ-modules. Using the categorical bar construction B (see Abstract 629 above) one may then construct a natural complex $M_\Lambda(C) = Q(C) \otimes_{Q(\Lambda)} B(Q(\Lambda), \eta)$ for each Λ-module C. By the same type of successive approximation, the induced map $M(E)/M(A) \to M(C)$ is again a chain equivalence. It follows that $M_\Lambda(C)$ is a standard and functorial projective (in fact, free) resolution of C. Hence M gives direct expressions for the functors Tor and Ext of Cartan-Eilenberg, and for the projective homotopy groups of Eckman-Hilton. (Received June 26, 1956.)

631t. D. W. Miller: *On uniform semigroups.*

A semigroup S is termed (i) *uniform* if $xS \cup Sx = yS \cup Sy$, all $x, y \in S$; (ii) *completely uniform* if $xS \cup Sx = S$, all $x \in S$; (iii) a *left* (*right*) *semigroup* if $xy = x(xy = y)$, all $x, y \in S$ (iv) a *zero semigroup* if S^2 has order 1. The following structural results are obtained. A completely uniform semigroup S is isomorphic to the direct product of a left (or right) semigroup and a group if and only if S contains at least one idempotent. A uniform semigroup containing exactly one idempotent is isomorphic to a subdirect product of a zero semigroup and a group; conversely, if Z is a zero semigroup and G a group then every subdirect product of Z and G is a uniform semigroup with just one idempotent. The class of uniform semigroups containing more than one idempotent is characterized by means of a semigroup constructed from an arbitrary group, two arbitrary sets, and two single-valued mappings. A completely uniform semigroup without idempotents is shown to admit division on one side, and hence is imbeddable in a generalized Baer-Levi semigroup (Teissier, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 1120–1122); if S is a uniform semigroup without idempotents, then S^1 admits division on one side. (Received July 5, 1956.)

632. C. N. Moore: *Patterns and prime pairs.*

If we stop the process known as the sieve of Eratosthenes at the end of a finite number of steps we obtain a sequence of integers relatively prime to each of the primes used in forming the sieve. The integers in this sequence less than the square of the following prime are absolutely prime. The set of integers which are less than or equal to $P_k = 2 \cdot 3 \cdot 5 \cdot \cdots \cdot p_k$, where p_k is the last prime used, form a pattern for all following integers, since any following sequence of length P_k can be found by adding a proper multiple of P_k to the integers of the original sequence. Such a sequence, or its successive differences, form a pattern for the whole distribution. It is the purpose of the present discussion to develop certain theorems concerning these patterns and to apply them to problems relative to the distribution of primes, in particular to the problem of prime pairs. The pattern obtained by deleting 3 and its successive multiples contains all the prime pairs except the first one. As we delete succeeding primes and their multiples, we obtain what may be termed patterns of prime pairs. The relationship between successive patterns of this sort gives us information concerning the asymptotic behavior of the prime pair distribution. (Received July 3, 1956.)

633. T. S. Motzkin, K. E. Ralston, and J. L. Selfridge (p): *Minimal overlapping under translation.*

Let the integers of the interval $(1, 4n)$ be divided into any two equal disjoint classes $\{a_i\}$ and $\{b_i\}$, $i=1(1)2n$. Must there exist an integer t such that $a_i+t=b_j$ has at least n solutions (i, j)? (P. Erdos, *Some remarks on number theory*, Riveon Lematematika vol. 9 (1955) pp. 45–48.) A k-tuple $c_1c_2 \cdots c_k$ shall mean: the first rc_1 integers are a's, the next rc_2 are b's, the next rc_3 are a's, etc., alternating, where $r=4n/\sum c_i$. A negative answer to the above question is furnished by the quintuple 35123, which has the property that there are $\leqq 6m/7$ solutions for any t; we write $\rho(35123)=6/7$. This is the smallest ρ for any subdivision with $k<7$, but $\rho(5314155)=5/6$ (e.g. with $n=6$). After Selfridge found the above examples, he and Ralston coded SWAC to search for others. The smallest ρ found was $4/5=\rho(4124111612214)$. P. Scherk has shown that $\rho>2-2^{1/2}$. (Received August 16, 1956.)

634t. R. B. Reisel: *A note on the Malcev theorem.*

Let A be as associative algebra over the field F having Jacobson radical N such that A/N is locally separable, $\bigcap_{k=1}^{\infty} N^k=\{0\}$, and A is complete with respect to the topology in which the powers of N form a fundamental system of neighborhoods of zero. Suppose that A contains subalgebras S and S^* such that $A=S\oplus N=S^*\oplus N$ (vector space direct sums). This note is concerned with finding a condition which will insure that S can be mapped onto S^* by an inner automorphism of A generated by an element of the form $(1-r)$, r in N. The condition obtained here is that the set N be complete with respect to a topology defined as follows: for each finite dimensional separable subalgebra E of S, let $N(E)$ be the set of elements r in N such that the inner automorphism of A generated by $(1-r)$ is the identity on E; for each s in N a fundamental system of neighborhoods of s is defined to be the collection of the sets $N(E) \circ s$, where $r \circ s=r+s-rs$, for E running through all finite dimensional separable subalgebras of S. (Received June 29, 1956.)

635t. Alex Rosenberg: *On the structure of the infinite general linear group.*

Let L be the ring of all linear transformations on a vector space of countably infinite dimension over a division ring. Let F be the (unique) ideal of the elements of L with finite dimensional range, Z the center of L, and G the group of invertible elements of L. Theorem: *There is no proper normal subgroup of G containing the group H of elements of the form $s+f$, s in Z and f in F; i.e. G/H is simple.* The main part of the proof consists in showing that G is generated by its elements of order two, or equivalently by the elements of the form $1+T$, $T^2=0$. (Received June 29, 1956.)

636. H. J. Ryser: *Combinatorial properties of* 0, 1 *matrices.*

Let A be a 0, 1 matrix of size m by n, with r_i denoting the sum of row i and s_j of column j. Define the *row* and *column sum vectors* of A by $R=(r_1, \cdots, r_m)$ and $S=(s_1, \cdots, s_n)$, respectively. Let $\bar{A}$ with column sum vector $\bar{S}$ be a 0, 1 matrix of size m by n such that the 1's in each row of $\bar{A}$ are shifted to the left as far as possible. Let S be a vector with non-negative integral components and let S be majorized by $\bar{S}$ (Hardy, Littlewood, Pólya, *Inequalities*, Cambridge, 1952). Then by rearranging 1's in the rows of $\bar{A}$, one may construct a matrix A having column sum vector S. An *interchange* is a transformation of four elements of A which replaces a two by two identity submatrix of A by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ or vice versa. Let A and A^* be two m by n matrices composed of 0's and 1's, possessing equal row sum vectors and equal column sum vectors. Then A is transformable into A^* by a finite number of interchanges. Applications to latin rectangles and to systems of distinct representatives are studied. (Received June 27, 1956.)

637. R. L. San Soucie: *Weakly standard rings.*

Albert introduced the topic of *standard rings*, defined by (1) $(wx, y, z)+(xz, y, w)+(wz, y, x)=0$ and (2) $(x, y, z)+(y, z, x)-(y, x, z)=0$, and showed that simple, finite dimensional algebras are standard if and only if they are either associative or Jordan algebras. Kleinfeld (Bull. Amer. Math. Soc. Abstract 62-1-127) remarked that (1) implies (3), $((w, x), y, z)=0$ and generalized Albert's result by proving that simple rings satisfy (2) and (3) if and only if they are either associative or commutative. Set $z=x$ in (2) and see that the generalized standard rings of Kleinfeld are *flexible*. In a flexible ring R, let $S=[s\in R \mid (s, x, y)=0]$ and let $M=[m\in R \mid (x, m, y)=0]$. Then (2) implies that $S\subset M$ and (3) states that $(x, y)\in S$. Call a ring R *weakly standard* if R is flexible and commutators are in $S\cap M$. The author proves that weakly standard rings are simple if and only if they are either associative or commutative, thus generalizing Kleinfeld's result. Finally, let R be *primitive* if R contains a maximal regular right ideal A which contains no two sided ideal of R other than the zero ideal. Then the author proves that a primitive weakly standard ring is either commutative or associative. (Received March 30, 1956.)

638. A. Sklar: *Dirichlet series and summation formulas.*

A class of meromorphic functions is considered, each member of which can be represented, in an appropriate half-plane, as an absolutely convergent Dirichlet series of the form $\sum_{n=1}^{\infty} a(n)\lambda_n^{-s}$ $(0<\lambda_1<\lambda_2<\cdots)$. For every function of the class an identity involving sums of the form $\sum_{\lambda_n\leq x} a(n)(x-\lambda_n)^q$ $(x>0, q$ complex) is obtained. The identities so obtained lead, in a quite simple fashion, to summation formulas which give expressions for sums of the form $\sum_{n=1}^{\infty} a(n)f(\lambda_n)$ where f is any function satisfying certain not very stringent conditions. The results include the summation formulas of Poisson, Voronoi, Hardy-Landau and certain of those of Ferrar and Guinand as special cases and lend themselves to a variety of application in analytic number theory. (Received July 2, 1956.)

639t. D. W. Wall: *Sub-quasigroups of finite quasigroups.*

Lagrange's theorem for finite groups does not hold for finite quasigroups, i.e., the order of a sub-quasigroup need not divide the order of the quasigroup. However, certain relationships can be obtained between the order of the quasigroup and the orders of its sub-quasigroups. Let Q be a quasigroup of order n. If S is a sub-quasigroup of order s then $n\geqq 2s$. Let R and S be sub-quasigroups of order r and s, respectively. If R and S intersect their intersection is a sub-quasigroup. If R and S intersect in a sub-quasigroup P of order p, then the following can be proved: I. $n\geqq r+s+\max(r, s)$

$-2p$. II. If $n=r+s+\max(r, s)-2p$ then $r=s$ if and only if the subset $P\cup[Q\backslash(R\cup S)]$ is a sub-quasigroup of Q. If R and S do not intersect the results take the following form: I. $n\geqq r+s+\max(r, s)$. II. If $n=r+s+\max(r, s)$ then $r=s$ if and only if $Q\backslash(R\cup S)$ is a sub-quasigroup of Q. (Received July 6, 1956.)

640. J. H. Walter: *Invariants for nonsemisimple algebras*. Preliminary report.

Let K be an algebraically closed field and $\mathfrak{A}$ be an algebra of finite rank over K which possesses an identity. Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$ and assume that $\mathfrak{A}-\mathfrak{N}$ is separable. Designate by $F_1, F_2, \cdots, F_h$ the irreducible representation modules of $\mathfrak{A}$. Let U_i be a principal indecomposable representation module with $U_i/\mathfrak{N}U_i$ isomorphic to F_i. Designate by V_{ij} the completely reducible representation space $\mathfrak{N}^j U_i/\mathfrak{N}^{j+1}U_i$. Let $H_{i,j,k}$ be the vector space $\mathrm{Hom}_{\mathfrak{A}}(F_i, V_{jk})$ of $\mathfrak{A}$-homomorphisms of F_i into V_{jk} and by $H^*_{i,j,k}$ its dual. Then if $C^1(\mathfrak{A}, \mathrm{Hom}_K(F_i, F_j))$ and $Z^1(\mathfrak{A}, \mathrm{Hom}_K(F_i, F_j)$ are the groups of 1-chains and 1-cocycles defined by Hochschild, bilinear functions from $H^*_{i,j,k}\times H_{m,j,n}$ to $Z^1(\mathfrak{A}, \mathrm{Hom}_K(F_m, F_i))$ and to certain factor groups of $C^1(\mathfrak{A}, \mathrm{Hom}_K(F_m, F_i))$ are used to give a set of invariants which distinguish up to isomorphism $\mathfrak{A}$ from any other algebra $\mathfrak{L}$ with radical $\mathfrak{M}$ such that $\mathfrak{A}-\mathfrak{N}$ is isomorphic to $\mathfrak{L}-\mathfrak{M}$. (Received July 9, 1956.)

641*t*. Y. K. Wong: *On matrices with nonnegative elements*.

Consider a square matrix $A=(a_{ij})$ with elements in an ordered field. Assume that $a_{ij}\geqq 0$ for $i, j=1, 2, \cdots, n$. If $u'A\leqq u'$ for some u' with positive components, then the adjoint $B=(b_{ij})$ of $I-A$ is nonnegative, and $b_{ij}\leqq u_j u_i^{-1} b_{ii}$ where $i\neq j$, and $i, j=1, \cdots, n$. Let A_k be the principal submatrix consisting of the elements in the first k rows and columns of A. Then $\det(I-A_k)$ is nonnegative and is nonincreasing relative to k. By $\det(I-A_n)\geqq(1-a_{11})\prod_k(1-a_{kk}-\sum_{i<k} u_i u_k^{-1} a_{ik})$, where $k=2, \cdots, n$, gives a lower bound. Thus a sufficient condition for $\det(I-A_n)>0$ is that $u'A<u'$ for some $u>0$. By Gauss' method of elimination, we can reduce $I-A$ into the form $I-C$, where $C=(c_{ij})$ is a triangle matrix with zeros below the principal diagonal. Then all the proper values of A are less than unity in absolute value if and only if $1-c_{ii}>0$ for $i=1, 2, \cdots, n$. (Received July 5, 1956.)

ANALYSIS

642*t*. D. R. Anderson: *On a generalization of the notion of a measurable group*.

In A. Weil's definition of *measurable group* (*groupe mesuré*) (P. Halmos, *Measure theory*, pp. 257–277) the requirement of left-translation invariance of the measure is equivalent to the weaker requirement that translation preserves measurability and sets of measure zero. (It is called a *generalized measurable group* if the weaker property is assumed.) It is shown that if the latter holds then an equivalent left-translation invariant measure exists. A new proof of the uniqueness of the left-invariant measure associated with a measurable group is also given. From these results follows a generalization of a theorem of MacKey and Loomis on representations of locally compact groups (Duke Journal of Mathematics vol. 19 (1952) pp. 641–645) where the topological assumptions are replaced by measure-theoretic assumptions. Also, one can show that a topological group is locally compact if and only if (1) the group uniform structure is complete, (2) the group is a *generalized measurable group*, and (3) all members of some fundamental system of neighborhoods of the identity are measurable. (Received August 17, 1956.)

643. Richard Arens: *Removing ideals by Banach algebra extension.*

An element z of a commutative Banach algebra A is called subregular if there is any extension B of A which has an inverse to z; this property implies that z is not a zero divisor, even in the generalized sense. One establishes here the converse, by constructing an extension $B(t)$ in the following way. Having performed a scaler modification of z (if needed) to insure that $\|az\| \geq \|a\|$ for all a in A, one forms the set $A(w;\ t)$ of all $f=a_0+a_1w+a_2w^2+\cdots$ where $\|f\|_t=\|a_0\|+\|a_n\|t^n+\cdots$ converges; and let $B(t)=A(w;\ t)/J$ where J is the closed ideal generated by $1-zw$. This $B(t)$ contains A as soon as $t\geq 1$, and thus solves the problem. This result can be reformulated in terms of removing points from spectra. When A is semisimple and z generates A, then (one can show that) $B(t)$ is semisimple, provided $t>\|\hat{z}\|_\infty$. (Received June 10, 1956.)

644. D. G. Aronson: *The first boundary value problem for a linear parabolic equation containing a small parameter.*

Let $\bar{R}$ be a closed region bounded by horizontal line segments S_1, S_4 and nowhere horizontal curves S_2, S_3. Consider in $\bar{R}$ the boundary value problem (*) $L_\epsilon(u)=\epsilon u_{xx}+a(x,\ y)u_x-b(x,\ y)u_y+c(x,\ y)u=d(x,\ y)$, u specified on $S_1+\bar{S}_2+\bar{S}_3$, $\epsilon>0$, where the data is in C^{IV} and $b>0$. Let $\overline{\mathfrak{M}}\subset\bar{R}$ be bounded by $I<S_i$ $(i=1,\ 2,\ 3)$, $J\subset S_2+S_3+\bar{S}_4-I$, and the characteristics of (**) $L_0(u)=au_x-bu_y+cu$ through their end points, where I is a closed connected set such that no characteristic of (**) starting there is tangent to $\bar{R}-R$ and J is the image of I under the characteristic mapping. *Theorem*: In any $\bar{M}$ the solution of (*) can be written $u(x,\ y,\ \epsilon)=v(x,\ y)+z_2(x,\ y,\ \epsilon)+z_3(x,\ y,\ \epsilon)+w(x,\ y,\ \epsilon)$, where v is the solution of $L_0(u)=d$ which assumes the given values of u on I, $w=0$ on $\bar{M}\cap\{S_1+\bar{S}_2+\bar{S}_3\}$, and $w=O(\epsilon)$ uniformly as $\epsilon\to 0+$. If $J\cap S_j=0$ $(j=2,\ 3)$, then $z_j=0$; otherwise z_j is a boundary layer term corresponding to $J\cap S_j$ and has the properties of the analogous term found in the elliptic case by Levinson (Annals of Math. vol. 51, pp. 428–445). The proof is based on Levinson's construction of the boundary layer term, the maximum principle for L_ϵ, and the fact that $L_\epsilon(w)=O(\epsilon)$, $w=0$ on $\bar{M}\cap\{S_1+\bar{S}_2+\bar{S}_3\}$, and w uniformly bounded in $\bar{M}$, implies $w=0(\epsilon)$ inside M. (Received June 15, 1956.)

645. M. G. Arsove: *Some remarks on potentials of linear mass distributions.*

The remarks deal with properties of continuity and differentiability and their dependence on both the set (assumed to have finite positive linear measure) and on the mass distribution (assumed to be absolutely continuous with respect to the measure). Generalizations of the classical theorems are given, along with a number of counterexamples (e.g. there exists a rectifiable simple curve which at each point admits a tangent and has linear density one, but on which a uniform distribution of the unit mass gives rise to a discontinuous potential). (Received July 5, 1956.)

646. J. H. Barrett: *A Prufer transformation for matrix differential equations.*

Let capital letters denote $n\times n$ square matrices and consider the matrix differential equation (I): $(PY')'+QY=0$, where on $a\leq x<\infty$, $P(x)$ is nonsingular and $P(x)$

and $Q(x)$ are real, symmetric and continuous. It is shown that for every nontrivial solution $Y(x)$ of (I), for which $Y(0)=0$, there exists a nonsingular differentiable matrix $R(x)$ and a symmetric continuous matrix $Q(x)$ such that $Y(x)=S^*[a, x; Q]R(x)$ and $P(x)Y'(x)=C^*[a, x; Q]R(x)$, where $Z(x)=S[a, x; Q]$ and $W(x)=C[a, x; Q]$ are solutions of the system $Z'=QW$, $W'=-QZ$, $Z(a)=0$, $W(a)=E$ (identity). It is noted that for $n=1$, S and C are $\sin\int_a^x Q$ and $\cos\int_a^x Q$, respectively. For $n\geq 1$ various "trigonometric" identities and inequalities are established for the matrices S and C. (Received July 5, 1956.)

647*t*. H. S. Bear and Bertram Yood: *Multiplicative functionals on semi-groups of continuous functions.*

Let $S(X)$ be the multiplicative semi-group of all real continuous functions on a compact Hausdorff space X. Let $M(X)$ be the semi-group of all real continuous multilicative functions defined on $S(X)$ to the real number system. Let X_1, X_2 be compact Hausdorff spaces satisfying the first axiom of countability. It is shown that X_1 is homeomorphic to X_2 if and only if there exist a semi-group isomorphism of $M(X_1)$ onto $M(X_2)$. The case where the X_i do not satisfy the countability axiom is also studied. (Received April 23, 1956.)

648*t*. Stefan Bergman: *On singularities of solutions of certain differential equations in three variables.* I.

In order to study real solutions of differential equations (1) $\hat\psi_{xx}+\hat\psi_{yy}+\hat\psi_{zz}+\widehat{F\psi}=\psi_{XX}-\psi_{ZZ^*}+F(Z, Z^*)\psi=0$, $\hat F(y, z)=F(Z, Z^*)$, the author introduces a special class $\mathfrak{K}$ of complex solutions of (1) which possesses the property that $\psi(X, 0, Z^*)$ is a function of X alone. (Here $X=x$, $Z=(z+iy)/2$, $Z^*=-(z-iy)/2$; x, y, z are the Euclidean coordinates.) He defines an integral operator $\mathrm{P}[g(x, z)]$ transforming functions $g(X, Z)$ into solutions of (1). If in the domain D of the (X, Z, Z^*) space $|F(Z, Z^*)|\leq C<\infty$, $|g^{(k)}|=|\partial^k g(X, Z)/\partial X^k|\leq Ak!a^k$ then a solution of (1) possessing the property that (2) $\psi(X, Z, 0)=g(X, Z)$, $\psi(X, 0, Z^*)=g(X, 0)$, exists provided that $\sum_{n=0}^{\infty}|ZZ^*|^{n/2}C^{-n/2}a_nB_n[2C^{1/2}|ZZ^*|^{1/2}]<\infty$. ($B_n$ are modified Bessel functions.) Let $J_1^{(k)}(X, Z, Z^*)=\int_0^Z\int_0^{Z^*}F(Z_2, Z_2^*)\int_0^{Z_2}\int_0^{Z_2^*}F(Z_1, Z_1^*)g^{(k)}(X, Z_1)dZ_1dZ_1^*dZ_2dZ_2^*$ and $J_n^{(k)}$ be analogously formed sums of $2n$-ply integrals. If $\psi=\sum_{\nu=0}^{\infty}\sum_{k=0}^{\nu}J_\nu^{(k)}$ converges uniformly in a domain D, it represents a solution of (1) satisfying the conditions (2). The author derives the representation $\psi=\sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}\int_0^Z H^{(\nu,k)}(Z, Z^*, Z_1)g^{(k)}(X, Z_1)dZ_1$ for solutions of (1). $\mathrm{H}^{(\nu,k)}$ depends only on F. (Received June 4, 1956.)

649*t*. Stefan Bergman. *On singularities of solutions of certain differential equations in three variables.* II.

This paper continues the author's investigations about the connection between the location and character of the singularities of a solution ψ of a partial differential equation and the properties of a subsequence of coefficients of the series development of ψ. Let $L=[X-\sum_{m=0}^{M}a_mZ^m=0]$ where a_m are constants. Choosing $g(X, Z)=(X-\sum_{m=0}^{M}a_mZ^m)^{-1}$ the author obtains solutions $\psi(X, Z, Z^*)=\mathrm{P}[(X-\sum_{m=0}^{M}a_mZ^m)^{-1}]=\mathrm{P}[\sum_{\nu=0}^{M-1}a_\nu(X)(Z-Z_\nu(X))^{-1}]$ of (1) of the class $\mathfrak{K}$. They are singular along L. Necessary and sufficient conditions for the subsequence $\{A_{nm0}\}$, n, $m=0, 1, 2, \cdots$ of the coefficients of a function element $\tau(X, Z, Z^*)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}A_{nmr}X^nZ^mZ^{*r}$ are determined in order that for $X_0\leq X\leq X_1$, $\tau=\psi+\phi$, where ϕ is an entire solution of (1). (The results obtained for solutions of the class $\mathfrak{K}$ can be easily interpreted as analogous results for real solutions.) The author determines further the singularity manifolds $Z=Z_\nu(X)$, $\nu=0, 1, \cdots, M-1$ of τ in terms of the $\{A_{n,m,k}\}$. These results can be generalized in many instances to the case of solutions ψ $P[(\sum_{\nu=0}^{m} b_\nu(X)Z^\nu)^{-1}]$ e.g., if $b_\nu(X)$ are linear functions of X. (Received June 4, 1956.)

650. G. U. Brauer: *Sets of divergence of ordinary Dirichlet series.*

If E is a set of type F_σ on the τ-axis M, and of logarithmic measure zero, then there exists an ordinary Dirichlet series $\sum_n a_n n^{-s}$ which diverges on E and converges on $M-E$ (here $s=\sigma+i\tau$). The series has bounded partial sums on any bounded subset of M, and the function represented by this series is continuous on M. (Received July 2, 1956.)

651. T. S. Chihara: *On a class of polynomials related to the Hermite polynomials.*

The polynomials $R_n^\nu(x)=(-1)^n \exp(x^2)(d/dx)^n x^\nu \exp(-x^2)$ $(n, \nu=0, 1, 2, \cdots)$ are considered. For $\nu=0, 1$, these reduce to Hermite polynomials, while for $\nu\geqq 2$, $R_n^\nu(x)$ can be expressed as a simple linear combination of Hermite polynomials. This latter relation leads to the following "quasi-orthogonality" relation: $\int_{-\infty}^{\infty} \exp(-x^2) \cdot R_n^\nu(x)R_m^\nu(x)dx=0$ for $m\neq n\pm 2j$ $(j=0, 1, \cdots, [\nu/2])$. A simple generalization of this property to polynomials $q_n(x)$ satisfying a relation of the form $\int_a^b q_n(x)q_m(x)d\alpha(x)=0$ for $m\neq n\pm rj$ $(j=0, 1, \cdots, k)$ is briefly considered. Generating functions, recurrence formulas, differential equations, various representations, and simple bounds for the $R_n^\nu(x)$ are also obtained. (Received June 28, 1956.)

652. P. C. Curtis, Jr.: *Bounded directed families of self-adjoint elements in AW* algebras.*

A C* algebra A will be called a BW* algebra if every bounded directed family of self adjoint elements in A has a least upper bound in A: It is shown that BW* algebras are a sub-class of AW* algebras, and as a partial converse, an AW* algebra of type I is a BW* algebra. Let A and B be BW* algebras with $A\subset B$. A will be called a BW* subalgebra of B if A is a closed * subalgebra of B and if the least upper bound (in B) of every bounded directed family of self adjoint elements of A belongs to A. Let A be a commutative subalgebra of B and let C be a maximal commutative subalgebra of B containing A. By a suitable comparison between the structure spaces of A and C, it can be shown that a necessary and sufficient condition for A to be a BW* subalgebra of B is for it to be an AW* subalgebra of B. (Received July 5, 1956.)

653t. M. M. Day: *Means and ergodicity.* II.

The result of Abstract 606 can be used to strengthen and improve Theorem 4 of M and E, I (Trans. Amer. Math. Soc. vol. 69 (1950) pp. 276–281) to Theorem: If Σ is an amenable semigroup and if F is a bounded representation of Σ over a Banach space B, then $F(\Sigma)$ is uniformly ergodic. (Received June 29, 1956.)

654t. S. P. Diliberto: *Periodic surfaces* I.

The notion of a *periodic k-surface* (a k-dimensional torus composed of trajectories is introduced in the study of systems of first order ordinary differential equations $\dot{x}=f(x, t, \cdots, t, \lambda)$ where $x=(x_1, \cdots, x_n)$, t, λ are real and $f(x, \theta_1, \cdots, \theta_e, \lambda)$ has period ω_i in θ_i. This notion includes that of "periodic solution" and the Kryloff-Bogilinboff "invariant curve"; and is shown to be fundamental in describing the physical phenomena of *resonance*, *entrainment* of *frequency*, and *Plaat's phenomena*.

Several perturbation problems are solved. For the $K-B$ invariant curve problem for $\ddot{x}+\varpi_1^2x=\lambda f(x, \dot{x}, t)$ where f has period $2\pi/\varpi_2$ in t; $a(r, \theta, t)=\cos \varpi_1\theta f(r \sin \varpi_1\theta, \varpi_1 r \cos \varpi_1\theta, t)$; $b(r, \theta, t)=\partial a(r, \theta, t)/\partial r$; $c(r, \theta, t)=\tan \varpi_1\theta a(r, \theta, t)/r$; $d(r, \theta, t)=c(r, \theta, t) -(\varpi_1\varpi_2/(2\pi)^2) \int_0^{2\pi/\varpi_2}\int^{2\pi/\varpi_1}c(r, \theta, t)d\theta dt$ we sharpen their theorem by reducing hypothesis to: smoothness of f; and for some $\rho_0(\neq 0)$ $M_t a(\rho_0, t+\theta, t)=0$, $M_t d(\rho_0, t+\theta, t)=0$, and $M_t b(\rho_0, t+\theta, t)\neq 0$ (each for all θ) where $M_t(\ )=\lim_{T\to\infty} (1/T)\int_0^T(\ )dt$. (Received July 3, 1956.)

655*t*. S. P. Diliberto: *Periodic surfaces* II.

Let $\mathfrak{C}$ be the Banach space of all continuous periodic (period ϖ_1) real functions of the real variable θ_1. If $\alpha_i\in\mathfrak{C}$ $(i=1, 2)$ define $(\alpha_1+\alpha_2)(\theta_1)=\alpha_1(\theta_1)+\alpha_2(\theta_1)$; $\|\alpha\| =\max_\theta |\alpha(\theta)|$; and if L a linear operator (on $\mathfrak{C}$ into $\mathfrak{C}$) let $\|L\|=\sup \|L\alpha\|$ for $\|\alpha\|=1$. Let $b(\theta_1, \theta_2)$ be continuous (θ_1, θ_2) and periodic ϖ_i in θ_i $(i=1, 2)$; $B(\theta_1, t)=\int_0^t b(s+\theta_1, s)ds$ and $B(\theta_1)=B(\theta_1, \varpi_2)$. *Theorem*: If L is the linear operator *on C defined for* $\alpha\in\mathfrak{C}$ *by* $(L\alpha)(\theta_1)=\alpha(\theta_1+\varpi_2)-\alpha(\theta_1) \operatorname{Exp} [B(\theta_1)]$, then L^{-1} *exists and is bounded if and only if* $|\lim_{T\to\infty} (1/T)B(\theta_1, T)| >0$ *for all* θ_1. This result shows that the above strengthening of the $K-B$ theorem is sharp. (Received July 3, 1956.)

656*t*. Jim Douglas, Jr.: *A uniqueness theorem for the solution of a Stefan problem.*

A Stefan problem is a free boundary problem associated with a parabolic differential equation. Numerous existence theorems have been given for such problems, but no complete uniqueness theorem has been proved. The object of this paper is to demonstrate the uniqueness of the solution of the following problem: $u_{xx}=f(u)u_t$, $0<x<x(t)$; $u_x(0, t)=-a$, $t>0$; $u(x(t), t)=0$, $t>0$; $x(0)=0$; $dx(t)/dt=b-cu_x(x(t), t)$, $t>0$; where $x(t)$ is the position of the moving boundary at time t, $f(u)>0$, $f'(u)\geq 0$, and a, b, c are positive constants. The paper will appear in the Proceedings. (Received July 5, 1956.)

657. A. C. Downing, Jr.: *Convergence of finite difference solutions of the multi-region diffusion problem.*

The classical method of Courant, Friedrich, and Lewy for demonstrating the convergence of finite difference solutions of the Dirichlet problem is extended in this paper to the multi-region diffusion problem. Under appropriate uniqueness conditions, the prescribed discontinuities of the solution are shown to be satisfied almost everywhere in the limit as the mesh size of the finite difference solutions tend to zero. (Received July 9, 1956.)

658. B. J. Eisenstadt and G. G. Lorentz (p): *Boolean rings and Banach lattices.*

Let $\Phi(e)\geq 0$ be a function defined on a Boolean σ-ring B. Necessary and sufficient conditions are given under which B may be imbedded into a Banach lattice X so that $\Phi(e)$ coincides with the norm of X on B. Among such X there is one with the largest norm; its elements are certain functions $x=x(t)$ from reals to B. x, y are called covariant if all $x(t)$, $y(t_1)$ are comparable. B can be imbedded in an X with the norm additive for positive covariant elements if and only if Φ is concave, i.e. satisfies $\Phi(e_1\cup e_2) +\Phi(e_1\cap e_2)\leq\Phi(e_1)+\Phi(e_2)$. In this case, the norm in the extremal extension $X=\Lambda_\Phi$ of B is given by $\|x\|=\int_0^\infty\Phi(|x(t)|)dt$, so that these X are natural generalizations of Λ-spaces of one of the authors. These results allow a complete characterization of Banach lattices with a unit which are isomorphic to a general Λ-space. (Received July 5, 1956.)

659*t*. B. J. Eisenstadt and G. G. Lorentz: *Functions of measures and Λ spaces.*

A Banach lattice may be realized as a collection of functions from the reals to a σ-Boolean algebra. [Freudenthal, Proc. Amsterdam Academy vol. 39 (1936) pp. 641–651]. The norm is an extension of a positive (vanishing only at zero) increasing function Φ on the algebra. Necessary and sufficient conditions are given for Φ to be a real valued function of a positive measure if Φ is multiply subadditive [Lorentz, Canadian Journal of Math. vol. 4 (1952) pp. 455–462)] and if Φ is concave. Let μ be a fixed positive measure on the algebra. The notion of equimeasurability (with respect to μ) is defined for positive measures. If Φ is concave and of the form $F(\mu)$, it is shown that $\Phi = \sup \psi_r$ where ψ_r runs through a fixed class of equimeasurable positive measures. Furthermore, $\psi_r(e) = \int_e f_r(t)d\mu(t)$ where the f_r are equimeasurable (in the usual sense) positive functions on a representation space E. If, in addition, the Banach lattice is a Λ_Φ space, then under this representation an element x of the lattice becomes a real function $x(t)$ on E with the norm $\sup_{f_r} \int_E |x(t)| f_r(t)d\mu(t)$. (Received July 5, 1956.)

660*t*. Evelyn Frank: *A new class of continued fraction expansions for the ratios of Heine functions.*

A new class of continued fraction expansions for the ratios of two Heine functions is described in detail. A number of ratios of Heine functions are expanded, and the convergence of the corresponding continued fractions is determined as well as the functions to which these continued fractions converge in certain regions. Recurrence relations for the numerators and denominators of the approximants of these continued fractions are derived. Certain special continued fraction expansions are found, and many special relations are obtained between functions, finite continued fractions and infinite continued fractions. (Received July 6, 1956.)

661. W. B. Fulks: *Heat conduction in general regions.*

Let R be a bounded region in the xt plane, and f a bounded function defined on the boundary B of R. The method of O. Perron is applied to the problem of finding a solution to the Dirichlet problem for the heat equation: to find a solution of $u_t = u_{xx}$ in R and $u = f$ on B. Sub- and super-parabolic functions are defined and used in a manner analogous to the use of sub- and super-harmonic functions. (See also Sternberg, Math. Ann. (1929) and Petrowsky, Comp. Math. (1934).) The concept of thermal capacity zero is defined and the following theorem is proved: if u_1 and u_2 be two solutions of $u_t = u_{xx}$ in R and have the same boundary values on B except possibly for a set E of thermal capacity zero, then $u_1 \equiv u_2$. (Received July 5, 1956.)

662*t*. I. S. Gál: *On the fundamental theorems of the differential and integral calculus.*

Let the real function f be defined in the finite closed interval $[a, b]$ and let $\Lambda^-(x)$ and $\Lambda^+(x)$ denote the limit superior of f at x from the left and from the right respectively. Denote by $\mu(t)$ the exterior or the interior measure of the set where $D^+f < t$ according as $t<0$ or $t>0$. A property is said to hold nearly everywhere if it holds everywhere except possibly on a countable set of points. It is proved that: I. *If* $\Lambda^-(x) \leq f(x) \leq \Lambda^+(x)$ *everywhere and* $D^+f > -\infty$ *nearly everywhere then* $f(b) - f(a)$

$\geq \int t d\mu$ *with* $-\infty < t \leq 0$. Hence if $D^+f \geq 0$ almost everywhere then f is increasing in $[a, b]$. II. *If f is continuous in $[a, b]$ and $D^+f > -\infty$ nearly everywhere then $f(b) - f(a) \geq t d\mu$ with $-\infty < t < +\infty$ whenever $\int t d\mu$ with $-\infty < t \leq 0$ converges.* De la Vallée Poussin's decomposition theorem is a simple consequence of I and II. Only the elements of the theory of exterior measure and the compactness of $[a, b]$ are used in the proofs. (Received June 29, 1956.)

663*t*. B. R. Gelbaum: *Conditional and unconditional convergence in Banach spaces.*

The countable direct product G_∞ of the group of order two with itself is used to investigate conditional and unconditional convergence in Banach spaces: (i) If $\{x_n; X_n\}$ is a basis, then $\sum_1^\infty \epsilon_n(g) X_n(x) x_n$, $(\epsilon_n(g) = \pm 1)$, converges for every x for all g in G_∞ or for a set of g of measure 0; (ii) Let $R(x) = \{y \mid y = \sum_1^\infty \epsilon_n(g) X_n(x) x_n, g \text{ in } G_\infty\}$. Then $\{x_n; X_n\}$ is an unconditional basis if and only if $R(x)$ has a compact closure in either the weak or norm topologies; (iii) Let $H = \{x \mid \sum_1^\infty \epsilon_n(g) X_n(x) x_n$ converges for all g in $G_\infty\}$. Then H is the whole Banach space or H is of the first category. (Received May 14, 1956.)

664*t*. B. R. Gelbaum: *Notes on Banach spaces and bases.*

A collection of theorems on the phenomena associated with bases in Banach spaces is offered. Among other results are the following, some of which are stated in Banach's book without proof or reference: (i) Every separable Banach space has a basis if and only if every Banach space which is the linear image of a Banach space with a basis has a basis. (ii) Every separable Banach space is the linear image of a Banach space with a basis. (iii) If $\{x_n; X_n\}$ is a basis, there are constants a_n such that the $\rho_n = \sum_1^n a_m x_m$ form a basis. Examples and special cases are discussed. (Received May 14, 1956.)

665*t*. Casper Goffman: *Convergence in norm of integral means.*

Let X be a Banach lattice of summable functions on the closed interval $[0, 1]$, such that if $x = x(t) \in X$ then, for every u, $x_u = x(t+u) \in X$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that $|u - v| < \delta$ implies $\|x_u - x_v\| < \epsilon$. It is shown that the integral means $x^h = h^{-1}\int_0^h x(t+u)du$ belong to X and that $\lim \|x - x^h\| = 0$, for every $x \in X$. This furnishes a rigorous version, for this special kernel system, of a heuristic theorem stated by S. Bochner (Harmonic Analysis and the Theory of Probability, Berkeley, 1955, p. 9). The lattice property seems to overcome the absence of an analogue to the Hölder inequality, which is used in the proof for L_p. (Sponsored by National Science Foundation grant no. NSF G-2267). (Received June 6, 1956).

666*t*. Casper Goffman: *Compatible semi-norms in a vector lattice.* I.

A locally convex topology τ for a vector space X is compatible with a lattice ordering ω for X if τ may be given by semi norms P_α for which $|x| \geq |y|$ implies $P_\alpha(x) \geq P_\alpha(y)$. With the notations X_τ for the topological vector space, X'_τ for its dual, and X_ω for the vector lattice, X'_ω for its dual of bounded linear forms, the following is obtained: If τ is compatible with ω then $X'_\tau \subset X'_\omega$ and if τ is the finest topology compatible with ω then $X'_\tau = X'_\omega$. Hence, if X_τ is a Mackey space (τ compatible with ω) for which $X'_\tau = X'_\omega$ then τ is the finest topology compatible with ω. For Banach spaces, the following is then obtained: A given vector lattice may be normed as a Banach lattice in at most one way. In particular, the classical norms used for various real function spaces thus appear as the only "reasonable" choices. (Sponsored by National Science Foundation Grant No. NSF G-2267). (Received May 14, 1956.)

667. A. A. Goldstein: *On the Tchebycheff and least (2q)th approximation of an overdetermined system of linear equations.*

Let $A=\|a_{ij}\|$ be an $m\times n$ matrix $m>n$ which has no vanishing $n\times n$ determinants and let $X=\|x_j\|$ be a vector whose domain is a closed bounded subset E of the n-dimensional Euclidean vector space E_n. Let $B=\|b_i\|$ and $Y=\|y_i\|$ be vectors in E_m. If the transform $AX=Y$ of X lies closest to B in the metric 1. $\sup_i |y_i-b_i|$ or 2. $[\sum_i (y_i-b_i)^{2q}]^{1/2q}$ then the solution vector will be called Tchebycheff or least $(2q)$th and denoted by $\bar{X}$ or X' respectively. We have that the $X\in E$ which satisfy $\sup_i |\sum_j a_{ij}x_j-b_i|\leq K$ and $\sum_i (\sum_j a_{ij}\, x_j-b_i)^{2q}\leq K^{2q}$ are convex bodies, and that for q sufficiently large the $\bar{X}$ and X' lie arbitrarily close. This suggests the method of descent to find $\bar{X}$ or X'. Let $g(x)=k$, $m<k\leq K$ describe a convex body, where m is the minimum of $g(x)$. Define a sequence $g(x_p)$ as follows: given $Z_p\in E$ an arbitrary vector let x_{p+1} be the point on which $g(x)$ assumes a minimum along the ray through x_p parallel to Z_p. Let E' be the set of all rays through x_p which cut the assemblage of points $g(x)<g(x_p)$. If Z_p is any sequence such that the rays through x_p parallel to Z_p are frequently in E' then $g(x_p)$ converges to the minimum of $g(x)$ and there exists a unique x for the minimum of $g(x)$. (Received July 9, 1956.)

668t. J. K. Hale: *On a class of linear differential equations with periodic coefficients.*

Consider the system of linear differential equations (1) $y''+A(\lambda)y=\lambda\phi(t,\lambda)y+\lambda\psi(t,\lambda)y'$, where λ is a real parameter, $y=(y_1,\cdots,y_n)$, $A(\lambda)=\text{diag}\,(\sigma_1^2,\cdots,\sigma_n^2)$, ϕ and ψ are $n\times n$ matrices whose elements are real, periodic functions of t of period $T=2\pi/\omega$, are L-integrable in $[0, T]$, are analytic in λ and have mean value zero. Further, suppose that each $\sigma_j^2(\lambda)$, $j=1, 2, \cdots, n$, is a real positive analytic function of λ with $\sigma_j(0)\not\equiv\sigma_k(0)$, (mod ωi), $j\neq k$, $j, k=1, 2, \cdots, n$. By employing a method of successive approximations introduced by L. Cesari [Atti Accad. Italia, (6) vol. 11 (1940) pp. 633–692], the following theorem is proved: If $\phi=\|\phi_{ij}\|$, $\psi=\|\psi_{ij}\|$ where ϕ_{ij}, ψ_{ij}, $i, j=1, 2$, are matrices with ϕ_{11} and ψ_{11} of dimension $\mu\times\mu$, $0\leq\mu\leq n$, and if ϕ_{11}, ϕ_{22}, ψ_{12}, ψ_{21} are even in t, ϕ_{12}, ϕ_{21}, ψ_{11}, ψ_{22}, are odd in t, then, for $|\lambda|$ sufficiently small, all the absolutely continuous solutions of (1) are bounded in $(-\infty, +\infty)$. (Received July 2, 1956.)

669. J. K. Hale: *A new sufficient condition for periodic solutions of a class of weakly nonlinear differential equations.*

A vector function $f(x, t)$ belongs to $A[\omega]$ if, in a neighborhood U of the origin (U independent of t), each component of f has a power series expansion convergent in U whose coefficients are L-integrable functions of t of period $2\pi/\omega$. Consider a system of equations (1) $y''+Dy=\epsilon f(y, y', \epsilon, t)$, where $y=(y_1,\cdots,y_n)$, $f=(f_1,\cdots,f_n)$, $D=\text{diag}\,(d_1^2,\cdots,d_n^2)$, ϵ real, $d_1,\cdots,d_n$ positive, and $f\in A[\omega]$. Let m be any integer, $0\leq m\leq n$, and $y=(u, w)$, $u=(y_1,\cdots,y_m)$, $w=(y_{m+1},\cdots,y_n)$, $a=(a_1,\cdots,a_n)$, $b=(b_1,\cdots,b_n)$, $c=(c_1,\cdots,c_n)$, $d=(d_1,\cdots,d_n)$, where the numbers a_j, b_j are positive integers. By employing a method which has been successively developed by L. Cesari, R. A. Gambill, and J. K. Hale, the following theorem is proved: If $f_j(u, -w, -u', w', \epsilon; -t)=\lambda_j f_j(u, w, u', w', \epsilon; t)$, $\lambda_j=+1$ if $j=1,\cdots,m$, $\lambda_j=-1$ if $j=m+1,\cdots,n$, then real functions $B_k(a, b, c, d, \omega, \epsilon)$, $k=1,\cdots,n$, analytic in ϵ for $|\epsilon|$

small, are determined such that, for the system of equations (2) $a_k\omega/b_k+\epsilon B_k=d_k$, $k=1, \cdots, n$, has solutions for real non-zero d, c, and $|\epsilon|$ small, then system (1) has a solution $y(\epsilon, t)\in A[\omega/b_1 \cdots b_n]$ of the form $y_j(0, t)=c_j \cos (a_j\omega t/b_j), j=1, 2, \cdots, m$, $y_j(0, t)=c_j \sin (a_j\omega t/b_j), j=m+1, \cdots, n$, and $y_j(\epsilon, -t)=\lambda_j y_j(\epsilon, t), j=1, \cdots, n$. Solutions for system (2) can be assured even in cases where classical known conditions fail. (Received July 2, 1956.)

670. John Harton, Jr.: *Extremal problems for real star mappings.*

Let $S=$ the family of functions $f(z)=z+a_2z^2+a_3z^3+ \cdots$ which are regular and univalent for $|z|<1$, map the unit circle onto star-shaped regions and for which $a_2, a_3, \cdots$ are real. Let B, C and α be complex numbers such that $|B|+|C|>0$ and $0<|\alpha|<1$. Consider the maximum possible value of the real part of $B \log f'(\alpha) +C \log [f(\alpha)/\alpha]$ for $f\in S$. A function of S which maximizes the above expression is called an extremal function. It is shown that every extremal function is a member of S which maps the unit circle onto the ω-plane with, at most, four radial slits. That this theorem is sharp is shown by taking particular values of B, C and α for which extremals with one up to four radial slits are found. For α real explicit formulas are obtained for the extremal functions and in this case each has at most two radial slits. The problem is attacked by means of an approximation theorem which reduces the problem to that of maximizing a certain sum. This method is similar to that used by Robinson (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 364–377). (Received June 27, 1956.)

671. L. J. Heider: *The Myers' F_T functionals as integrals.*

A Banach space B is susceptible of an ordering under which it becomes an abstract (AL) L space, if and only if it contains a subset T closed under addition, and such that for each element a in B there exist uniquely elements a^+, a^- in T with $a=a^+-a^-$ and $\|a\|=\|a^+\|+\|a^-\|$, while $a=b-c$ with b, c in T implies $\|b-a^+\|=\|b\|-\|a\|$. The resulting AL space has a weak unit if and only if T contains an element e such that $\|a-e\|<\|a\|+\|e\|$ for all a in T. A subset T of B with the above properties is a Myers' T-set, and if one T-set of B has the above properties, so does any other. Let T_s, $s\in S$, denote the collection of all T-sets in a potential AL space B, and let F_s denote the F_T functional corresponding to T_s. In the representation of the AL space (B, T_{s_0}), T_{s_0} fixed, as the space of integrable functions over a measure space (Y, S, μ), for all a in (B, T_{s_0}) one has $F_{s_0}(a)=\int a d\mu$, while $(F_{s_0}[a]+F_s[a])/2=\int_s a d\mu$ under the 1-1 correspondence between measurable subsets $s\in S$ of Y and T-sets T_s, $s\in S$, of B. In the representation of (B, T_s), $s\neq s_0$, the same space (Y, S, μ) is used, but the functions on Y representing a given element of B first as in (B, T_{s_0}) and then as in (B, T_s) differ by a change of sign at each point of the measurable subset $s\in S$ of Y corresponding to the T-set T_s. (Received July 2, 1956.)

672t. J. A. Hummel: *A counterexample to the Poincaré inequality.*

If D is a bounded plane domain with a sufficiently regular boundary, then there exists a constant K such that for any continuous real function ϕ with $\iint_D\phi dxdy=0$, the Poincaré inequality $\iint_D\phi^2dxdy\leq K\iint_D(\phi_x^2+\phi_y^2)dxdy$ holds (cf. Courant and Hilbert, ***Methoden der Mathematischen Physik***, vol. II). It is shown that this inequality does not hold for all bounded plane domains even if ϕ is restricted to be harmonic. In the w-plane $(w=u+iv)$, let $\Delta=\{w:1<u<\infty,\ 1/(u+2\pi)^2<v<1/u^2\}$. Then the counterexample is given by the domain D which is the image of Δ under the mapping

$z = e^{iw}$ and $\phi = u - c$, where c is to be chosen to make $\iint_D \phi\, dx\, dy = 0$. (Received July 5, 1956.)

673. J. A. Hummel: *Complete orthonormal sequences of functions uniformly small on a subset.*

If G is a region of the complex plane and if f and g are two complex valued functions, set $(f, g)_G = \iint_G f\bar{g}\, dx\, dy$, $\|f\|_G = (f, f)_G^{1/2}$. Let D be a bounded plane domain and $L^2(D)$ the Hilbert space of all analytic functions with $\|f\|_D < \infty$. Let G be a domain in D with $\bar{G} \subset D$ and let $\{\psi_n\}$ be a doubly orthogonal sequence in G and D (cf., Bergman, *The kernel function*, Mathematical Surveys, no. 5, Amer. Math. Soc., 1950, pp. 14–17), i.e., $\{\psi_n\}$ is complete in $L^2(D)$, $(\psi_n, \psi_m)_D = \delta_{nm}$, $(\psi_n, \psi_m)_G = \lambda_n \delta_{nm}$, and $\sum \lambda_n = M^2 < \infty$. Let (ϵ_{ij}) be an $N \times N$ matrix (where $N = 2^p$ for some integer p) with each $\epsilon_{ij} = \pm 1$ and $\sum_j \epsilon_{ij}\epsilon_{kj} = 0$ for $i \neq k$. Set $\phi_i = (1/N)^{1/2} \sum_{j=1} \epsilon_{ij}\psi_j$ for $i \leq N$ and $\phi_i = \psi_i$ for $i > N$. Then $\{\phi_n\}$ is a complete orthonormal set in $L^2(D)$, but given $\epsilon > 0$, for N sufficiently large, $\|\phi_n\|_G < \epsilon$ for all n. Using this result, it is easily proved that: *If K is a compact subset of D, then given $\epsilon > 0$ there exists a complete orthonormal set $\{\phi_n\}$ in $L^2(D)$ such that $|\phi_n(z)| < \epsilon$ for all n and all $z \in K$.* This shows in particular that if the Bergman kernel function is to be approximated by a finite number of terms of its series expansion, the number of terms required cannot be given independently of the sequence used. (Received July 5, 1956.)

674. Meyer Jerison: *The set of all generalized limits of bounded sequences.* II.

Let L denote the set of all generalized (Banach) limits defined on the space (m) of bounded sequences. L is a weak*-compact convex subset of the conjugate space of (m), and in an earlier note [Bull. Amer. Math. Soc. Abstract 59-6-625] a set Q of elements of L was constructed with the property that the closed convex hull of Q is L. For any $x = (\xi_1, \xi_2, \cdots)$ in (m), it is possible to find $q(x) = \sup_{f \in L} f(x)$ by calculating $\sup_{f \in Q} f(x)$. It turns out to be $q(x) = \lim_{n \to \infty} \limsup_{k \to \infty} n^{-1} \sum_{i=0}^{n-1} \xi_{k+i}$, which is much simpler than the formula given by Banach (Théorie des opérations linéaires, pp. 33–34). (Received July 6, 1956.)

675t. E. S. Johnson and C. T. Taam: *On the solutions of nonlinear differential equations* II.

Let $p(t)$ and $q(t)$ be real Lebesgue-measurable functions having positive lower and upper bounds on $0 \leq t < \infty$. Denote by $x = x(t, E)$ the solution of (NL) $x'' + px - 2qx^3 = 0$ satisfying (*) $x(0) = 0$, $x'(0) = E > 0$. The starting point in this paper is this result: (I). If $0 < E_1 < E_2$, then $x(t, E_1) < x(t, E_2)$ for all positive t as far as $x(t, E_1)$ remains non-negative. (I) is proved by an argument resembling the proof of Sturm's comparison theorem. (I) is used to prove (II): there exists a positive constant E_0 such that for $0 < E < E_0$, $x(t, E)$ has a first positive zero $T(E)$ which increases with E and tends to ∞ or to a positive limit according $E \to E_0-$ or $E \to 0+$. The solution $x(t, E_0)$ exists and is bounded for all $t \geq 0$ and has no positive zero. For $E_0 < E < \infty$, $x(t, E)$ has no positive zero and tends to ∞ as t tends to some finite positive number. Under the additional assumption that p and q are even periodic functions with a common period then, as a consequence of (II), equation (NL) possesses various oscillatory periodic solutions. Solutions satisfying different initial conditions also are discussed. A comparison theorem of Sturmian type is also obtained. This work was supported by the Office of Ordnance Research. (Received July 9, 1956.)

676. R. V. Kadison (p) and I. M. Singer: *Three test problems in operator theory.*

The following three problems were suggested to us in conversation by I. Kaplansky as test problems for the adequacy of a theory of unitary equivalence of operators on a Hilbert space (rephrased, these are the test problems of his *Infinite Abelian groups*): (1) With A and B operators on H and K, does the unitary equivalence of diag (A, A) and diag (B, B) acting in the usual way on $H\oplus H$ and $K\oplus K$ imply the unitary equivalence of A and B? (2) Does the unitary equivalence of diag (A, B) and diag (A, C) imply that of B and C? (3) If A and B are unitarily equivalent to direct summands of each other are they unitarily equivalent? Using the theory of rings of operators, it is shown that the answer to Problems 1 and 3 is "Yes." Problem 2 has a negative answer, in general, but has an affirmative answer when the operators discussed generate rings of finite type with finite commutants. (Received July 5, 1956.)

677. N. D. Kazarinoff (p) and R. W. McKelvey: *A note on the asymptotic behavior of Laguerre polynomials and functions for large n.*

Tricomi [Ann. Mat. Pura Appl. (4) vol. 28 (1949) pp. 263–289] has derived a series of the form $n!e^{-x/2}L_n^\alpha(x)=\sum_{m=0}^\infty A_m(\nu)J_{\alpha+m}[(\nu x)^{1/2}](x/\nu)^{m/2}$, $\nu=4n+2\alpha+2$, which is an asymptotic series in n for $L_n^\alpha(x)$ (x complex and α real) when $|x|<Cn^{1/3}$. Recent results of McKelvey yield a different expansion which asymptotically represents $L_n^\alpha(x)$ (x and α complex) at least when $|x|<4(2-3^{1/2})^2n-\epsilon$, $\epsilon>0$ and small. It is of the form $c_1(n)J_\alpha(\xi)\sum_{j=0}^\infty a_j(x)n^{-j}+c_2(n)J'_\alpha(\xi)\sum_{j=0}^\infty b_j(x)n^{-j}$, where $\xi(x, n)$, $a_j(x)$, and $b_j(x)$ are elementary functions or integrals of elementary functions. The result may also be stated for Laguerre functions. (Received July 3, 1956.)

678t. C. E. Kerr: *On the Tchebycheff approximation of a continuous function of M-variables by a linear combination of continuous functions of M-variables.*

Define $\mathfrak{C}$ to be the family of continuous functions with continuous first derivatives with at most a finite number of oscillations on the closed interval $[\alpha, \beta]$. Let $a_i(t)\in\mathfrak{C}$ $i=1, 2, 3, \cdots, n$ and $b(t)\in\mathfrak{C}$ be $n+1$ linearly independent functions and let $x=(x_1, x_2, \cdots x_n)$ be a vector from the n-dimensional Euclidean vector space, E_n. Define the Tchebycheff solution vector $\bar{x}$ as the $\bar{x}\in E_n$ such that $\sup_t\left|\sum_i a_i(t)\bar{x}_i-b(t)\right|$ is minimized. It is shown that the problem of finding $\bar{x}$ is equivalent to minimizing a convex function $M(x)$ defined on a bounded closed convex set $E\rightarrow E_n$; a constructive proof can be made using the method of descent to show that this function has an absolute minimum given at a unique vector $\bar{x}\in E_n$. It is shown that when an analogous class of functions B is defined for a continuous variable of a higher dimension and the same problem is proposed the method of descent again implies the existence and uniqueness of a solution vector $\bar{x}$. (Received July 9, 1956.)

679t. Erwin Kreyszig: *Coefficient problems of solutions of partial differential equations of the second order* I.

Solutions (1) $u(z, z^*)=\sum_{m,n=0}^\infty u_{mn}z^mz^{*n}$ of (2) $u_{zz^*}+a(z, z^*)u_z+b(z, z^*)u_{z^*}+c(z, z^*)u=0$ (a, b, c entire) can be represented by Bergman operators of the first kind: (3) $u(z, z^*)=p(g)=\exp\left(-\int_0^{z^*}a(z, t)dt\right)g(z)+\sum_{n=1}^\infty q_n(z, z^*)\int_0^z\int_0^{z_1}\cdots\int_0^{z_{n-1}}g(z_n)dz_n\cdots dz_1$ where $q_n(z, 0)=0$, $n=1, 2, \cdots$ and $g(z)$ is an analytic function regular at the origin, cf. S. Bergman, Trans. Amer. Math. Soc. vol. 57 (1945) p. 299 ff. By means of (3) theorems on analytic functions can be formulated as theorems on solutions of (2). In this way various properties of $u(z, z^*)$ can be obtained if the sequence (u_{m0}) is known, cf. (1). In consequence of the properties of the above Bergman operators other coefficients u_{mn} do not occur in theorems of this type. Obviously, information on the behavior of $u(z, z^*)$ should be obtained also from other sequences (u_{mn}), $n>0$ and fixed. Indeed by inserting (4) $u(, z, z^*)=\sum_{n=0}^{\infty} u_n(z)z^{*n}$, $u_n(z)=\sum_{m=0}^{\infty} u_{mn}z^m$, into (2) a system S_n of ordinary differential equation is obtained which yields relations between the sequences (u_{mn}), $n>0$ and fixed, and (u_{m0}). Cf. the following abstract. (Received July 23, 1956.)

680t. Erwin Kreyszig: *Coefficient problems of solutions of partial differential equations of the second order* II.

The ordinary differential equations of the system S_n (cf. the preceding abstract) are linear and of the first order. They involve the functions $u_0(z), u_1(z), \cdots, u_n(z)$ and thus yield relations between (u_{m0}) and (u_{mn}), $n>0$ and fixed. There are two possibilities: (A) The system S_n may be considered in its original form. (B) It may be transformed into a single ordinary linear differential equation which involves only the functions $u_0(z)$ and $u_n(z)$ and whose order does not exceed n. Since (B) leads to relatively complicated conditions, (A) is preferable in many cases of partial differential equations (1) (cf. the preceding abstract) which are of practical importance. At a singular point z_0 of $u_n(z)$ the function $u_0(z)$ is singular while $u_\alpha(z)$, $0<\alpha<n$, may or may not be singular at z_0. Conversely, a singular point of $u_0(z)$ may correspond to a regular point of $u_n(z)$. However conditions can be obtained in order that a singular point of $u_0(z)$ be a singular point of $u_n(z)$. In this way the domain of regularity and other properties of solutions $u(z, z^*)$ of (1) (cf. the preceding abstract) can be determined if a sequence (u_{mn}), $n>0$ and fixed, of the coefficients of the corresponding power series development is known. (Received July 23, 1956.)

681t. Erwin Kreyszig: *On solutions of partial differential equations generated by Bergman operators.*

In a recent paper (Journal of Rational Mechanics and Analysis vol. 4 (1955) pp. 907–923) necessary and sufficient conditions were obtained in order that solutions $u(z, z^*)$ of (1) $u_{zz^*}+B(z, z^*)u_{z^*}+C(z, z^*)u=0$ can be generated by a Bergman operator (cf. Bergman, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155) in the form (2) $u(z, z^*)=\int_{-1}^{1}E(z, z^*, t)f(z(1-t^2)/2)(1-t^2)^{-1/2}dt$ where the "generating function" E is of the form (3) $E(z, z^*, t)=\exp(Q(z, z^*, t))$, $Q=\sum_{\alpha=1}^{m} q_\alpha(z, z^*)t^\alpha$ and the "associated function" $f(z)$ is an analytic function regular at the origin. Theorem: Solutions (2) of (1) corresponding to generating functions (3) and meromorphic associated functions $f(z)=(z-z_0)^{-n}$, $z_0\neq 0$, $n=1, 2, \cdots$ satisfy also a linear ordinary differential equation in $z_1=(z-z^*)/2$ with algebraic coefficients. The order k of this equation is independent of n; there always exists such an equation for which $k\leq m+3$. In particular, if Q is an even function of t then $k\leq m/2+2$. This result and the method of the proof is different from that in the case of regular associated functions $f(z)=z^n$, $n=0, 1, \cdots$ which was investigated previously (Journal of Rational Mechanics and Analysis, in press). Since explicit expressions of the coefficients of the above ordinary differential equation are obtained, relations between properties of the coefficients of (1) and those of the ordinary differential equation can be derived. (Received June 28, 1956.)

682*t*. Erwin Kreyszig: *On singularities of partial differential equations with analytic coefficients.*

The equation (1) $\Delta\phi+\alpha(x_1, x_2)\phi_{x_1}+\beta(x_1, x_2)\phi_{x_2}+\gamma(x_1, x_2)\phi=0$ can be transformed into (2) $u_{zz^+}+B(z, z^+)u_{z^+}+C(z, z^+)u=0$ where $z=x_1+ix_2$, $z^+=x_1-ix_2$ are independent variables if x_1 and x_2 are complex. Let B, C be such that the functions (3) $u(z, z^*)=\int_{-1}^{1}\exp\left(\sum_{\alpha=1}^{m} q_\alpha(z, z^*)t^\alpha\right)(z(1-t^2)/2-\beta)^{-n}(1-t^2)^{-1/2}dt$ are solutions of (2). Then $u(z, z^*)=U(x_1, x_2)$, considered as a function of x_1, satisfies an ordinary linear differential equation (4) $\sum_{\rho=0}^{k} G_\rho(x_1, x_2)(d^\rho U/dx_1^\rho)=0$ of degree k not exceeding $m+3$; in (4), $G_\rho(x_1, x_2)=g_\rho(z, z^*)$. Theorems: I. If B, considered as a function of z for any finite $z^*=z_0^*=$const., has a pole of rth order at a point $z=a$, then g_ρ, considered as a function of z, has at $z=a$ a pole of order $w(m, \rho)r$ where $w(m, \rho)=m-\rho+3$. In particular, if $r=1$ then (4) is of Fuchs' type at $z=a$. II. If $q_1\neq 0$, cf. (3), and C, considered as a function of z^* for any finite $z=$const., has a pole of order $2s-1$, $s>1$, at a point $z^*=a^*$ then g_ρ, considered as a function of z^*, has at $z^*=a^*$ a pole of order $w(m, \rho)s-(\delta_{1\rho}+\delta_{2\rho})(s-1)$ (if $m=1$) or $w(m, \rho)+\sum_{\nu=0}^{[(m+2)/2]}\delta_{2\nu,\rho}(s-1)$ (if $m>1$), where δ_{ik} is the Kronecker symbol. Using these theorems the behavior of the above solutions $u(z, z^*)$ in the neighborhood of the singularities of (2) can be investigated by means of the theory of ordinary differential equations. (Received June 28, 1956.)

683. Stephen Kulik: *A note on the Laguerre method for separating the roots of an algebraic equation.*

Let $f(x)=0$ be an algebraic equation with all its roots real and let D_n be a determinant of order n, the kth line of which is $\left| f_n^{(k)}/(k-1)!\ f^{(k-1)}/(k-1)!\ f^{(k-2)}/(k-2)! \cdots f(x) \cdots 0\right|$. Any two numbers u and v, satisfying the equation $(u-x)(v-x)D_n+(u+v-2x)f(x)D_{n-1}+f^2(x)D_{n-2}=0$ with n even and x an arbitrary real number, separate the roots of $f(x)=0$. If x is closer to a root a than to any other root of the equation and v is fixed, then u converges to a as $n\rightarrow\infty$. If v and x are not separated by any root of $f(x)=0$, then u and x will be separated by the root a starting with some definite value of n. (Received June 11, 1956.)

684*t*. R. A. Kunze: *The Hausdorff-Young theorem for locally compact unimodular groups*. Preliminary report.

Let $\Gamma=(\mathcal{H}, \mathcal{A}, m)$ be a gage space determined by a ring of operators $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a regular gage m on the projections in $\mathcal{A}$. A measurable operator T has for $1\leq p\leq\infty$ a norm $\|T\|_p$, and the collection $L_p(\Gamma)$ of all measurable operators with $\|T\|_p$ finite is a Banach space. A locally compact unimodular group G determines a gage space $\Gamma(G)=(L_2(G), \mathcal{L}, m)$ where $\mathcal{L}$ is the ring generated by left translations on $L_2(G)$ by elements of G. By an extension of the Riesz-Thorin interpolation theorem the following Hausdorff-Young theorem is obtained. For $1\leq p\leq 2$ and $f\in L_p(G)$ the operator L_f of convolution by f in $L_2(G)$ is measurable and $\|L_f\|_{p'}\leq\|f\|_p$ where $1/p+1/p'=1$. For compact groups the above global result can be stated as follows: Suppose G has Haar measure 1 and let $\{\phi_\lambda\}$ be any collection of inequivalent continuous irreducible unitary representations of G where ϕ_λ has degree d_λ. Put $F_\lambda=\int_G f(a)\phi_\lambda(a)da$ and let $U_\lambda|F_\lambda|$ be the canonical polar decomposition of F_λ. Then $(\sum_\lambda(\text{trace }|F_\lambda|^{p'})d_\lambda)^{1/p'}\leq\|f\|_p$. (Received April 26, 1956.)

685*t*. J. G. de Lamadrid: *On a function space.* Preliminary report.

Let Λ^2 be the vector space of all real valued functions f defined on the whole finite real axis, such that the Lebesgue integral $p_a^b(f)_b^2 = \int_a^b f^2(x)dx < +\infty$ exists for every finite interval $[a, b]$. p_a^b is a semi-norm of Λ^2 and the system $\{p_a^b\}$ defines a locally convex linear topology of Λ^2. The purpose of this note is to study the topology and geometry of Λ^2. Let H_i be the Hilbert space of all functions of Λ^2 vanishing outside of $[-i, i]$ $i = 1, 2, 3, \cdots$, with the inner product defined in an obvious way. Clearly, $H_1 \subset H_2 \subset H_3 \cdots$. It is shown that Λ^2 is the projective limit (Dieudonné, Bull. Amer. Math. Soc. vol. 59 (1953) p. 502) of the sequence $\{T_i, H_i\}$, where $T_i f = e_i f$, e_i is the characteristic function of $[-i, i]$ and $(e_i f)(x) = e_i(x) f(x)$. From this and other properties it follows that Λ^2 is an $\mathfrak{F}$-space (metrizable and complete), non-normable, and reflexive. Its dual space is found to be $\bigcup_{i=1}^{\infty} H_i$, the space of functions in Λ^2, each of which vanishes outside some finite interval, and that the dual with its strong topology is the strict inductive limit of the sequence $H_1 \subset H_2 \subset H_3 \subset \cdots$. These results are generalized from Λ^2 to a class of spaces defined in the paper and termed monotone projective limits of Hilbert spaces. (Received July 6, 1956.)

686*t*. P. D. Lax: *An abstract stability theorem*, II.

Let H be a Hilbert space and D an unbounded operator with the following property: there exists an infinity of lines parallel to the imaginary axis with arbitrarily large negative real parts on which the resolvent of D is uniformly bounded by some constant d^{-1}. For D selfadjoint this is equivalent to the presence of infinitely many gaps of length $2d$ in the spectrum of D on the negative axis. Let $u(t)$ be a solution of $u_t = (D + K(t))u$, $\|K(t)\| \leq k < d$. Conclusion: unless identically zero, $u(t)$ tends to zero no faster than some exponential. The restriction on the constant k is the mildest possible. This result is used in studying the local behavior of solutions of elliptic equations. (Received June 4, 1956.)

687*t*. Karel deLeeuw: *Almost periodic functions on the half-line.*

Let B be the space of all bounded continuous functions on the half-line $\{x : x \text{ real and } x \geq 0\}$. For each f in B and each real $x \geq 0$, define ${}_xf$ in B by ${}_xf(y) = f(y + x)$. Let A be the subspace of B that consists of all those f with the set of all ${}_xf$ totally bounded (that is, having compact closure) in B supplied with the topology induced by the norm $\|h\| = \text{Sup } |h(x)|$. Let A_p be the subspace of A that consists of all functions that are the restriction to the half-line of continuous almost periodic functions on the entire line. Let A_∞ be the subspace of A that consists of all functions in B that are zero at infinity. *Theorem*: A is algebraically and topologically the direct sum of A_p and A_∞. A similar decomposition is obtained for a large class of sub-semigroups of abelian groups, the first factor being as above restrictions of almost periodic functions on the entire group, but the second factor is not easily identified. (Received June 25, 1956.)

688*t*. J. J. Levin: *The asymptotic behavior of the stable initial manifolds of a system of nonlinear differential equations.*

It is shown that under suitable hypothesis the stable initial manifolds associated with the system of differential equations (1) $dx/dt = f(t, x, y, \epsilon)$, $\epsilon dy/dt = g(t, x, y, \epsilon)$ approach, as $\epsilon \to 0+$, the stable initial manifold associated with the system of differential equations (2) $dy/d\gamma = g(0, x(0), y, 0)$. The variables x and y are vectors of m and n components respectively; (2) is the boundary layer equation for (1). It is also shown that the solutions of (1) are very well approximated for small t and ϵ by corresponding solutions of (2). (Received July 3, 1956.)

689t. W. S. Loud: *Some growth theorems for linear ordinary differential equations.*

Liapounoff (Problème général de la stabilité du mouvement, Princeton, 1949) has shown that if $A(t)$ is a matrix of real, piecewise continuous, bounded functions for all $t \geqq 0$, there exists a finite real number λ_0 such that if $\lambda > \lambda_0$, and $x(t)$ is any nontrivial vector solution of $x' = A(t)x$, then $x(t)e^{-\lambda t} \to 0$ as $t \to \infty$. Here it is shown that if $A(t) = \{a_{ij}(t)\}$ $i, j = 1, 2, \cdots, m$, and if $\sum\sum |a_{ij}(t)| \leqq A^n$ $(A > 0)$ for all $t \geqq 0$, then $\lambda_0 \leqq A$ for the system $x^{(n)} = A(t)x$. If in $x'' + a(t)x = 0$, $a(t)$ is real and piecewise continuous, and $\alpha \leqq a(t) \leqq \beta$ for all $t \geqq 0$, then $\lambda_0 \leqq (-\alpha)^{1/2}$ if $3\alpha + \beta \leqq 0$, and $\lambda_0 \leqq 2^{-1/2}(\beta - \alpha)(\beta + \alpha)^{-1/2}$ if $3\alpha + \beta \geqq 0$. If in $x'' + cx' + a(t)x = 0$, $a(t)$ is as above and $\alpha > 0$, then for any solution, $x(t)$ and $x'(t)$ approach zero exponentially if $c > \beta^{1/2} - \alpha^{1/2}$. This work was supported in part by the Office of Ordnance Research. (Received June 15, 1956.)

690t. W. S. Loud: *Behavior of certain forced nonlinear systems with large forcing.*

In the equation (*) $x'' + [c + f(x)]x' + kx + g(x) = Ae(t)$, let the following hypotheses be satisfied. (1) $e(t)$ has period L; (2) c and k are constants such that $x'' + cx' + kx = 0$ has no solution of period L; (3) $g(x)$ is bounded and satisfies a Lipschitz condition for which the constant approaches zero for large $|x|$; (4) $f(x)$ and $F(x) = \int_0^x f(u)du$ are bounded, and $f(x) \to 0$ as $|x| \to \infty$; (5) the unique solution $\phi_0(t)$ of period L of $x'' + cx' + kx = e(t)$ is not zero in any interval. Then if A is large enough, (*) has a unique solution $x_0(t)$ of period L such that $|A^{-1}x_0(t) - \phi_0(t)| = O(A^{-1})$. A similar result holds when $k = 0$ under certain additional hypotheses. This generalizes a result of the author for systems with limiting (Bull. Amer. Math. Soc. Abstract 62-1-102). If a proper subharmonic of order n for (*) is a solution having nL but no smaller multiple of L as a period, the result shows that for any particular n there are no proper subharmonics of order n for large enough A. The proofs are based on equivalent integral equations and use the near linearity of (*) for large $|x|$. This work was supported in part by the Office of Ordnance Research. (Received June 25, 1956.)

691. E. B. McLeod, Jr.: *The solution of an extremal problem in the complex domain by the variation of Schiffer.*

It is a well-known classical result that within a family of domains, each of which has a fixed outer mapping radius and is bounded by a closed rectifiable curve, the one which has a boundary of minimum length is a circle. A generalization of this problem is to impose the constraint that the complement of each of these competing contains a fixed linear segment. Once the existence of an extremal domain has been established, a method of comparison devised by M. Schiffer may be used. This device of Schiffer has the advantage that only interior points of the domain are involved and no hypothesis which concerns the smoothness of the boundary is necessary. The analyticity of the boundary is deducible by this method. The use of the Schiffer Variation leads us to a quadrature representation of the analytic function which maps the exterior of the unit circle onto the extremal domain. (Received May 21, 1956.)

692t. T. S. Motzkin and J. L. Walsh: *Infrapolynomials.*

Let the compact set E contain if finite $m > n > 0$ points. A polynomial $f(z) = z^n + \cdots$ is an *infrapolynomial* on E if no $g(z) = z^n + \cdots$ exists with $|g(z)| < |f(z)|$ on E where $f(z) \neq 0$, $g(z) = f(z)$ on E where $f(z) = 0$; an equivalent condition is that no $g(z) \neq f(z)$ exists with $|g(z)| \leqq |f(z)|$ on E. The class F_n of such $f(z)$ is closed and connected, but convex for $n=1$ or $m-1$ only. Any $f(z)$ in F_n is the limit of infrapolynomials on finite subsets of E. A polynomial $f(z)$ is in F_n if and only if $f(z)=g(z)h(z)$, where $g(z)$ has only, and the infrapolynomial $h(z)$ on E has no, zeros of the third kind (i.e., at limit points of E); also, if and only if $f(z)=g(z)h(z)$ where $g(z)$ (necessarily in some F_ν) has only zeros of the third kind and $h(z)$ is a factor of some infrapolynomial $\sum_1^{r+1} \lambda_i\omega(z)/(z-z_i)$ on $\{z_1, \cdots, z_{r+1}\}$, with $\omega(z)=\prod_1^{r+1} (z-z_i)$, $\lambda_i \geq 0$, $\sum \lambda_i=1$, z_i on E. If E is real, a necessary and sufficient condition that $f(z)$ be in F_n is that the μ zeros not of third kind of $f(z)$ separate $\mu+1$ distinct points of E. (Received July 6, 1956.)

693t. J. C. C. Nitsche: *On the solutions of the differential equation* $rt-s^2=1$.

Author presents an elementary proof of the following theorem (c.f. K. Joergens, Math. Ann. vol. 127 (1954): Let the function $z(x, y)$ be of class C^2 and satisfy $rt-s^2=1$, $r>0$ for all values of x and y. Then $z(x, y)$ is a quadratic polynomial. This proof is based on the facts: (i) The transformation $\xi=x+p(x, y)$, $\eta=y+q(x, y)$ is a one-to-one mapping of the x, y-plane onto the $\xi\eta$-plane (cf. H. Lewy, Trans. Amer. Math. Soc. vol. 41 (1937)). (ii) The function $f(\sigma)=(x-p)-i(y-q)$ is a regular analytic function of the variable $\sigma=\xi+i\eta$. (iii) $f'(\sigma)=[(t-r)+2is]\cdot(2+r+t)^{-1}$ and $|f'(\sigma)|<1$. Thus $f'(\sigma)$ must be a constant and hence $r=$const., $s=$const., $t=$const. An immediate consequence of the proved theorem (cf. a remark of E. Heinz in the quoted paper by Joergens) is the famous theorem by S. Bernstein: A minimal surface which can be represented in the form $z=z(x, y)$ where $z(x, y)$ is of class C^2, for all x, y must be a plane. Applying Schwarz's lemma to the function $f(\sigma)$ one can derive bounds in terms of the second order derivatives for the third and higher order derivatives of a solution $z(x, y)$ of $rt-s^2=1$. Also further results concerning the behaviour of $z(x, y)$ can be obtained. (Received June 18, 1956)

694t. J. C. C. Nitsche: *An analytic function connected with the minimal surface equation and Bernstein's theorem.*

Let the function $z=z(x, y)$ be twice continuously differentiable and satisfy the minimal surface equation in a domain G of the x, y-plane. Consider the transformation $\xi=x+\int_{(x_0,y_0)}^{(x,y)} W^{-1}[(1+p^2)dx+pqdy]$, $\eta=y+\int_{(x_0,y_0)}^{(x,y)} W^{-1}[pqdx+(1+q^2)dy]$. Here $W=(1+p^2+q^2)^{1/2}$. Because of $(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2 \geq (x_2-x_1)^2+(y_2-y_1)^2$ this is a one-to-one mapping of G onto a domain Γ in the ξ,η-plane. If $z(x, y)$ is defined in the whole x, y-plane then Γ coincides with the whole ξ,η-plane. Now observe that the function $F(\sigma)=(q+ip)(1+W)^{-1}$ is a regular analytic function of the variable $\sigma=\xi+i\eta$, defined in Γ. One has the inequality $|F(\sigma)|<1$. Thus, if $z(x, y)$ is an "entire" solution, that is if G coincides with the whole x, y-plane then $F(\sigma)$ must be a constant. Hence $p=$const. and $q=$const. and hence $z=a+bx+cy$. This proves Bernstein's theorem on minimal surfaces. Using Schwarz's lemma, the fact that $|F(\sigma)|<1$ in Γ, and the "shrinking property" of the mapping $\xi,\eta\rightarrow x$, y one can infer bounds in terms of the first order derivatives for the second and higher order derivatives of $z(x, y)$ and further results. (Received June 25, 1956.)

695t. J. C. C. Nitsche: *On the solutions of the differential equation* $\Delta\phi=\exp \phi$.

Suppose $\phi(z, \bar{z})$ is a real solution of $(^*)\Delta\phi=\exp (2\phi)$, twice continuously differentiable in a domain G. Then the function $w(z)=\phi_{zz}-\phi_z^2$ is a regular analytic function of z

defined in G (see L. Bieberbach, Math. Ann. vol. 77). Conversely, given a regular analytic function $w(z)$ and the initial conditions $\phi=\log 2\alpha$, $\phi_z=\beta+i\gamma$ in $P_0[z=z_0, \bar{z}=\bar{z}_0]$, the corresponding solution ϕ of (*) can be determined in the following way: Let $\psi^1(z)$ and $\psi^2(z)$ be a fundamental system of $(^{*}_{*})\ \psi_{zz}+w\psi=0$ satisfying $\psi^1=1$, $\psi^1_z=-(\beta+i\gamma)$; $\psi^2=0$, $\psi^2_z=1$ in P_0. Then $\phi=\log 2\alpha-\log Q$ where $Q=|\psi^1(z)|^2-\alpha^2|\psi^2(z)|^2=|\psi^1(z)|^2\cdot\{1-\alpha^2|\int_{z_0}^{z}[\psi^1(u)]^{-2}du|^2\}$. From this representation a proof for the nonexistence of "entire" solutions of (*) follows (for another proof see Wittich, Math. Zeit. vol. 49): Otherwise $w(z)$ and $\psi^k(z)$ would also be entire functions of z. Now $Q=1$ in P_0 and Q also must assume negative values. Because if everywhere the inequalities $|\psi^1|^2>0$ and $Q>0$ would hold then the integral in the brackets would be bounded which is a contradiction due to Liouville's theorem. If the point $P_1[z=z_1, \bar{z}=\bar{z}_1]$ is an isolated singularity of ϕ, and if ϕ_z is single valued in a neighborhood of P_1, and if $(^{*}_{*})$ has a regular singularity there then this singularity of ϕ is of either one of the two types $c\cdot\log|z-z_1|$ or $c\cdot\log|\log|z-z_1||$. (Received July 3, 1956.)

696t. J. C. C. Nitsche: *On harmonic mappings and a lemma of Radó-Bers.*

Let $x=\text{Re } F(\zeta)$, $y=\text{Re } G(\zeta)$, $\zeta=\xi+i\eta$ be a one-to-one harmonic mapping of the unit disc $\Gamma[|\zeta|<1]$ onto a domain C of the x, y-plane. T. Radó (Math. Zeit. vol. 26) and L. Bers (Annals of Math. (2) vol. 53) have proved that C cannot coincide with the whole x, y-plane. This can easily be shown using the theorem that an entire solution of the differential equation $rt-s^2=1$ must be a quadratic polynomial (for a proof which makes no use of the properties of harmonic mappings see a preceding abstract of the author). According to a theorem of H. Lewy (Bull. Amer. Math. Soc. vol. 42) $\text{Im }(F'\overline{G'})=x_\xi y_\eta-x_\eta y_\xi\neq 0$ in Γ. Form the expressions (see K. Joergens, Math. Ann. vol. 129, Lemma 3) $r=|G'|^2\cdot[\text{Im } F'\overline{G'}]^{-1}$, $s=-[\text{Re } F'\overline{G'}]\cdot[\text{Im } F'\overline{G'}]^{-1}$, $t=|F'|^2\cdot[\text{Im } F'\overline{G'}]^{-1}$ and consider r, s, t as functions of x and y defined in C. Because of $r_y-s_x=s_y-t_x=0$ there exists a function $z(x, y)$ such that $r=z_{xx}$, $s=z_{xy}$, $t=z_{yy}$ and $z_{xx}z_{yy}-z_{xy}^2=1$ in C. Now assume that C is the whole x, y-plane. Then r, s, t must be constants. Thus the equation $F(\zeta)=(\alpha+i\beta)G(\zeta)+(\gamma+i\delta)$ with real constants α, $\beta\neq 0$, γ, δ would hold true. After the linear transformation $u=y$, $v=(\gamma-x)\beta+y\alpha/\beta$ one then gets $w=u+iv=G(\zeta)$. This is a contradiction. (Received July 16, 1956.)

697t. Robert Osserman: *On the solution of a functional equation.*

In answer to a problem posed by S. Chowla on the existence of an entire solution of $f(f(z))=e^z-1$, the following theorem is proved. Let S be a strip $-b<\text{Im }\{z\}<b$ for some $b>\pi$. Let $f(z)$ be a function defined in a domain D such that $S\subset D$ and $f(S)\subset D$. If $f(z)$ satisfies the equation $f(f(z))=e^z-1$ throughout D then $f(z)$ cannot be analytic throughout S. Both Thron (Canadian Journal of Mathematics vol. 8 (1956) pp. 47–48) and Baker (Math. Ann. vol. 129 (1955) pp. 174–180) have proved that there exist no entire solutions of $f(f(z))=F(z)$ for a wide class of functions $F(z)$ including $F(z)=e^z-1$ and $F(z)=e^z$. The latter case was investigated by Kneser (J. Reine Angew. Math. vol. 187 (1950) pp. 56–67) who showed that there exists a solution analytic on the whole real axis. Our proof also applies to show that Kneser's solution cannot be extended analytically throughout S. (Received July 23, 1956.)

698. O. O. Pardee: *On representations for certain variational problems.* Preliminary report.

On sets of functions closed in L_2 whose derivatives or integrals are also closed and which satisfy certain auxiliary conditions suitable for use in an explicit representation of J. Indritz (Pac. J. Math. vol. 5 (1955) pp. 765–797) for the function u^* (assumed) which minimizes the functional $I(u)=\iint_R(au_x^2+bu_y^2+cu^2)dxdy$. Here R is a plane bounded region with a smooth boundary curve C; the functions a, b, c are bounded, integrable on $R+C$ and $a>0$, $b>0$, $c\geqq 0$. The minimal function is represented by $u^*=g(x, y)+\sum f_k(x, y)$ where $g(x, y)$ is a continuously differentiable function defined on $R+C$ attaining prescribed boundary values on C, and the functions $f_k(x, y)$ are defined as follows: Let $G(x, y)$ be a function continuously differentiable on $R+C$, positive in R, vanishing on C, with positive normal derivative on C, and let $P_k(x, y)$ be a dense set of polynomials defined over a region containing $R+C$. Then $f_k(x, y)$ are the functions obtained by orthonormalizing the set (GP_k) where the square of the norm is $I(u)$. The present paper determines more general sets of functions than polynomials for the functions P_k and removes some additional smoothness conditions required on the boundary C and the functions g and G. (Received July 2, 1956.)

699. R. N. Pederson: *A new inequality for the unique continuation theorem*. Preliminary report.

Let $\hat{A}=a^{ij}(x)\partial^2/\partial x_i\partial x_j+b^i(x)\partial/\partial x_i+c(x)$ be an elliptic operator defined in a neighborhood of the origin. Let A be the Laplace-Beltrami operator corresponding to the metric $ds^2=a_{ij}dx_idx_j$, where (a_{ij}) is the inverse matrix of (a^{ij}). Let r be the geodesic distance from the origin. We prove that there exist constants R_0, α_0 such that $\int\exp(\alpha/r)(Au)^2dV\geqq\alpha\int\exp(\alpha/r)(\Delta u\cdot\Delta u+u^2)dV$ for $R\leqq R_0$, $\alpha\geqq\alpha_0$. Here dV denotes the volume element induced by the metric, the integration is performed over the geodesic sphere $r\leqq R$, and $u\in C^2$ is assumed to vanish both near zero and near R. The proof follows easily after substituting $u=r^p\exp(-\alpha/r)v$ with $p=(6-n)/2$. The unique continuation theorem for the operator $\hat{A}$ can be shown to be a consequence of the above inequality by an argument similar to the one employed by Heinz (Nachr. Ges. Wiss. Gottingen (1955) pp. 1–12) and Aronszajn (Comptes rendus, vol. 242 (1956) pp. 723–725). (Received June 12, 1956.)

700. F. W. Perkins: *On the Almansi-Nicolesco theorem*. Preliminary report.

The Almansi-Nicolesco Theorem asserts that if u_p is p-harmonic (that is, if $\Delta^p u_p=0$) in a domain D and if M is a fixed point of D then (in a certain subdomain of D) $u_p=V_0+V_1r^2+V_2r^4+\cdots+V_{p-1}r^{2(p-1)}$ where r is the distance from M and $V_0, V_1, \cdots, V_{p-1}$ are harmonic functions uniquely determined by u_p and M. (M. Nicolesco, Ann. École Norm. vol. 52 (1935) pp. 183–220; see particularly pp. 182–192.) The present paper contains a discussion of a number of topics connected with this theorem, including an alternative method of determining the functions V_0, $V_1, \cdots, V_{p-1}$, once their existence is known. (Received July 3, 1956.)

701. R. S. Phillips: *Dissipative hyperbolic systems.*

This paper is concerned with the Cauchy problem for hyperbolic systems in only one spatial variable of the form $Ey_t=(Ay)_x+By$, $a<x<b$, $t>0$; here y is a vector-valued function of (x, t) and E, A, B are matrix-valued functions of x alone, E being hermetian positive definite, A hermetian of constant rank, and B satisfying the condition $B+B^*+A_x\leqq\Theta$ for all $x\in(a, b)$. It is assumed that E and A are absolutely continuous on each compact subinterval of (a, b), and that the elements of E_x, A_x, and B are square integrable on each compact subinterval of (a, b). By applying methods from the theory of semi-groups all possible semi-group solutions are found for which $(Ay, y)^b-(Ay, y)^a \leq 0$ for all functions in the domain of the generator: here the base space is the hilbert space with norm $[\int_a^b(Ey, y)dx]^{1/2}$. In order to include all of the "dissipative" boundary conditions commonly associated with hyperbolic systems, the above differential system has been coupled at the end points to finite degree of freedom dissipative systems and all possible semi-group solutions found, again subject to the condition that no energy enters through the boundary. (Received July 13, 1956.)

702. R. T. Prosser: *C^* algebras with trace.*

This paper presents a systematic study of those C^* algebras which admit a densely-defined faithful effective trace. Such algebras admit an abstract integration theory which is entirely analogous to that of the commutative case. Moreover, every such algebra is strongly semi-simple, and there is a natural one-one correspondence between the set of maximal two-sided ideals, and (modulo a scalar factor) the set of extreme traces. These extreme traces may be densely embedded in a certain locally compact Hausdorff space over which the algebra may be decomposed into factors which are almost all simple. An analysis of this decomposition shows that there are essentially two types of these algebras: The CCR algebras of type I and the FR algebras of type II. (Received July 6, 1956.)

703*t*. M. H. Protter: *Uniqueness theorem for the Tricomi problem.*

Let $K(y)$ be a monotone increasing twice continuously differentiable function with $K(o)=0$. Let D be a domain bounded by an arc Γ_1 lying in $y>o$ with end points on the x-axis at A and B and by the characteristics Γ_2 and Γ_3 issuing from A and B which intersect. *Theorem: Any solution of $K(y)U_{xx}+U_{yy}=0$ which vanishes on Γ_1 and on one of the characteristics vanishes throughout D provided that $K'(y)>0$ for $y<0$.* The condition $K'(y)>0$ may be weakened by assuming that $(3K'^2-2KK'')/K'^2 \neq -\infty$ for $y<0$. This contains as special cases the previous results of Frankl, Ou, Ding, and the author (Received May 4, 1956.)

704. Edgar Reich: *On schlicht functions with real coefficients.* II.

Let S_r be the class of functions $f(z)=z+a_2z^2+\cdots$ (a_k real) which are regular and schlicht for $|z|<1$ (Cf. Bull. Amer. Math. Soc. Abstract 61-5-613). By transforming the Basilewitsch-Tammi equation qualitative results regarding $m(z_0)=\inf f(z_0), f\in S_r$, in the "unknown" region $|R(z_0+1/z_0)|\leq 2$ are obtained; e.g., $m(z)$ has an absolute maximum at an interior point of $|z|<1$. Let $s(\alpha)$ be the class of functions $w=h(z)=a_1z+a_2z^2+\cdots$ (a_k real), $0<a_1\leq\alpha$, $|h(z)|<1$, regular and schlicht for $|z|<1$. The variability region of $h(z)$ is the set of values w with $|w|<1$, such that $w/(1+w^2)=\beta z/(1+2\gamma z+z^2)$, as (β, γ) range over all real pairs with $0<\beta\leq\alpha$, $|\gamma|\leq 1-\beta$. (Received June 27, 1956.)

705*t*. P. V. Reichelderfer: *A covering theorem for transforms.*

Given a continuous transformation T from a bounded domain in Euclidean n-space R^n into a bounded portion of R^n Assume that T is essentially of bounded variation. Let $\mathfrak{D}$ be a family of domains D for each of which TD is a domain whose boundary has Lebesgue measure zero, D is a component of $T^{-1}TD$, and there exists a point x in R^n such that the topological index of x with respect to T and D is defined and not zero. Consider a nonempty set U for each point u of which there exists a sequence of domains D_j in $\mathfrak{D}$ containing u and such that their images TD_j have parameters of regularity uniformly bounded away from zero and diameters tending to zero as j becomes large. Then there exists a subset U_0 of U such that the Lebesgue measure of TU_0 is zero and the set of points in U and not in U_0 is covered by a countable sequence of pairwise disjoint domains in $\mathfrak{D}$. (Received March 12, 1956.)

706t. Walter Rudin: *Continuous functions on compact spaces without perfect subsets.*

Let Q be a compact Hausdorff space which has no nonvacuous perfect subset (examples show that Q may be uncountable) and let $C(Q)$ be the Banach algebra whose elements are the continuous complex-valued functions on Q. It is proved that every member of $C(Q)$ has an at most countable set of values; it follows that every closed subalgebra of $C(Q)$ is self-adjoint. Consequently, every subalgebra of $C(Q)$ which separates points on Q is dense in $C(Q)$. This provides a marked contrast between the structure of $C(Q)$ and that of $C(X)$, where X is a compact Hausdorff space which has a subset homeomorphic to the Cantor set (see Bull. Amer. Math. Soc. Abstract 61-4-486.) It is also proved that every regular Borel measure on Q is concentrated on an at most countable set of points. (Received June 19, 1956.)

707t. Walter Rudin: *Subalgebras of group algebras.*

The group algebra $L(G)$ of the locally compact abelian group G has as its elements the complex-valued Haar-integrable functions on G; multiplication in $L(G)$ is convolution. A subset S of $L(G)$ is said to be separating if the Fourier transforms of the members of S separate points on the dual group. It is shown that the maximal ideals of $L(G)$ are the only separating proper closed subalgebras of $L(G)$ if G is compact. On the other hand, if G is the real line, then there is a separating maximal closed subalgebra A of $L(G)$ (not an ideal) and a bounded totally disconnected perfect set P of real numbers, with the following property: if $f \in A$ and if its Fourier transform F is not constant on P, then the set consisting of the numbers $F(y)$ $(y \in P)$ has dimension at least 1. This seems to be the first example of a maximal subalgebra of the group algebra of the real line which is not a maximal ideal and which is separating. A similar example can be constructed for the case in which G is the additive group of all integers. (Received June 19, 1956.)

708t. P. P. Saworotnow: *Diagonalisation of a complemented algebra.*

The following is a consequence of the spectral theorem and the result recently obtained by the author [Bull. Amer. Math. Soc. Abstract 61-5-614t]. Every simple semi-simple complemented algebra can be realized as an algebra of all functions $a(s, t) \in L^2(S \times S)$ such that $\int K(s)|a(s, t)|^2 ds dt < \infty$, where S is some measure space and $K(s)$ some positive, measurable, function such that $K(s) \geq 1$ for all $s \in S$. (Received March 29, 1956.)

709t. H. M. Schaerf: *On the existence of linear functionals invariant under operators.* Preliminary report.

Let L be a real linear space and let G be a set of mappings of L into L such that $g \in G$, $l \in L$ implies $g(\alpha l) = \alpha g l$ for every scalar α. Let J be an arbitrary subadditive positively homogeneous functional defined on L. Then (L, G, J) is called *perfectible* if there exists an additive homogeneous function I which is defined on L, invariant under G and such that $-J(-l) \leq I(l) \leq J(l)$ for every $l \in L$. *Theorem* 1. Let (L, G)

be the set of all $\sum_{i=1}^{n} (g_i h_i - h_i)$ with $g_i \in G$, $h_i \in L$. In order that (L, G, J) be perfectible it is necessary and sufficient that we have $J(l) \geqq 0$ for every l in (L, G). *Theorem* 2. If G is an abelian semi-group such that we have $g(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 g h_1 + \alpha_2 g h_2$ for all $g \in G$, $h_i \in L$, α_i real $(i = 1, 2)$ and if, for every $l \in L$, the set $E(l)$ of all $J(gl)$ with $g \in G$ is bounded, then (L, G, J^*) is perfectible where $J^*(6)$ is the least upper bound of $E(l)$. Theorems 1 and 2 generalize results of E. Folner and the author, respectively. Some applications of Theorem 2 are stated in another abstract. (Received July 5, 1956.)

710*t*. H. M. Schaerf: *On the existence of invariant measures*. Preliminary report.

Let (X, S, m) be a totally finite measure space and let G be an abelian semi-group of measurability preserving mappings of X into X. Then there exists a finitely additive measure μ on S which is invariant under G and such that, for every $E \in S$, $\mu(E)$ is between the bounds of the set of all numbers $m(g^{-1}E)$ with $g \in G$. Therefore, μ is countably additive under each of the following conditions: (a) For every decreasing sequence $\{E_n\} \subset S$ whose intersection is empty, we have $\lim_{n \to \infty} [\sup_{g \in G} m(g^{-1}E_n)] = 0$; (b) $m(E) = 0$ implies $m(g^{-1}E) = 0$ for all $g \in G$ and $m(E) > 0$ implies that the set of all numbers $m(g^{-1}E)$ with $g \in G$ has a positive lower bound. In case (b), μ has the same null sets as m. These results follow from Theorem 2 of the preceding abstract. (Received July 5, 1956.)

711*t*. I. E. Segal: *Ergodic subgroups of the orthogonal group on a real Hilbert space.*

A subgroup of the orthogonal group on a real Hilbert space is ergodic with respect to the isotropic normal (resp. Clifford) distribution, in the sense that there exist no invariant square-integrable functionals (resp. operators), if it is any of the following: (a) non-compact, simple, and finite-dimensional; (b) irreducible; (c) n-parameter abelian with pure continuous spectrum. Similar results hold also for the direct product of the normal and Clifford distributions, showing the nonexistence of a true S-operator for certain quantum fields. (Received July 9, 1956.)

712*t*. V. L. Shapiro: *Localization on spheres.*

Let Ω be the unit $(k-1)$-dimensional sphere in Euclidean k-space, $k \geqq 3$, and let $S = \sum_{n=1}^{\infty} Y_n(x)$ be a series of surface spherical harmonics defined on Ω with $Y_n(x) = O(n^{\beta})$ uniformly for x on Ω, $\beta \geqq 0$. To S can be associated a Riemann function, $F(x) = \sum_{n=1}^{\infty} Y_n(x) [n(n+p)]^{-w}$, where w is the smallest integer greater than $(\beta+1)/2$ and $p = k-2$. It is then shown in this paper that if $F(x)$ is of class C^{2w} on a domain D of Ω, S is uniformly (C, α) summable, $\alpha > p + 2w$, to $\Delta^w F(x)$ in any domain D_1 whose closure is contained in the interior of D, where $\Delta^w = \Delta(\Delta^{w-1})$ and Δ is the Laplace operator on Ω. This result extends to spheres the localization theorems obtained by Riemann and Zygmund for the circle. (Received July 3, 1956.)

713. L. S. Shapley: *Deviation of nonadditive set functions.*

The departure from additivity of real-valued functions on a Boolean ring of sets B is considered. A *tree partition* of a set $S \in B$ is a collection $\{S_\alpha | \alpha \in A\}$, A a tree, with $S_0 = S$ and $\{S_{\beta i}\}$ partitioning S_β, where $0 = \text{glb } A$ and the βi are the immediate successors of β in A. The *deviation* of f on S is defined $\Delta(f, S) = \sup \sum_\beta |f(S_\beta) - \sum_i f(S_{\beta i})|$, where the supremum is taken over all finite tree partitions of S, and β is restricted to be nonterminal in A. Every function of finite deviation can be decomposed into superadditive, subadditive, and additive parts: $f = f^{+} + f^{-} + f^{0}$. The first two are defined in terms of "upper" and "lower" deviation, in analogy with the upper and lower variation of a real function of bounded variation. The third may be defined $f^{0}(S) = \lim \sum_i f(S_i)$, where the limit is taken over the directed set of finite partitions $\{S_i\}$ of S, ordered by refinement. If the functions f for which f^{0} is well-defined and everywhere finite are called *integrable*, then the functions of finite deviation may be characterized as the subspace of the linear space of integrable functions spanned by the superadditive functions. (Received July 6, 1956.)

714t. I. M. Sheffer: *Some theorems on asymptotic series.*

Let $J(\alpha, \beta)$ be the sector $\alpha < \arg z < \beta$, $|z| > 0$. A class $\mathfrak{F}$ of analytic functions is called an asymptotic family for the sector $J(\alpha, \beta)$ if every series $\sum A_n/z^n$ is the asymptotic series of at least one $F(z) \in \mathfrak{F}$. Carleman showed that the class $\mathfrak{F}$ of functions analytic in $\alpha < \arg z < \beta$, $|z| > R$ is an asymptotic family. It is here shown that for $J(0, 2\pi)$ an asymptotic family is the class of all functions single-valued throughout the complex plane, whose only finite singularities consist of a denumerable number of simple poles on the positive real axis with no finite limit point, together with a possible singularity at $z = 0$. A corresponding result holds for the sector $J(0, 2k\pi)$, taken on the Riemann surface for w defined by $w^k = z$. (Received July 6, 1956.)

715t. Walter Strodt: *Approximate factorization of the homogeneous linear ordinary differential operators.*

Let $A = \sum_{j=0}^{n} a_j(x)D^j$, with the $a_j(x)$ in a logarithmic domain E^* and $a_n(x)$ nontrivial, (cf. author's Memoirs of the Amer. Math. Soc. no. 13). $(V, w_1, w_2, \cdots, w_n)$ is called a principal factorization sequence for A if (1) V and w_j are logarithmic monomials, (2) $A(V) \sim 1$, (3) $A = F_n F_{n-1} \cdots F_1 V + \sum_{j=0}^{n} \delta_j(x) F_j F_{j-1} \cdots F_1 V$, with $\delta_j \prec 1$, and $F_j = 1 - (1/w_j(x))D$, and (4) $(V, w_1, \cdots, w_n)$ is minimal for (1) (2) (3) under lexicographic $\prec$. Every A has a principal factorization sequence and all sequences are obtainable by algebraic methods. V is the principal monomial of $A(y) - 1$. Let E^* and V have logarithmic ranks $\leq p$. With the transform of $A(y)$ under $y = Vz$ written $\sum_{j=0}^{n} c_j(x)[x \log x \operatorname{loglog} x \cdots \log_p xD]^j z$, $w_1, \cdots, w_n$ are logarithmic monomials, (determined by a Newton polygon), to which the zeros $r_1(x), \cdots, r_n(x)$ of $\sum_{j=0}^{n} c_j(x)[x \log x \operatorname{loglog} x \cdots \log_p x]^j r^j$ are asymptotically equivalent, numbered so that w_{j+1} is never $\prec w_j$. Every w_j is in the divergence class. The proof applies the principal monomial algorithm to the generalized Riccati equation for A, i.e. to the transform of $y^{-1}A(y)$ under $h = y^{-1}Dy$. If all w_j have nonexceptional indices, (3) leads to a determination of the principal solutions of $A(y) = g$, whenever $g \sim a$ logarithmic monomial. (Received June 11, 1956.)

716t. Walter Strodt: *Uniformly quasi-linear algebraic ordinary differential equations.*

Let $w_1(x), \cdots, w_n(x)$ be logarithmic monomials in the divergence class, with indices $(C_1, k_1, t_1), \cdots, (C_n, k_n, t_n)$. Let $F_j = 1 - (1/w_j)D$, $G_j = F_j F_{j-1} \cdots F_1$, $\delta_j(x) \prec 1$ in $\bar{F}(a, b)$, $(j = 0, 1, \cdots, n)$. Let $Q(x, Y_0, \cdots, Y_n)$ be a polynomial in the Y_i, each term being of degree ≥ 2, and each coefficient being $\prec 1$ or ≈ 1. Let $f \prec 1$. If $P(x, y, Dy, \cdots, D^n y) = f(x) + G_n y + \sum_{j=0}^{n} \delta_j(x) G_j y + Q(x, G_0 y, G_1 y, \cdots, G_n y)$, P is called uniformly quasi-linear. If every index is nonexceptional, a complete description is obtained of those solutions of $P = 0$ which are $\prec 1$. Let $s_j = t_j$ if $k_j = 0$, $s_j = 0$ if $k_j > 0$. Let

$h_j(\theta)=\cos(s_j\theta+\arg C_j)$. For every u in (a, b) there exists $\delta>0$ such that $P=0$ has a nonempty family S^* of solutions $\prec 1$ in $\overline{F}(u-\delta, u+\delta)$. The number of arbitrary constants in S^* equals the number of j having $h_j(u)<0$. Every element of S^* has an extension which is $\prec 1$ in $\overline{F}(u_1, u_2)$, where $u_1=\max\{a, \theta: \theta<u; h_j(\theta)=0$ and $h_j'(\theta)<0$ for at least one $j\}$, and $u_2=\min\{b, \theta: \theta>u; h_j(\theta)=0$ and $h_j'(\theta)>0$ for at least one $j\}$. The proof uses successive approximations $F_{j+1}Y_{j,k+1}=Y_{j+1,k}$, $(j=0, 1, \cdots, n-1)$, $Y_{n,k+1}=-f(x)-\sum_{j=0}^{n} \delta_j(x)Y_{jk}-Q(x, Y_{0k}, \cdots, Y_{nk})$, $(k=0, 1, \cdots)$. (Received June 11, 1956.)

717t. Walter Strodt: *Asymptotically quasi-linear algebraic ordinary differential equations.*

Let $P(x, y, Dy, \cdots, D^ny)$ be a nonvanishing differential polynomial with coefficients in a logarithmic domain L^*; let P be nontrivially of order n. Let M be a principal monomial of P. Let $E\prec 1$. Let $R(x, z, Dz, \cdots, D^nz)$ be the transform of P under the substitution $y=M(1+E)+z$. P is called asymptotically quasi-linear at M if M is simple, P is asymptotically nonsingular at M, and the homogeneous linear part of R has a principal factorization sequence $(V, w_1, \cdots, w_n)$ independent of E. In a certain precise sense, almost every P is asymptotically quasi-linear at every principal monomial M, with every w_j of nonexceptional index. Whenever this occurs, $P=0$ has at least one solution $\sim M$ in a suitable CS. Let L^* and M have logarithmic ranks $\leqq p$. Let $\theta=x\log x\log\log x\cdots\log_p xD$. The proof uses operators H_j, $(j=0, 1, \cdots, n)$, where H_jQ is obtained from any differential polynomial Q, written in terms of θ, by omitting every nonlinear term whose order in θ exceeds j. A finite sequence $Q_0, Q_1, \cdots, Q_n$ of differential polynomials is constructed, with H_jQ_j uniformly quasi-linear, and Q_j equivalent to R under $z=a_j(x)Z+b_j(x)$, where $a_j(x)$ and $b_j(x)$ are constructed using solutions $\prec 1$ of $H_{j-1}Q_{j-1}=0$. (Received June 11, 1956.)

718t. Walter Strodt: *On a problem of R. Bellman.*

R. Bellman (Bull. Amer. Math. Soc. vol. 61 (1955) p. 192) posed the following question: "Does the equation $u'+u=(u'')^2$ have a solution which approaches zero like e^{-t} as $t\to\infty$ for suitably small values of $u(0)$ and suitably chosen $u'(0)$?" A partial answer to this question, and other information about the given equation G, can be found in the following way: Let the subsitution $t=\log x$, $u=Cx^{-1}+y$, (where C is any nonzero complex constant), carry G into $P=0$. The principal monomial M of P is $-C^2x^{-2}$. Under $y=M+Mz$, P is transformed into a uniformly quasi-linear equation $Q=0$ for which a description of the solutions $\prec 1$ is available. Each such solution yields a solution $\sim Ce^{-t}$ of G. Among other results, for every C, $Q=0$ has a solution $\prec 1$ in $\overline{F}(-\delta, +\delta)$, for some $\delta>0$. Hence G has a solution u with $u\sim Ce^{-t}$ in a complex neighborhood system of $+\infty$ consisting of horizontal half-strips $S_\epsilon=\{t: |\mathcal{I}(t)|<\delta-\epsilon;\ \mathcal{R}(t)>A_\epsilon\}$, $(0<\epsilon<\delta)$. A continuity argument now shows that for every sufficiently small nonzero real I there is a solution u with $u(0)=I$, and $u(t)e^t\to K$ for some nonzero real K. (Received June 11, 1956.)

719. Vikramaditya Singh: *Some extremal problems in univalent functions.*

Schiffer's method of interior variation has been adapted to symmetric univalent functions, bounded univalent functions and bounded symmetric univalent functions. In particular, the following theorems are proved: *Theorem* 1. Let S_0 be the class of regular univalent functions $f(z)$ in $|z|<1$ of the form $f(z)=z+\sum_{n=2}^{\infty} a_nz^n$ and let R be its subclass with real coefficients. Let $F(a_2, a_3, \cdots, a_n, \bar{a}_2, \cdots, \bar{a}_n)$ be a real-valued function of $(n-1)$ variables a_v and their complex conjugates $\bar{a}_v$, $v=2, \cdots, n$. Then the functions $f_R(z) \subset R$ and $f_S(z) \subset S$ which maximize $F(a_2, \cdots, a_n, \bar{a}_2, \cdots, \bar{a}_n)$ satisfy the same functional differential equation. *Theorem* 2. Let S be the family of bounded regular univalent functions $f(z)$ in $|z|<1$ such that $|f(z)|<1$, $f(0)=0$ and $f(\xi)=\omega$, $|\xi|<1$. Then the maximum and minimum values of $|f'(\xi)|$ are given respectively by $f'(\xi)=(\omega(1+|\xi|)/\xi(1\mp|\xi|))((1\mp|\omega|)/(1\pm|\omega|))$. The boundaries of the extremal domains consist of the unit circumference together with the radial slits starting respectively at the points $\mp|\omega|/\bar{\omega}$. *Theorem* 3. Let S_1 be the family of bounded symmetric univalent functions $f(z)$ in $|z|<1$ satisfying $|f(z)|<1$, $f(0)=0$, $f(\xi)=\omega$, $-1<\xi<1$. Then the functions which give the maximum and the minimum of $f(\eta)$, $-1<\eta<1$ are respectively given by $f(z)/(1\pm f(z))^2=(\omega/\xi)((1+\xi)^2/(1\pm\omega)^2)(z/(1\pm z)^2)$. (Received July 6, 1956.)

720. W. B. Stenberg: *On sequences with divergent total variation*. II.

These results are an extension of the author's paper (Proc. Kon. Nederl. Akad. v. Wetensch. Ser. A vol. 58 (1955) pp. 178–190.) A sequence $a_1, a_2, \cdots$ is called "admissible" if it possesses the properties: (I) $a_n>0$ for $n=1, 2, \cdots$, (II) $\lim_n a_n=0$, (III) $\sum_n a_n=\infty$. A sequence is called "singular" if it possesses together with properties (I), (II), (III), the property: (IV) for each subsequence $c_1, c_2, \cdots$, one has $\sum_n c_n=\infty$ if and only if $\sum_n |c_n-c_{n+1}|=\infty$. A necessary and sufficient condition that one can reorder all of the terms of an admissible sequence $a_1, a_2, \cdots$ to form a singular sequence is that $N(1/x)=o(x)$ as $x\to\infty$. Here $N(1/x)$ denotes the number of terms of $a_1, a_2, \cdots$ which are $>1/x$. It is further shown that given any admissible sequence $a_1, a_2, \cdots$ one may construct a reordered subsequence $b_1, b_2, \cdots$ having the property: if $c_1, c_2, \cdots$ is a subsequence of $b_1, b_2, \cdots$ and $\alpha_1, \alpha_2, \cdots, \alpha_p$ are numbers, then for all n, $\sum_{k=1}^{n} |\alpha_1 c_k+\cdots+\alpha_p c_{k+p-1}| \geq (|\alpha_1|+\cdots+|\alpha_p|)\cdot\sum_{k=1}^{n+p-1} c_k-Kb^*$. Here K depends only on p and the numbers $\alpha_1, \alpha_2, \cdots, \alpha_p$, and b^* denotes the largest term in the sequence $b_1, b_2, \cdots$. (Received July 9, 1956.)

721. C. A. Swanson: *Differential operators with perturbed domains*.

Consider a second order ordinary linear differential operator L on a half-open real interval $(0, b]$ $(b>0)$ which contains no singular points. The *basic operator* A_0 is defined on this interval as a restriction of L to a suitable domain $\mathfrak{D}_0$ of the Hilbert space $\mathfrak{L}^2(0, b)$. Let $[a, b]$ be a closed subinterval of $(0, b]$, where a is a small positive number. A *perturbed operator* A_a on $[a, b]$ is a restriction of L to a perturbed domain $\mathfrak{D}_a$. It is proved under certain assumptions on the basic operator that for each characteristic value Λ of A_0 there exists a characteristic value $\lambda(a)$ of A_a which is developable in an asymptotic expansion with leading term Λ, valid as $a\to 0$. Furthermore, the characteristic function $y(x, \lambda)$ of A_a possesses an asymptotic expansion valid as $a\to 0$ uniformly in the interval $[a, b]$. These expansions are not asymptotic power series, but are asymptotic expansions of a more general type. (Received July 5, 1956.)

722. R. C. Thorne: *Asymptotic solutions of differential equations with a turning point and singularities*.

Let u be a large positive parameter and $f(\zeta)$ a regular function of ζ in a domain D. Olver (Phil. Trans. Roy. Soc. London, Ser. A. vol. 249, p. 95) has shown that if $f(\zeta)=O(|\zeta|^{-1/2-\sigma})$ $(\sigma>0)$ as $|\zeta|\to\infty$ in D, asymptotic expansions of solutions of $d^2w/d\zeta^2=\{u^2\zeta+f(\zeta)\}w$ exist which are valid uniformly with respect to ζ in unbounded sectors of D and over $\zeta=0$. By a preliminary $z-\zeta$ transformation and application of this theory one can derive asymptotic expansions for solutions of (*) $d^2w/dz^2=$

$=\{u^2z^{-1}(z_0-z)p(z)+z^{-2}q(z)\}$, valid over the turning point $z=z_0$ but not valid over $z=0$ since $z=0$ corresponds to a point $|\zeta|=\infty$ for which $f(\zeta)=O(|\zeta|)$; $p(z)$ and $q(z)$ are regular functions of z in D. The author shows that if (*) is modified, the asymptotic solutions are valid over $z=0$. When (*) has a regular singularity at infinity or an irregular singularity of rank one at infinity, (*) can be modified so that the asymptotic solutions of (*) are valid over all the singularities. The theory is relevant to the Legendre and Bessel equations. (Received June 18, 1956.)

723*t*. W. R. Utz: *A note on second-order nonlinear differential equations.*

The equations considered are $x''+\alpha(x)x'+\beta(x)x=0$, (1); $x''+g(x')+cx=0$, (2); wherein $x'=dx/dt$. It is shown that if $\alpha(x)$ and $\beta(x)$ are real functions such that for all real x, $\alpha(x)\leqq 0$, $\beta(x)>0$ and if $x(t)$ is a solution of (1) valid for all large t, then $x(t)$ oscillates or, for all large t, $x(t)$ is monotone. In case $x(t)$ is monotone increasing, $\lim_{t\to\infty} x(t)>0$ and in case $x(t)$ is monotone decreasing, $\lim_{t\to\infty} x(t)<0$. As a corollary to this theorem one can show that when c is a positive constant, $g(0)=0$, and for all real z, $g(z)\leqq 0$, then if $x(t)$ is a solution of (2) valid for all large t, then $x(t)$ is oscillatory, $\lim_{t\to\infty} x(t)=\infty$ or $\lim_{t\to\infty} x(t)=-\infty$. (Received July 5, 1956.)

724*t*. C. J. Wallen and Francis Regan: *A generalized Hansen series.*

A series of the type $H(z)=\sum a_n z^{rn+t}/(1-z^{rn+t})$, where r and t are fixed positive integers, is a generalized Hansen series. If $\sum a_n$ diverges the H-series and the associated power series $\sum a_n z^{rn}$ converge at the same points within the unit circle, but if $\sum a_n$ converges the H-series converges for every z of modulus not equal to unity. Further, the convergence is uniform in every closed sub-region completely contained in one of the regions of convergence. Therefore the function represented by an H-series within its region of convergence can be expanded as a power series having coefficients which are sums of coefficients of the H-series; if t is a multiple of r, relations exist which give the coefficients of the power series as unique sums of the coefficients of the H-series, and conversely. Employing a radial approach to a boundary point, the unit circle is established as a natural boundary for the function represented by an H-series with certain restrictions on the coefficients; imposing an added restriction the same result is obtained when a complex approach is employed. A condition is determined sufficient to establish the unit circle as a natural boundary when the H-series has positive real coefficients and converges within the unit circle. (Received April 26, 1956.)

725*t*. W. R. Wasow: *Asymptotic development of the solution of Dirichlet's problem at analytic corners.*

Let the boundary C of a bounded plane region R consist of a finite number of closed analytic arcs. Let C be, furthermore, composed of a finite number of closed arcs on each of which there is prescribed an analytic function of the arc length s such that the resulting function $f(s)$ is continuous on C. The solution $u(x, y)$ of the Dirichlet problem $\Delta u=0$ in R, $u=f(s)$ on C, has, in general, singularities at the corners, i.e. at the points where C or $f(s)$ is not analytic. By means of methods introduced by H. Lewy and extended by R. S. Lehman it is shown that u is the real part of an analytic function of $z=x+iy$ which possesses at every corner $z=a$ an infinite asymptotic series development. Let $\pi\alpha$, $0<\alpha\leqq 2$, be the interior angle at the corner, then this expansion is a power series in $(z-a)$ and $(z-a)^{1/\alpha}$, if α is irrational, and in $(z-a)$,

$(z-a)^{1/\alpha}$, $(z-a)\log(z-a)$, if α is rational. The partial derivatives of u have at a the same order of magnitude as those of $u_1(x, y)+\operatorname{Re}[(z-a)^{1/\alpha}]$ or as those of $u_1(x, y)+\operatorname{Re}[(z-a)^{1/\alpha}\log(z-a)]$, according as $\alpha\neq 1/m$ or $\alpha=1/m$ (m an integer). Here, $u_1(x, y)$ is a function all of whose derivatives are bounded at $z=a$. (Received May 2, 1956.)

726t. Y. K. Wong: *A generalization of Toeplitz's theorem in Hilbert space.*

Toeplitz proved that if A is bounded, then a bounded left inverse L exists if and only if $\sum_1^n \bar{x}_i x_i \leq c \sum_1^n \bar{x}_i g_{ij} x_j$ where $g_{ij}=\sum_1^\infty \bar{a}_{ti} a_{tj}$ for all n, and a similar equivalent condition holds for the existence of a bounded right inverse R, and $L=R$. As A is bounded, $u^*A=0$ implies $u=0$ and $Av=0$ implies $v=0$. In this paper we consider the elements in a locally compact, connected topological field. We generalize Toeplitz's theorem to unbounded infinite matrices A where the columns of A and of its conjugate-transpose A^* are Hilbert vectors. The graphs of the linear transformations induced by such matrices are not closed; consequently a new method has to be developed for our purpose. Although bounded left and right inverses L, R exist, they are unequal. However, we can construct the inverse A^{-1} by $A^{-1}=R+(I-F)L+D=L+R(I-E)+D$, where $AD=DA=0$ and E, F are projections such that $EA=AF=A$. The inverse A^{-1} is not unique, as $u^*A=Av=0$ for some Hilbert vectors $u\neq 0$, $v\neq 0$. (Received July 5, 1956.)

727. F. M. Wright: *On C-fractions belonging to a set* Γ.

In this paper, the author considers the set Γ consisting of all C-fractions $C(w)$ having the property that the corresponding power series $Q(w)=\sum_{n=0}^\infty \mu_n w^{n+1}$ of $C(w)$ is such that $P(z)=Q(1/z)$ has a J-fraction expansion. It is first noted that a method discussed by the author for essentially absolutely regular elements of Γ can also be used to give a fairly simple and direct approach to formulas corresponding to results of W. T. Scott for the entire set Γ. It is then shown that a technique employed by the author for absolutely regular elements of Γ in order to obtain from the above formulas certain determinant formulas for the coefficients of $C(w)$ as well as results on the usual persymmetric determinants $\Delta_{1,n}$ can also be used to obtain similar information for more general elements of Γ. It is to be noted that this information corresponds to results obtained for absolutely regular C-fractions by W. T. Scott and H. S. Wall (Annals of Mathematics, 1940) as well as E. Frank (American Journal of Mathematics, 1946). An additional result is obtained by the techniques mentioned relative to conditions under which the determinants $\Delta_{1,n}$ may equal 0 for two successive integers n. (Received July 3, 1956.)

APPLIED MATHEMATICS

728. C. M. Ablow: *Wave refraction at an interface.* Preliminary report.

A plane wave in one of two perfect gases moves toward the parallel plane interface between the gases. The wave is either continuous or headed by a shock front weak enough that entropy changes may be neglected. Using Riemann's solution of the appropriate hyperbolic partial differential equation, four equations are derived giving the details of the reflected and refracted wave motions. The equations are of first order integro-differential or implicit functional form depending on the boundary conditions and must be solved simultaneously for four functions of a single independent variable. The equations are suitable for numerical step-by-step solution. (Received June 25, 1956.)

729. W. Band, J. L. Brenner and P. A. Clement (p): *The eigenvectors of a matrix and the second quantization.*

A certain matrix of Jacobi ($a_{ij}=0$, $|i-j|>1$) arises in calculating rate processes by second quantization. It is proved that its eigenvalues are the same as those of a matrix due to Sylvester (Muir and Metzler, *Theory of determinants*, 1933, p. 546), and some properties of the eigenvectors are established. Each eigenvector corresponds to a certain mixed complexion of a physical system. The entropy of each pure complexion would be zero. Certain weighted sums of the components of the eigenvectors give the expected value of the energy and of the dispersion of the populations. By computing the latter quantities in closed form, it is determined what strategy the system should employ to enjoy maximum stability subject to constancy of the rate process in question. (Received May 28, 1956.)

730. E. H. Bareiss: *Numerical solution of the multigroup Boltzmann transport equation for nuclear reactor design.*

A code for the numerical solution of the one-dimensional multigroup Boltzmann Transport Equation is described. Large scale computations are being performed on the NORC (Naval Ordnance Research Computer) for the Westinghouse Atomic Power Division. The method for the numerical solution of the system of integro-differential equations, arising in this problem will be discussed. Some actual but unclassified numerical results obtained by this code will be presented. (Received July 6, 1956.)

731t. H. W. Becker: *Pythagorean off-resonance systems.*

Useful in teaching electronics fundamentals, the Diophantine system (1) $R^2+X_L^2=Z_L^2$, $R^2+X_C^2=Z_C^2$, $R^2+X^2=Z^2$, $X=|X_L-X_C|\neq 0$ requires (2) $R=2abcdef$, X_L, $Z_L=cdef(a^2\mp b^2)$; X_C, $Z_C=abef(c^2\mp d^2)$; X, $Z=abcd(e^2\mp f^2)$; $X=(ac+bd)(bc-ad)$, $ef/(e^2-f^2)=abcd/(ac+bd)(bc-ad)$. This has the parametric solution (3) a, $c=g(g^2\mp h^2)$; b, $d=h(g^2\pm h^2)$; e, $f=g\pm h$. Each such solution yields a transformed solution (4) a, $b\rightarrow(a\pm b)(c\pm d)$; c, $d\rightarrow e(a-b)(c+d)$, $f(a+b)(c-d)$; e, $f\rightarrow e$, f. Thus R, X_L, Z_L, X_C, Z_C, X, $Z=60$, 11, 61, 91, 109, 80, 100→3003, 26596, 26765, 30600, 30747, 4004, 5005. But numerical solutions are best obtained empirically: of the 25 known with $R \not> 3003$, only 6 come under (3) or (4), an algebraic virility or coverage of only 24%. The problem is equivalent to Alliston's problem: to find a Pythagorean and a Hero triangle with a side and the area in common, Math. Snack Bar (Cambridge, 1936) p. 122. On the previous page, he proposed the problem of a $P\triangle$ and $2H\blacktriangle$ with side and area in common; his solution is in numbers of 5 and 6 digits. But the simplest solution is the set of $\blacktriangle$ with sides 120, 209, 241; 123, 209, 218; 169, 209, 150. A twin solution of (1) satisfying (5) $X=X_L\pm X_C$ with either sign is $R=120$, $X_L=209$, $X_C=182$. (Received April 24, 1956.)

732. J. S. Bendat: *Zero crossings of a sine wave plus random noise.*

A new simple formula is derived for the expected number of zeros per second of a sine wave plus random noise. The main difference between this work and a previous study by Rice (BSTJ, vol. 27, Jan. 1948) is that whereas Rice considers a single sine wave of fixed amplitude, phase and frequency, the present formulation allows the sine wave to be a representative member of a "partially ergodic" stationary random process, an ensemble of records varying in amplitude and phase, fixed in frequency only. Permitting a random variation in amplitude and phase agrees more closely than before with many physical problems. The final result is expected on the average even if the process is not "partially ergodic." The formula obtained in this paper agrees with the known Rice result for random noise alone; it also coincides with the required identity for a sine wave alone. In combined situations, the formulas disagree. Procedures for measuring the expected number of zeros per second in a finite time period are discussed, as well as the question of how this information, together with other statistical properties, may be used to determine the fixed frequency of the sine wave. (Received June 27, 1956.)

733. R. L. Brock: *Mathematics as applied to the analysis of electronic detection systems.*

The analysis of electronic detection and control systems is discussed from the standpoint of the applied mathematician. Time dependent input functions for such systems consist of the sum of an unwanted random noise term and a term which represents the information desired for detection or control. Because of the random character of the input functions the analysis of such systems is necessarily statistical in nature. An effort is made to trace briefly the mathematical development in this relatively new field of application. Particular attention is given to the way in which methods of mathematical statistics enter into these studies along with such analytic topics as differential equations, special functions, integral equations and integral transforms. Three approaches to the analysis and optimization of systems subject to random input functions are considered. These are each in turn based on statistical decision theory, theory of probability distributions and autocorrelation analysis. The present state of development is indicated for each and important areas for extended mathematical effort are noted. Typical are the areas of nonlinear systems, nonstationary random processes and non-Gaussian processes. (Received July 5, 1956.)

734t. F. H. Brownell: *A simplified model for a mathematical study of the perturbation theory of the Lamb shift.*

It has long been questioned whether the divergences arising in quantum electrodynamics using formal perturbation expansions are due to the invalidity of these perturbation expansions or to other mathematical difficulties glossed over in the present quite formal formulation used by the physicists. In order to study this question, we consider a radically simplified model of quantum electrodynamics in which spins and negative energy states of electrons and polarization of photons are omitted and in which only a finite number of photon oscillators, depending on b and R, are allowed. b here is the linear spacing between photon oscillators and R the cut-off radius in phase space. Under this model we study the shift in energy levels of the bound electron states of the hydrogen atom due to interaction with the radiation field as computed by formal perturbation series, and take the limit as $b\to 0^+$ and $R\to +\infty$. For fixed b and R, it is clear that the theorems of Rellich dealing with the perturbation of isolated point spectra of finite multiplicity for self-adjoint operators justify these perturbation expansions. It is also easy to prove that although the lowest nontrivial order energy term diverges as $b\to 0$ and $R\to +\infty$ (which the physicists remove by dubious renormalization) the difference in this term for two different unperturbed states converges, agreeing with the experimental measurement of the Lamb shift. It remains to determine what happens to the sum of the series, which equals the total energy, as $b \to 0^+$ and $R \to +\infty$. Also we investigate the modifications arising upon adding the complications of electron spins and photon polarization, and positive-negative energy states of electrons. (Received July 9, 1956.)

735. J. W. Carr: *Languages for digital computers.*

Languages for the description of the solution of problems using general purpose digital computers have thus far been informal and generally unsatisfactory. For "automatic programming," descriptions are needed which allow manipulation of algorithms by other algorithms, and can be easily used by both machine and human. Simple methods of abstraction for retention of commonly used algorithms (subroutines) are required. The language must be "regular," in the sense of preventing incorrect numerical operations on syntax and incorrect syntactical operations on numbers. A survey of the methods possible from the flow diagram through the Pitts-McCulloch-Kleene-von Neumann "nerve net theory" to the language of recursive function theory indicates certain possible directions of usage. (Received July 25, 1956.)

736. P. L. Chambré: *Chemical surface reactions in flow systems.*

A general basis for the analysis of catalytic surface reactions of arbitrary complexity in both external and internal hydrodynamic flows is presented. The mathematical formulation leads to a system of singular integral equations of the Volterra type for the thermodynamic state parameters of the reactants. The kernels depend on the specific hydrodynamic character of the flow. Applications to boundary layer and fully-established flows of laminar as well as turbulent character have been made. (Received July 5, 1956.)

737t. Jim Douglas, Jr.: *A note on the numerical solution of parabolic differential equations.*

The object of this paper is to discuss the relation between two techniques of demonstrating convergence of solutions of difference equations to those of the parabolic differential equation $a(x)u_{xx}+b(x)u_x=u_t$. The two methods are based on, respectively, a maximum principle and stability analysis. It is shown that, for reasonable analogues of the differential equation, a difference satisfying the maximum principle must be stable. An example of a stable difference equation which does not satisfy the maximum principle is constructed; hence, the stability analysis technique is properly more general than the maximum principle method. (Received July 5, 1956.)

738. Jim Douglas, Jr.: *The application of stability analysis in the numerical solution of quasi-linear parabolic differential equations.*

It has been shown previously (Douglas, Pacific Journal of Mathematics, 1956, to appear) that, for reasonable initial and boundary conditions, the solution of the first boundary value problem for the quasi-linear parabolic differential equation $u_{xx}=f(x, t, u)u_t$ can be approximated by the solution of the backward difference equation $\Delta_x^2 w_{i,n+1}=f(x_i, t_n, w_{in})(w_{i,n+1}-w_{in})/\Delta t$, where Δ_x^2 is the centered, second x-difference. As the error is $O((\Delta x)^2+\Delta t)$, more computation is required than is desirable. The object of this paper is to prove that the solution of the Crank-Nicolson type difference equation $\Delta_x^2(w_{i,n+1}+w_{in})=2f(x_i,\ t_{n+1/2},\ w_{in}+\Delta t\Delta_x^2 w_{in}/2f(x_i,\ t_n,\ w_{in}))(w_{i,n+1}-w_{in})/\Delta t$ converges L_2-wise to u with an error which is $O((\Delta x)^2+(\Delta t)^2)$. The proof is based on an extension of a stability analysis argument (Douglas, Journal of Society for Industrial and Applied Mathematics, 1956, to appear). Generalizations to other difference equations and higher dimensions are given. (Received July 6, 1956.)

739. T. C. Doyle: *A method of matrix inversion applicable to a digital computer.*

Associated with any real nonsingular $n\times n$ matrix A is the related symmetric matrix $a=A^TA$, to be regarded as the coefficient matrix of a positive-definite quadratic form y^Tay. Since $A^{-1}=a^{-1}A^T$, the inversion of A is reduced to finding a^{-1}. To form a^{-1}, the linear system $ax=b$ is transformed into $H^TaH\xi=H^Tb$ by $x=H\xi$, where H is a triangular matrix determined so that $H^TaH=d^{-1}$ is a diagonal matrix. Then $\xi=dH^Tb$ and $x=HdH^Tb$ so that $a^{-1}=HdH^T$. H is readily machine constructed and the elements of a^{-1} appear as summations over the ascending ranges (1) (1, 2) (1, 2, 3), $\cdots$, (1, 2, $\cdots$, n). A machine storage plan and addressing scheme is illustrated for $n=3$. (Received March 28, 1956.)

740. M. L. Friberg and C. R. Cassity (p): *A rational description of explosive-induced shock waves in metals.* Preliminary report.

Although the equations describing the motion of elastic waves are well-known, the conditions under which these equations may be applied are not well-established and very few calculations which might be used to compare with the extensive test data available have been made heretofore. Some new solutions of the wave equation are compared with the results of a series of tests. (Received July 2, 1956.)

741. Bernard Friedman: *Two theorems on wave propagation.*

(1) It is shown that the equation $u''+(\lambda-q(x))u=0$ over the interval $(0,\ \infty)$ with the boundary condition $u(0)=0$ must have virtual eigenvalues if $\int_0^\infty x|q(x)|dx<\infty$ and $q(x)\neq 0$. (2) A result of Rellich (Jbr. Deutschen Math. Verein vol. 53 (1943) pp. 57–64) is generalized to prove the uniqueness of the solution of $\Delta u+(\lambda-q)u=0$ with Dirichlet boundary conditions if $\int\int|q|r^{-1}drd\Omega<\infty$. (Received July 9, 1956.)

742. W. T. Guy, Jr.: *Integral transform solution of a nonlinear differential equation.*

An approximate solution to a Duffing differential equation was obtained using integral transforms and the convergence of the process was presented. (Received July 9, 1956.)

743. Isidore Heller: *On linear systems with integral valued solutions, I.*

The constraint matrix of the Hitchcock-Koopmans transportation model has the property: whenever a column is a linear combination of linearly independent columns, the coefficients are integers (and therefore necessarily ±1 or 0). This property permits a simplification in the computational procedure of the "simplex method" as was shown by G. Dantzig (*Activity analysis of production and allocation*), and assures that all basic solutions are integral valued whenever one integral valued solution exists. C. Tompkins and the author (Annals of Mathematics Studies, no. 38) have shown the mentioned property to hold for an extended class of matrices. J. Kruskal and A. Hoffman (ibid.) have related this property, which they termed "unimodular property," to partly ordered sets. The present note, unifying and extending the cited results, proves the following statements (see below). (Received May 31, 1956.)

744. Isidore Heller: *On linear systems with integral valued solutions, II.*

1. If $b_1, b_2, \cdots, b_k$ are linearly independent vectors in a linear vector space, then the set $D=\{\pm b_i, b_i-b_j\}$ is unimodular. 2. D is maximal for its dimension, in the sense that the span of D does not contain a unimodular set properly containing D. 3. Theorem: If a unimodular set of dimension k contains at least $k(k+1)$ vectors, not counting the nullvector, then it is of type D (and hence contains exactly $k(k+1)$ vectors). 4. For $k\geqq 4$ there are k-dimensional maximal unimodular sets which are not of type D (they necessarily have less than $k(k+1)$ elements). The unimodular sets of type D containing the unit vectors, that is the representations of D in coordinates relative to the different bases in D, are easily described by associating a tree of $k+1$ vertices and k oriented and numbered segments to each basis in D; this yields 1-1 correspondence between representations and trees. (Received May 31, 1956.)

745. William Karush: *On a minimisation problem involving convex functions.*

Let $f(x_1, x_2, \cdots, x_n)=\sum_{i=1}^{n} f_i(x_i)$, where each f_i is a real-valued convex continuous function over the real numbers. In various applications, e.g., in mathematical economics, one encounters the problem of minimizing f with the x_i subject to linear restraints. Let A, B with $A\leqq B$ be arbitrary and restrain x_i by the conditions $A\leqq x_1 \leqq x_2\leqq \cdots \leqq x_n\leqq B$. In this paper, the function $F(A, B)=\min f$ over such x_i is studied. It is shown that F satisfies the functional equation $F(A, C)=F(A, B)+F(B, C)-F(B, B)$, $A\leqq B\leqq C$. From this is obtained the decomposition $F(A, B)=M(A)+N(B)$, where M, N are explicitly determined monotonic convex functions. These properties of F also hold in the continuous case where $F(A, B)=\min \int_0^1 f(t, x(t))dt$ over all monotonic functions $x(t)$, $0\leqq t\leqq 1$, satisfying $A\leqq x(0)\leqq x(1)\leqq B$; here $f(t, x)$ is convex continuous in x for each t, and f is sufficiently smooth as a function of t and x. (Received July 9, 1956.)

746t. Erwin Kreyszig: *Stable oscillating systems having quadratic damping.*

The equation (1) $\ddot{q}-(\beta/2)(\operatorname{sgn} \dot{q})(1-\alpha^2q^2)\dot{q}^2+(\gamma^2/2)f(q)=0$, $\beta>0$, where $q(t)$ is the elongation and t the time, may be considered as a generalization of the well known van der Pol equation. (1) can be transformed into (2) $V'-\beta(\operatorname{sgn} q)(1-\alpha^2q^2)V+\gamma^2f(q)=0$, where $V=\dot{q}^2$. Small oscillations increase while large oscillations decrease; in both cases the "maximum amplitude" $Q_n(t)$, $n=1, 2, \cdots$ (i.e. the value of $q(t)$ for which $\dot{q}(t)=0$) tends to a limit $q=Q_s$ if $n\to\infty$, i.e., $t\to\infty$. From the first integral of (1) bounds for Q_s can be obtained for every type of the restortion force which is of practical interest. For example, if $f(q)\geqq 0$ $(q>0)$, $f(q)\leqq 0$ $(q<0)$, $f(q)\neq 0$, then $|Q_s|>3^{1/2}/|\alpha|$. If $f(q)=q^{2m+1}$, $m=0, 1, \cdots$ the following results hold: I. $|Q_s|<q_0$ where q_0 is the smallest zero of $h(s)=\exp(2\beta q)P(2(m+1)/3, cq^3)-P(2(m+1)/3, -cq^3)$, $c=\alpha^2\beta/3$, and $P(m, x)=\int_0^x e^{-t}t^{m-1}dt$ is the incomplete Gamma function. II. The ratio $|Q_n|/|Q_{n+1}|$ depends on $u=\alpha/\beta$ only and is a monotone increasing function of u. Similar results can be obtained in the case where $f(q)$ is an arbitrary polynomial of q.

Approximate solutions of (1) can be derived by means of a refinement of the method of slowly varying amplitude and phase. (Received June 28, 1956.)

747. T. C. H. Li: *Tip effect of an oscillating finite thin wing in supersonic flow.*

The tip effect of a swept, oscillating, finite thin wing, with supersonic leading edges in supersonic flow, is determined by mapping the region on the imaginary diaphragm off the wing onto the disturbing region on the wing. This is accomplished by a linear transformation, independent of the coordinates of the receiving point. A set of aerodynamic influence coefficients for the disturbing region is derived which may be handled conveniently by any type of high-speed computer. Results obtained for the rectangular wing are in very good agreement with those of Acum and those of Watkins. (Received July 5, 1956.)

748. L. E. Malvern: *Eigenfunction series for beam-column with elastic end support.*

The use of series of eigenfunctions of a column to represent deflection under given lateral loads and sub-critical axial compressive load is well-known for the cases in which each end is simply-supported, fixed, or free. The usual orthogonality property of the second derivatives of the eigenfunctions, which facilitates determination of the series coefficients by use of the Principle of Virtual Displacements, is lacking when the ends are elastically supported against deflection and/or rotation. By properly including the virtual work of the elastic support forces, however, it is possible in this case also to obtain the coefficients without solving simultaneous equations. The formula obtained for the coefficient a_m in the series $\sum a_m v_m(x)$ then has added in the denominator the terms $A_0 v_m^2(0)+A_L v_m^2(L)+B_0[v_m'(0)]^2+B_L[v_m'(L)]^2$, but does not otherwise differ from the usual result. Here A_0 and A_L are the spring constants of the supports against deflection at the ends $x=0$ and $x=L$ of the beam-column, while B_0 and B_L are the spring constants of the restraints against rotation. (Received July 9, 1956.)

749. W. D. Maness: *A generalized Green's relation R for a $(n-1)$-dimensional Cartan hypersurface.*

An analytic variety which sustains asymptotic parametric curves is known as a Cartan variety. A generalization of Green's relation R is given for a line l and a linear $(n-2)$-space L in the tangent hyperplane to a $(n-1)$-dimensional Cartan hypersurface. The power series expansion in local nonhomogeneous coordinates for the equation of the hypersurface is obtained. The power series is simplified by choosing a suitable reference frame. The complete geometric characterization of this reference frame is given and a generalization of the edges of Green for n-dimensional space is obtained by use of the generalized relation R. The power series obtained and the reference frame chosen are shown to specialize to that obtained by G. M. Green using Wilczynski's normal coordinates (Trans. Amer. Math. Soc. vol. 20 (1919) pp. 79–153). (Received July 5, 1956.)

750. William Nachbar: *Note on exponential approximations for the rectangular plate.*

Deflections and stress resultants for a laterally-loaded rectangular plate with two opposite edges simply supported can be computed by Fourier series expansions. Many examples of this are given in S. Timoshenko *Theory of Plates and Shells*, McGraw-Hill, New York, 1940, Chapters V and VI. Slow convergence, particularly of the series representing the stress resultants, is sometimes a practical difficulty of this method, however. There are possible advantages to be gained from expanding portions of the Fourier coefficient of order n in a power series in the factor $e^{-\alpha n}$, where α is proportional to the ratio of the lengths of sides, and then rearranging the main series. This note discusses the advantages in using such a technique, in particular for obtaining simple formulas for approximating the sums of the Fourier series. The plate with one free edge and three edges simply supported is used as an example. (Received July 6, 1956.)

751. Walter Noll: *On isotropic functions of linear transformations.*

Isotropic functions are of great importance in modern continuum mechanics. They are of the form $f(x)$ or $F(x)$ ($f(x)$ real; x, $F(x)$ = linear transformations of a real Euclidean vector space) and satisfy $f(x)=f(RxR^{-1})$ and $RF(x)R^{-1}=F(RxR^{-1})$ for all orthogonal (proper orthogonal) transformations R. They are called symmetric if x and $F(x)$ are restricted to be symmetric. The properties of such functions and of their differentials (in the sense of Hamilton) are investigated. In particular, the functions $a^{(k)}(x)$ (=kth principal invariant of x), x^n, and $x^{1/n}$ are studied. Any symmetric isotropic $f(x)$ can be expressed in terms of the $a^{(k)}(x)$, and any symm. isotrop. $F(x)$ has a representation $F(x)=\sum_{k=0}^{n-1}\phi_k(x)x^k$, where the $\phi_k(x)$ are real valued and symmetric isotropic. The continuity and differentiability properties of these representations are discussed. (Received July 6, 1953.)

752t. F. Reza: *A qualitative theorem on the response of linear systems in the time domain.*

Let $f(t)$ be a continuous single-valued bounded function of a real variable t (time) whose derivative has a finite number of real roots. $f(t)=0$ for $t<0$. A shape function C_n is associated with $f(t)$ based on the following: (a) Denote by $t_1<t_2<t_3\cdots<t_n$ the real values of t_j for which $df(t_j)/dt=0$. (b) Let C_n be a finite sequence of $j=1, 2, \cdots, n$ so that the corresponding $f(t_j)$'s will be in order of their values, the smallest being at the left of this finite sequence. C_n gives some qualitative measure of the shape of $f(t)$ on the real axis. Two functions having same shape functions are said to have the same shape. *Theorem*: Let C_n be the shape function of a function $f(t)$ (as described in the above). (i) There exist finite linear passive lumped bilateral electrical networks N at rest at the time $t=0$, such that for an applied unit impulse stimulus, applied at $t=0$ the response of the system has C_n for its shape function. (ii) The networks N cannot contain less than $n+1$ reactive elements if N is confined to resistor-capacitor or resistor-inductor networks. (iii) The lower limit is possible, i.e., it is possible to synthesize a network of the type described in (i) with only $n+1$ reactive elements. (Received July 9, 1956.)

753t. L. S. Shapley: *Game solutions with arbitrary components.*

A class of solutions to the n-person simple game, $n\geq 4$, with minimal winning coalitions $\{2, 3, 4, \cdots, n\}$, $\{1, 3, 4, \cdots, n\}$, $\{1, 2, 4, \cdots, n\}$ is exhibited in which one component, J is an arbitrary closed subset of the $(n-3)$-dimensional set of imputations of the form $(0, a, a, \alpha_4, \cdots, \alpha_n)$, $a>0$ (in the $0-1$ normalization). The remaining component depends upon J, and is closed, connected, and has maximum dimension $n-2$. The arbitrariness of J permits the construction of "pathological" examples that dispose of several conjectures regarding the regular behavior of solutions. (Received July 6, 1956.)

754. H. E. Stelson: *Laplace transforms applied to interest functions.*

Interest functions with constant rate may properly be considered as step functions, so that difference equations may be determined and resolved by means of Laplace transforms. The analog is continuous interest where differential equations are used in evaluation. Generalized interest functions may be derived by use of Laplace transforms. Interest functions with constant time may use Laplace transforms to determine an unknown rate. (Received July 6, 1956.)

755*t*. R. H. Wasserman: *A formulation and solutions of the equations of ideal fluid flow.*

Following the work of N. Coburn (Michigan Mathematical Journal, vol. 1, no. 2, (1952)) the laws of steady ideal fluid flow are formulated intrinsically in terms of the unit trihedron of the stream lines for the two cases of incompressible fluids and polytropic gases subject only to the condition that the entropy be constant along a stream line. When the stream lines are taken as coordinate curves the equations governing the flow are obtained as a set consisting of two linear differential equations and one algebraic equation for a scalar function of the pressure. With these equations, given geometrical properties of the congruence of curves formed by the stream lines, additional general properties of the flows are obtained. In particular, the question of existence of fluid flows having stream surfaces on which the stream lines are geodesics or asymptotic curves can be answered in some cases. The answer is affirmative when the stream surfaces are concentric circular cylinders and the stream lines are geodesics on these cylinders. Moreover, the entire class of these helical flows is obtained explicitly in terms of several arbitrary functions, reflecting a considerable variety of geometrically and thermodynamically distinct flows. (Received July 6, 1956.)

756. H. F. Weinberger: *Forced vibrations and nonorthogonal projection.*

Let L be a differential operator with boundary conditions such that the bilinear form $\int uLu$ is positive definite. The forced vibration of a system on which the force corresponding to displacement u is Lu is governed by the equation $Lu-k^2\rho u=f$. For k sufficiently large the natural norm $\int u(Lu-k^2\rho u)$ is not definite. Hence the Schwarz inequality method of Diaz [Collecteanea Math. 4 (1951) fasc. 4] and the equivalent hypercircle method of Prager and Synge [Quarterly of Applied Mathematics vol. 5 (1947) pp. 241–269] do not work. Just as in these methods, we consider functions v satisfying the same differential equation and natural boundary conditions as u and functions w satisfying only the remaining boundary conditions for u. We use the norm $\|\phi\|^2=\int u(Lu+c^2\rho u)$ where c^2 is any positive constant. In this norm $(v-u)$ and $(w-u)$ are not orthogonal. It can be shown by direct computation that $\|w-u\|\leq K\|w-v\|$ for all v and w, where K depends on the distance of k^2 from the nearest eigenvalue of $Lu-\lambda\rho u=0$ and is singular when k^2 is an eigenvalue. Since $w-u$ satisfies only boundary conditions, this inequality is insufficient for a pointwise bound for $|w-u|$ if L is of second order. A geometric consideration in Hilbert space shows that the above inequality is equivalent to $\|v-u\|\leq K\|w-v\|$. Since w and v are arbitrarily chosen and since $v-u$ satisfies a differential equation, this inequality gives a pointwise bound for $|v-u|$ which can be made arbitrarily small. (Received July 9, 1956.)

757. Y. K. Wong: *Note on Hurwitz's condition on dynamic stability.*

The characteristic polynomial $\sum c_{n-i}\lambda^i$ of a real-valued square matrix A of order n can be determined in terms of traces t_p of the pth powers of A as follows: $c_k = \sum(-t_1)^\alpha(-t_2/2)^\beta \cdots (-t_k/k)^\nu/\alpha!\beta! \cdots \nu!$, where the summation extends to all $\alpha, \beta, \cdots, \nu$ such that $\alpha+2\beta+\cdots+k\nu=k$. We can prove that $\partial c_k/\partial t_p = -c_{k-p}/p$. Hurwitz's condition for dynamical stability is expressed in terms of a sequence of determinants d_m with its kth row of the form $(c_{2k-1}, \cdots, c_k, \cdots, c_2, c_1, 1, 0, \cdots, 0)$, where $k=1, 2, \cdots, m$ and $c_q=0$ if $q>n$. We show that d_m $(m=1, 2, \cdots, n)$ is independent of t_{2p}, the trace of the even power of A, provided that $2m-1 \leq n$. (Received July 5, 1956.)

GEOMETRY

758t. L. M. Blumenthal: *Global subsets of the n-sphere.*

A subset of the n-sphere S_n is global provided it is not contained in any hemisphere. The paper presents new necessary and sufficient conditions for globality of certain subsets of S_n. A mapping ϕ of S_n (minus its north and south poles) into itself is studied, for which the image of a point p of S_n is the point whose ith coordinate is $(\cos pp_i)/N$, $(i=0, 1, \cdots, n)$, where $p_0, p_1, \cdots, p_n$ are fixed points of the equatorial S_{n-1} and $N=\sum_{i=0}^{n} \cos^2 pp_i$. It is shown that ϕ is an isometry on S_{n-1} if and only if the basic $(n+1)$-tuple is either equilateral (and hence global on S_{n-1}) or becomes so upon reflecting one or more of its points in the origin (the center of S_n). (Received April 16, 1956.)

759. W. R. Cowell: *Ternary rings in relation to incidence matrices.*

Let the finite projective plane π be coordinatized by a ternary T and let π' be the corresponding affine plane. Then the rows and columns of the line-point incidence matrix M of π can be arranged so that permutation matrices corresponding to the latin square representation of π' appear as submatrices of M. It is then possible to interpret the ternary operation in terms of these permutation matrices. Various properties of T such as linearity, associativity of addition and multiplication, and the distributive laws correspond to matrix identities. The properties of linearity and associative addition are preserved under certain transformations of the coordinate system and linearity is shown to be equivalent to a particular isotopy relation among the latin squares representing π'. (Received July 5, 1956.)

760. K. D. Fryer (p) and I. Halperin: *The von Neumann coordinatization theorem for complemented modular lattices.*

The theorem of J. von Neumann, that coordinates can be introduced into any complemented modular lattice of order four or more is again considered. The proof of this theorem given previously by the authors is simplified and additional axioms are given under which this proof applies to complemented modular lattices of order three (the new axioms play the role of Desargues' theorem in projective geometry. (Received June 28, 1956.)

761t. Donald Greenspan: *On vertices of space curves.*

A new approach, which avoids differentiability assumptions and which has a potential topological generalization, is made in the study of vertices of n-dimensional space curves, $n \geq 3$. Classes of arcs and curves called base arcs and base curves, respectively, are defined. A fundamental theorem, the Reduction Theorem, is proved first. This theorem enables one to always employ the more useful "point of intersection" rather than the "point of support" in any discussion of the point set intersection of an arc (or curve) and a hypersphere (which term includes the concept of hyperplane). Subsequent to further definitions, remarks, and lemmas, the following major results are proved: (a) Contraction Theorem. Let hypersphere K be oriented and meet base arc B (or base curve C) in $(n+2)$ distinct points. Then it is possible to generate a descending sequence of $(n+2)$-tuples whose spans converge to zero. (b) If $P_1, P_2, \cdots, P_{n+2}$ are points of $B \cdot K$ in natural order, then the closed subarc T of B with initial point P_1 and terminal point P_{n+2} contains in its interior at least one K vertex. (c) Two Vertex Theorem. Every base curve C which meets hypersphere K in at least $(n+2)$ points has at least two vertices. (Received June 1, 1956.)

762*t*. Isidore Heller: *On the edges of a simplex.*

The edges of a simplex, interpreted as vectors (in both orientations), have the following property: if one of them is a linear combination of linearly independent ones then the coefficients are ± 1 or 0. The note proves the converse: If a k-dimensional set of at least $k(k+1)$ vectors $\neq 0$ has the mentioned property, then there is a k-simplex having the vectors as its edges (Comp. the author, *On linear systems with integral valued solutions*, Abstract 744 above.) (Received June 4, 1956.)

763*t*. Robert Hermann: *Global aspects of complete linear differential systems*. Preliminary report.

X is a C^∞ manifold of dimension n, $t_1, \cdots, t_n$ are everywhere independent first order linear differential operators (vector fields). V is a real vector space of dim. m, $a_1, \cdots, a_n$ are C^∞ maps $X \to E(V)$, the space of endomorphisms of V. Consider the differential system: $t_i(f) = a_i(f)$, to be solved for a C^∞ map $f: X \to V$. Assume the system has locally m linearly independent solutions. Let S denote the sheaf of germs of solutions. Then, (1) If X is simply connected, the system has globally m linearly independent solutions, and, in the language of the cohomology theory of sheaves, $H^i(X, S)$ is isomorphic to $H^i(X, R)$ for $i > 0$. (2) The system is elliptic in the sense of Schwartz. Corollary: On the homogeneous space of a Lie group, there is but one differential structure compatible with the action of the group. (Received June 28, 1956.)

764*t*. Shoshichi Kobayashi: *A generalization of bounded domains.*

Let M be a complex n-dimensional manifold with a holomorphic n-form ω which does not vanish at any point of M. Let F be the set of all holomorphic functions on M which are square integrable with respect to the volume element $\omega \wedge \bar{\omega}$. Then F is a separable Hilbert space. From a complete orthonormal basis for F, the generalized Bergman's kernel function can be constructed by the well known method. The generalized Bergman's metric (in general, positive semi-definite) can be defined if and only if (A) For every point z of M, there exists an f in F such that f is different from zero at z. The metric is positive definite if and only if (B) For every analytic tangent vector V to M at z, there exists an f in F such that $f(z) = 0$ and $V(f)$ is different from zero. The metric depends only on the complex structure of M. A similar class of complex manifolds was given by Calabi-Washnitzer independently. A manifold with the metric above defined can be imbedded isometrically in a natural manner into the projective space over F with the canonical metric. The group of holomorphic transformations of M satisfying (A) and (B) is a Lie group with compact isotropic subgroups. If the group is transitive on M, then the metric and the Ricci curvature coincide. (Received August 22, 1956.)

765. J. C. Morelock, N. C. Perry, and W. R. Hutcherson (p): *Fibonacci sequence applied to quadratic transformations.*

A method of finding perfect points in the neighborhood of a singular point by use of quadratic transformations and a cyclic homography is obtained by a Fibonacci sequence, provided only one quadratic transformation is used in the process. When two are used alternately a method is also obtained and a proof given for it. [W R. Hutcherson and J. C. Morelock, *Concerning a pattern for perfect points*, Revista Matematica Y Fisica Teorica (Tucuman) Serie A vol. 10 (1954) pp. 187–194]. (Received May 23, 1956.)

766. T. G. Ostrom: *Transitivities in finite projective planes and related topics.*

The author shows that if p, q, r, s are four points on a line L, then the perspectivities which leave p and q fixed and carry r into s all produce the same permutation of points on L. A similar result holds if p is the only fixed point on L. This amounts to a kind of uniqueness for addition and multiplication as defined by Veblen and Young. Combining results of Baer and Andre, the following theorem follows: If the plane is both $p-L$ transitive and $q-L$ transitive, where p is on L and q is not, then the plane can be coordinatized by a near field. Other results: If n is an odd power of 2 and the plane is doubly transitive, it is Desarguesian. For any n, if the plane is transitive on quadrangles, each quadrangle generates a Desarguesian subplane. Every finite plane contains many Desarguesian configurations. (Received July 5, 1956.)

767. T. K. Pan: *Exterior differential system and mapping of surfaces with associated vector fields in correspondence.*

Exterior differential systems have been successfully applied to the study of certain type of problems in differential geometry (Elie Cartan, *Les systèmes différentiels extérieurs et leurs applications géométriques*, Hermann & Cie., Paris, 1945). Analogous to the fifteen illustrative problems in the second part of the book, this paper investigates the surfaces in conformal correspondence which preserves associated vector fields, their indicatrices and their normal curvatures along corresponding curves. The general solution depends on six arbitrary functions of one variable. The characteristics are the curves and the indicatrices of the associated vector fields. The singular solution depends on four arbitrary functions of one variable. It consists of: equivalent or symmetrical surfaces associated with equivalent vector fields; similar developable surfaces with associated unit vector fields whose angular spreads along corresponding curves are inversely proportional to the arc lengths of these curves; similar surfaces with equal Gaussian curvatures at corresponding points and with associated vector fields whose curves are geodesics and whose angular spreads along corresponding curves are inversely proportional to the arc lengths of these curves. The problem of Cauchy is investigated. (Received June 6, 1956.)

768. R. R. Phelps: *Convex sets and nearest points.*

A well known theorem characterizes the convexity of a closed set S in Euclidean n-space E^n in terms of a "nearest point" map which assigns to each point of E^n a nearest point in S. We consider, in a more general normed linear space E, the following sets which correspond roughly to the inverse images of such a map: If $s \in S$, denote by S_s the set $\{x: \|x-s\| = \inf_{y \in s} \|x-y\|\}$ of all points having s as a nearest point in S. If the dimension of E is at least three, the statement "S_s is convex whenever S is convex" is equivalent to the existence of an inner product in E, while in two-dimensional spaces it is equivalent to strict convexity. We say that S is *proximinal* if for each $x \in E$ there is at least one $s \in S$ such that $x \in S_s$. If E is smooth and strictly convex a proximinal sets S is convex if and only if for each $s \in S$, S_s is a cone with vertex s. Finally, in an inner product space, a closed set T is convex if and only if there exist a set S and a point s in S such that $T = S_s$. (Received July 17, 1956.)

769t. Geert Prins and Frank Harary: *Some enumeration problems for trees.*

Using the methods of G. Pólya, Acta Math. vol. 68 (1937) pp. 145–254, and R. Otter, Annals of Math. vol. 49 (1949) pp. 583–599, several enumeration problems for trees are solved. The results include the derivation of generating functions for the number of rooted as well as free (or unrooted) trees of the following kinds: (a) labelled or multiply-rooted, (b) having a given vertex-degree partition, (c) oriented, (d) directed, (e) signed, (f) of given strength, (g) of given type, (h) having a given diameter, (i) having a given vertex-weight distribution, (j) homeomorphically irreducible, and also (k) having the identity group as their automorphism group. Results (a) and (c) above, as well as several others, have also been independently obtained in a recent unpublished manuscript by J. Riordan, *The enumeration of labelled trees*. (Received June 11, 1956.)

770. B. M. Stewart: *Asymmetry of convex curves with respect to centroids of area.*

Let the degree to which a point P serves as a center of symmetry for a closed plane curve C be measured by reflecting C in the point P to obtain the curve C' and considering the ratio $r(P)$ of the area common to C and C' to the area bounded by C. A. S. Besicovitch has shown when C is convex there exists a point P for which $r(P) \geq 2/3$; and for a triangle the maximum value for $r(P)$ is $2/3$ which is attained only when P is the centroid. We show by independent argument that if C is convex, the centroid G of the area bounded by C has the property $r(G) \geq 2/3$. We consider a conjecture of P. C. Hammer that if C is convex, there is a unique P^* (not usually G) for which $r(P)$ is maximal. We show when C is not convex that $r(P^*)$ may be arbitrarily small and that P^* need not be unique. (Received July 6, 1956.)

771. T. T. Tanimoto: *Minimal simple paths spanning a finite set.*

Given n points $P_1, P_2, \cdots, P_n$ in an arbitrary space, and the distance $d(P_i, P_j) = a_{ij} > 0$ for every pair of distinct points P_i and P_j, we define the weight W_i of the point P_i by $W_i = \sum_j a_{ij}$ and the weight w_{ij} of the segment (P_i, P_j) by $w_{ij} = (W_i + W_j)/2 - a_{ij}$. For any simple path Γ (homeomorphic to a Jordan arc) through all the points with P_1 and P_2 as end points, we show that $W = (T_1 + T_2)/2 - L$ where W is the sum of the weights of the segments of Γ, and L its length, and where $T_1 = \sum_i W_i - W_1$ and $T_2 = \sum_i W_i - W_2$ are constants L is minimized by maximizing the quotient W/L. To maximize the quotient we define q-numbers as ordered pairs of positive numbers (a, b) with addition defined componentwise and whose value is given by the quotient b/a. When given a finite set of q-numbers it is shown that the maximum-valued r-termed q-sum is found by a sequential maximization of partial q-sums starting with the maximum-valued term. The minimum simple path spanning the n points with the end points unspecified is determined by a sequential maximization of the values of the partial q-sums consisting of terms $q_{ij} = (a_{ij}, w_{ij})$ with the graphical restrictions. The segments appear in a not-necessarily-connected sequence. (Received July 6, 1956.)

772. F. A. Valentine: *A three point convexity property.*

A set $S \subset E_m$ is said to possess the three point convexity property P_3 if for each triple of points x, y, z in S at least one of the closed segments xy, yz, xz is in S. Theorem 1: *Suppose S is a closed connected set in E_2 (the Euclidean plane) having property P_3. Then S is expressible as the union of $n \leq 3$ closed convex sets having a nonempty intersection.* A point of S at which S is not locally convex is called a q-point, and the cardinality of the set of all q points of S is denoted by N. Theorem 2: *If N is not an odd integer greater than 1, then the set S in Theorem 1 can be expressed as the union of $n \leq 2$ closed convex sets having a nonempty intersection.* (Received July 1, 1956.)

773*t*. Donald Warncke and Fred Supnick: *On the covering of E_n by spheres.*

Let S_n denote the set of n-spheres with radii of length $(n^{1/2})/2$ and centers on the lattice points of a rectangular Cartesian coordinate system in E_n (Euclidean n-space). Since $(n^{1/2})/2$ is half the largest diagonal of the unit n-cube, every point of E_n falls on or within some of the spheres of S_n. For $n = 1, 2, 3$ if an n-sphere is removed from S_n, then certain points of E_n are not covered by the remaining spheres. However, for $n > 3$ proper subsets of S_n cover E_n completely. (Let $[x]$ denote the greatest integer less than or equal to x.) The following theorem is established: *Each point of E_n $(n \geq 4)$ is on or within some n-sphere with radius of length $(n^{1/2})/2$ and center at a lattice point $(y_1, \cdots, y_n)$ for which* $\sum_{i=1}^{n} y_i = 0 \bmod ([n/4]+1)$. (Received May 29, 1956.)

774. R. M. Winger: *On generalized Cotes' spirals.*

The curves considered have the polar equation $\rho = a \sec p\theta/q + k$, p, q integer, a, k real. The paper is the third of a trilogy by the present author concerning curves with analogous equations. The other cases were also discussed respectively by Moritz and Stratton, under the names cyclic-harmonic and polar tangent curves, employing mainly polar coordinates. Compact parametric equations are derived here, permitting the use of the technique for dealing with rational curves. The singularities and symmetry, together with the implied group properties, are studied. (Received July 5, 1956.)

LOGIC AND FOUNDATIONS

775*t*. C. C. Chang: *On the unions of chains of models.*

J. Łos and R. Suszko recently announced the following: I. An arithmetical class (for terminology and symbolism cf. Tarski, Indag. Math. vol. 16 (1954) pp. 572–588) K is determined solely by a set Σ of sentences σ each of which is logically equivalent to a sentence of the form (1) $\bigvee_{x_1 \cdots x_n} \bigvee_{y_1 \cdots y_m} \phi(x_1, \cdots, x_n, y_1, \cdots, y_m)$, where ϕ contains no quantifiers, if and only if K is closed under unions of chains of models from K (i.e., $U(K) \subset K$). (I) has been improved by the author to: II. If K is an arithmetical class, then the arithmetical closure L of the class $U(K)$ is also an arithmetical class and, furthermore, L is determined by sentences of the form (1). It is clear that (II) is a stronger result than (I). It is also known that in (II) L can not always be taken simply to be $U(K)$. The method used on proving (II) is essentially different from the method of Łoś and Suszko and consists in a mathematical characterization of classes K (of models) which are arithmetical classes determined by sentences of the form (1). The above quoted result of Łoś and Suszko can be found in Bull. de l'Acad. Polonaise des Sciences, Classe III, vol. 3 (1955) pp. 201–202. (Received July 5, 1956.)

776t. C. C. Chang: *On arithmetical classes which are closed under homomorphisms.*

It is known that an arithmetical class K of algebras (for terminology and symbolism see Tarski, Indag. Math. vol. 17 (1955) pp. 56–64) is closed under homomorphisms (i.e., $H(K) \subseteq K$) if the sentences σ determining K are all logically equivalent to sentences of the form (1) $Q\phi$ where Q is a sequence of quantifiers, ϕ contains no quantifiers, and ϕ is a conjunction of disjunctions of equalities. A sentence of form (1) is a ***universal*** (***existential***) ***positive*** sentence if Q is a sequence of universal (existential) quantifiers. This abstract presents a partial solution to the converse of the above result. I. If K is a universal (existential) class of algebras then the class $H(K)$ (the arithmetical closure of $H(K)$) is also a universal (existential) class and, furthermore, $H(K)$ (the closure of $H(K)$) is determined by universal (existential) positive sentences. II. If K is a universal (existential) class, then $H(K) \subseteq K$ if and only if K is determined by universal (existential) positive sentences. The proof depends on a mathematical characterization of universal and existential positive classes and the introduction of a notion of generalized subalgebras and congruence relations of an algebra. (Received July 5, 1956.)

777. P. C. Gilmore: *An interpretation for some set theories.* Preliminary report.

Let L be the first order theory with e and m as only primitive predicates and $(x)(y)(e(x, y) \supset m(x, y))$ as only axiom. Let $s(x)$ be $(u)m(u, x) \cdot (u)(v)((w)(e(w, u) \equiv e(w, v)) \cdot e(u, x) \cdot \supset e(v, x))$, $x \epsilon y$ be $e(x, y) \cdot s(x) \cdot s(y)$, $x \notin y$ be $\sim e(x, y) \cdot s(x) \cdot s(y)$, and $C(x)$ be $(Ez)(e(x, z) \cdot (u)(e(u, z) \supset s(u)) \cdot (u)(v)(e(u, v) \cdot e(v, z) \cdot \supset e(u, z)))$. Interpretating $C(x)$ as the range of the variables of the New Foundations of Quine (NF), $x \in y$ as set membership and $x \notin y$ its complement, a translation P' in L of any closed statement P of NF can be constructed. Let LNF be L with P' as added axiom for any P which is the closure of a member of the Aussonderungs scheme (R3) of NF. Then it is shown that for any closed P, P' is provable in LNF if and only if P is provable in NF. Similar results can be proved for several other first order set theories as well as for the monadic simple theory of types. In each case the extension of L in which the set theory is interpreted is an extension by the addition as axioms of statements which can be interpreted as asserting the existence of certain objects. Since e and m can be interpreted as relations between concrete objects (abstract Journal of Symbolic Logic, June, 1956) several set theories can be interpreted in terms of concrete objects and relations between them. (Received June 25, 1956.)

778t. Richard Montague: *Models of set theories.*

For terminology see Abstract 738t. Theorem 1: *If T is a set theory in which the axiom of extensionality is valid and which has a model of class $\mathfrak{S}_1$ having the power $\aleph_\alpha$, then T has a model, of class $\mathfrak{S}_2$, of any power $\aleph_\beta \leq \aleph_\alpha$.* Theorem 2. *If T is a set theory* (e.g., *ZF*) *in which the axiom of extensionality is valid and which has a model of class $\mathfrak{S}_1$, then $r_1T = r_2T$, and both are countable.* The proofs of Theorems 1 and 2 depend on Theorem

3 of Mostowski, Fund. Math. vol. 36 (1949) pp. 141–164, together with the generalized Löwenheim-Skolem Theorem. Assume the existence of ϑ, the first strongly inaccessible ordinal $>\omega$. Theorem 3. $r_3 ZF < \vartheta$. Thus ZF does not guarantee (in the sense of models of class S_3) the existence of all sets of accessible rank. (Received July 6, 1956.)

779. Steven Orey: *Regular classes of relational systems*. Preliminary report.

Let $(A; \bar{R}) = \langle A, R_0, \cdots, R_\xi, \cdots \rangle$ be a relational system in the sense of Tarski (for terminology see Tarski, Proc. Koninklijke Nederlandse Akad. van Wetenschappen, vol. 57, pp. 572–581) and let $F(\bar{R})$ be the union of the fields of $R_0, \cdots, R_\xi, \cdots$. If C is a class of relational systems the *regularisation of* C, $\widehat{C}$, is to consist of all $(A; \bar{R})$ such that for some B, $(B; \bar{R}) \in C$. It is proved that if C is an arithmetic class so is $\widehat{C}$. On the other hand, for some higher order systems L there are formulas ϕ (containing higher order bound quantifiers) corresponding to classes of relational systems C (the models of ϕ) such that $\widehat{C}$ does not correspond to any formula of L. (Received July 6, 1956.)

780. R. M. Robinson: *Restricted set-theoretical definitions in arithmetic*.

It is known that addition of natural numbers is set-theoretically definable in terms of successor, but not arithmetically definable. A. Tarski has proposed consideration of an intermediate type of definition, in which variables ranging over natural numbers and variables ranging over sets of natural numbers are allowed, and has raised the question whether addition is definable in terms of successor in this sense. As a contribution to this problem, it is shown that addition $(x+y)$ is definable in terms of successor $(x+1)$ and double $(x+x)$. (Received May 23, 1956.)

781. R. M. Smullyan: *Elementary formal systems*.

Elementary formal systems are a simplification of Post's canonical languages; "productions" are replaced by the rules of substitution and detachment (modus ponens). Let $a_1, \cdots, a_n, x_1, \cdots, x_k, b_1, \cdots, b_r, \rightarrow$ be a collection of symbols. The x_i are called "variables" (over the strings in the a_i); the b_i, "auxiliary letters"; the sign "$\rightarrow$" is for *logical implication*. An atomic string is one which lacks "$\rightarrow$". An atomic string F_1, or a string $F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_n$, where each F_i is atomic, is a w.f.f. (well formed formula). A set $\mathfrak{F}$ of w.f.f.s $A_1, \cdots, A_j$ is an *elementary formal system* over the signs $a_1, \cdots, a_n$; the A_i are the *axioms* of $\mathfrak{F}$. A *theorem* of $\mathfrak{F}$ is a string derivable from the axioms by rules R1—substitution of strings in the a_i for variables; R2—inferring X_2 from X_1 and $X_1 \rightarrow X_2$, providing X_1 is atomic. In $\mathfrak{F}$, an auxiliary letter b_i *represents* the set of all strings X (in the a_i) such that $b_i X$ is a theorem of $\mathfrak{F}$. Representability in these systems is provably equivalent to representability in Post's systems; this includes all "formal" or "finitary" languages. Elementary formal systems yield a very simple construction of an undecidable formal system—this leads readily to Godel's incompleteness theorem. The proof of its undecidability is vastly simpler than Post's proof for the Universal Canonical system; Post's normal form theorem is completely circumvented. (The work in this paper was sponsored jointly by the Army, Navy and Air Force under contract with Massachusetts Institute of Technology. The author is a Staff Member, at Lincoln Laboratory, Massachusetts Institute of Technology. (Received July 5, 1956.)

782*t*. R. M. Smullyan: *On definability by recursion.*

A new approach to recursive enumerability is considered based on the notion of "minimal models." A formula of the lower functional calculus of the form $F_1 \cdot F_2 \cdots F_{n-1} \cdot \supset F_n$ (or F_1 alone, if $n=1$) in which each F_i is atomic, and F_n contains no predicate constants, is termed *regular*. Let A be a finite set of regular formulae; Σ a collection of sets and relations, on some universe U; I an interpretation of the predicate constants (occurring in A) as elements of Σ. The ordered triple $\mathfrak{L}$ viz. (A, U, I) is a *recursive logic* over Σ. A *model* of $\mathfrak{L}$ is an interpretation of the predicate *variables* P_i in which each formula of A is valid. Let P_i^* be the intersection of all attributes assignable to P_i in some model; these P_i^* are called *definable* in $\mathfrak{L}$. If each P_i is interpreted as P_i^*, it can be proved that there is a model—this is the *minimal model*. Sets definable in some $\mathfrak{L}$ over Σ are termed *recursively definable* from Σ. It is proved: (1) the recursively enumerable sets are precisely those which are recursively definable from the successor relation and the unit set $\{0\}$; (2) Post's canonical sets in an alphabet $a_1, \cdots, a_n$, are those recursively definable from the concatenation relation and the unit sets $\{a_1\}, \cdots, \{a_n\}$. (The work in this paper was sponsored jointly by the Army, Navy and Air Force under contract with Massachusetts Institute of Technology. The author is a Staff Member at Lincoln Laboratory, Massachusetts Institute of Technology.) (Received July 5, 1956.)

783*t*. Alfred Tarski: *Notions of proper models for set theories.*

A set theory T is a theory with standard formalization (cf. Tarski-Mostowski-Robinson, *Undecidable theories*, 1953) and with binary predicate "$\in$" as the only nonlogical constant. *General set theory*, S, is based upon the axioms of extensionality, union, power set, pairs, and regularity, and the "Aussonderungsschema"; the axiom of choice may or may not be included. *Zermelo set theory*, Z, has, in addition, the axiom of infinity. *Zermelo-Fraenkel set theory*, ZF, has, further, the replacement schema. A model $\mathfrak{M} = \langle M, R\rangle$ of a set theory T is a *proper model of the first class* ($\mathfrak{M} \in \mathsf{S}_1$) if $R = E_M$, i.e., $R = \{\langle x, y\rangle / x \in y \text{ and } x, y \in M\}$; $\mathfrak{M} \in \mathsf{S}_2$ if, in addition, $x \in y \in M$ always implies $x \in M$; $\mathfrak{M} \in \mathsf{S}_3$ if, further, $x \subseteq y \in M$ always implies $x \in M$; $\mathfrak{M} \in \mathsf{S}_4$ if, for some ordinal α, M is of the form $\mathfrak{A}_\alpha = \langle R(\alpha), E_{R(\alpha)}\rangle$, where $R(\alpha) = \{x / \rho x < \alpha\}$ (cf. Tarski, Bull. Amer. Math. Soc. Abstract 61-5-628). The rank $r_i T$ of a set theory T ($i = 1, 2, 3, 4$) is the least ordinal α such that $\alpha = \rho M$ for some model $\langle M, R\rangle$ of T of class S_i. Clearly, $r_1 T \leq r_2 T \leq r_3 T \leq r_4 T$; $r_i S = \omega$ and $r_i Z = \omega + \omega$ for $i = 1, 2, 3, 4$. For deeper results see Abstracts 778 and 784 of Montague and Vaught. (Received August 13, 1956.)

784*t*. R. L. Vaught: *On models of some strong set theories.*

For terminology see Abstracts 778 and 783. A set A is *definable* in the model $\mathfrak{M} = \langle M, R\rangle$ of a set theory T if, for some formula ϕ of T with one free variable, A is the unique member of M satisfying ϕ in $\mathfrak{M}$. Theorem 1. *If* $\mathfrak{A}_\alpha = \langle R(\alpha), E_{R(\alpha)}\rangle$ *is a model of* ZF, *and* T_α *is the set theory whose valid sentences are all sentences true in* $\mathfrak{A}_\alpha$, *then the union of all sets definable in* $\mathfrak{A}_\alpha$ *is equal to* $R(\beta)$ *for a certain* $\beta \leq \alpha$, β *is confinal with* ω, *and* A_β *is a model of* T_α (indeed, A_α is an "arithmetical extension" of A_β—cf. Tarski-Vaught, *Arithmetical extensions of relational systems*, to appear in Comp. Math.). The proof uses a device employed previously by Montague (Bull. Amer. Math. Soc. Abstract 61-2-344). Corollary: $r_4 ZF < r_4 T_\vartheta < \vartheta$. (That $r_4 ZF < \vartheta$ also follows from an earlier result of Montague (Theorem 3 of Abstract 778) and the Lemma: *If* $\mathfrak{M} \in \mathsf{S}_3$ *and* $\mathfrak{M}$ *is a model of* ZF, *then* $\mathfrak{M} \in \mathsf{S}_4$.) Thus neither ZF nor T_ϑ insures (in the sense of models of class S_4) the existence of all sets of accessible rank. (Received August 13, 1956.)

Statistics and Probability

785t. G. A. Baker, Jr. and G. A. Baker, Sr.: *Peculiar speeds of stars.*

For the case of the single drift hypothesis of the distributions of velocities where the three space components of each of the observed stars are known without error and the three components of the velocity of the "local standard of rest" are known exactly, the distribution of the peculiar speeds is easily written down and is seen to depend on a single parameter. This case and successive generalizations about the local standard of rest, fundamental character of the distributions of space velocities, and knowledge of parameters are treated in this paper. The principal tools are modified maximum likelihood estimation, moment-generating functions, Laplace transforms, and expected values. The results consist of frequency distributions of peculiar speeds of stars under stated assumptions, confidence limits for estimates of parameters, and the effect of lack of exact knowledge of the local standard of rest on the distribution of the peculiar speeds of stars. (Received June 4, 1956.)

786. G. E. Baxter: *Some conditional probability distribution functions.*

Let $\{x(t), 0<t\leq\infty\}$ be a separable stochastic process with stationary, independent increments for which $x(0)=0$. Then for any $\sigma>0$ $(s, \lambda>0)$ define $N(\alpha, T, \sigma)$ by (1) $N(0, T, \sigma)=0$ where $0<T<\infty$, (2) $\int_0^\infty\int_0^\infty e^{-\lambda\alpha-sT}d_\alpha N(\alpha, T, \sigma)dT$ $=\exp[\int_0^\infty\int_0^\infty(e^{-\lambda\alpha-sT}-e^{-sT})/T\,d_\alpha P\{x(T)<\alpha\}dT]$ $-$ $\exp[\int_0^\infty e^{-sT}/T\,P\{x(T)\geq 0\}dT]$. Then, for any $\alpha\geq 0$, the conditional probability distribution function $P\{x(T)<\alpha\mid x(t)\geq 0 \text{ for all } 0\leq t\leq T\}=N(\alpha, T, \sigma)/N(\infty, T, \sigma)$. In the special case $P\{x(T)\geq 0\}=1/2$, it follows from this and the limiting form of Spitzer's identity (Bull. Amer. Math. Soc. Abstract 61-6-819) that $\int_0^\infty e^{-sT}N(\alpha, T, \sigma)dT=(\sigma s)^{1/2}\int_0^\infty e^{-sT}P\{\sup_{0<t\leq T} x(t)<\alpha\}dT$. A formula is also found for the distribution $P\{x(T)\leq\alpha\mid x(t)>0 \text{ for all } 0<t\leq T\}$, and calculations are made for certain special cases. (Received July 19, 1956.)

787t. A. T. Bharucha-Reid: *On random elements in Orlicz spaces.* Preliminary report.

Let $(\Omega, \mathfrak{A}, p)$ be a probability space, and let $x=x(\omega)$ be a function on Ω to $\mathfrak{X}$. $x(\omega)$ is called a random element in $\mathfrak{X}$. In this note we begin the study of random elements in Orlicz spaces, that is when $\mathfrak{X}=L_\Phi(\Omega, p)$. The expectation of x, $m=\mathcal{E}x$, is defined by the Pettis integral, and is that element of $\mathfrak{X}$ such that for all $x^*\in\mathfrak{X}^*$, $x^*(m)$ $=\int_\Omega x^*(x(\omega))dp$, where $x^*(x)=\int_\Omega x(\omega)y(\omega)dp$ is a bounded linear functional on $\mathfrak{X}$ to $\mathfrak{X}^*$, and $y\in L_\Psi$ (the conjugate space) is a uniquely determined random element. We consider in particular laws of large numbers for random elements in Orlicz spaces. Since L_Φ contains the same random elements as L_p when $\Phi(s)=\alpha s^p$, $\alpha>0$, $1\leq p<\infty$, these theorems yield laws of large numbers for random elements in L_p spaces as a special case. (Received June 4, 1956.)

788t. Aryeh Dvoretzky: *Almost sure everywhere divergence of random series.*

It is well known that if a_n $(n=0, 1, 2, \cdots)$ is a sequence of complex numbers and $r_n(t)$ $(n=0, 1, 2, \cdots)$ denote the Rademacher functions then the series $\sum_{n=0}^\infty r_n(t)a_n$

diverges for almost all t (in the usual Lebesgue sense) if and only if $\sum_{n=0}^{\infty} |a_n|^2$ is divergent. It follows immediately, by Fubini's theorem, that if $\sum_{n=0}^{\infty} |a_n|^2 + \infty$, then for almost all t the series $\sum_{n=0}^{\infty} r_n(t)a_n z^n$ diverges *almost* everywhere on the circle $|z|=1$ (again in the $(n=0, 1, 2, \cdots)$) is monotone and $|a_n|^{1+\epsilon}=\infty$ for some $\epsilon>0$ then the italicized *almost* in the preceding statement may be dropped; i.e. under the conditions $\sum_{n=0}^{\infty} r_n(t)a_n z^n$ is, for almost all t, nowhere convergent on $|z|=1$. Similar results obtain for other types of random power series as well as for trigonometric series, Dirichlet series, etc. (Received July 3, 1956.)

789. L. R. Ford, Jr.: *Solution of a ranking problem from paired comparisons.*

Given a square matrix $A=(a_{ij})$, with $a_{ij}\geq 0$, $a_{ii}=0$, where a_{ij} is interpreted as the number of times i has been preferred to j, weights are assigned to each index by a maximum likelihood method. The ratio $w_i:w_j$ is interpreted as the odds in a single ij comparison, and the w_i are determined to maximize the a priori probability of A. Under a weak restriction on A, existence and uniqueness of the maximizing w are shown in the region $\{w_i>0; \sum w_i=1\}$; an iterative scheme for computing w is given and shown to converge to the desired solution. (Received June 28, 1956.)

790. L. L. Helms: *A mean martingale convergence theorem.*

Let $(E, \mathfrak{B}, \mu)$ be a measure space consisting of a set E, a σ-algebra $\mathfrak{B}$ of subsets of E, and a finite measure μ defined on $\mathfrak{B}$, and let $(A, >)$ be an arbitrary directed set. L^p, $1\leq p<\infty$, will denote the totality of functions which are measurable relative to $\mathfrak{B}$ and pth power integrable with respect to μ. A net $\{x_a, a\in A\}\subset L^1$ is called a martingale if there is a monotone increasing net $\{\mathfrak{B}_a, a\in A\}$ of σ-subalgebras of $\mathfrak{B}$ such that x_a is measurable relative to $\mathfrak{B}_a$ and $\int_S x_a d\mu=\int_S x_b d\mu$ for $S\in\mathfrak{B}_a$ and $b>a$. The directed set $(A_\infty, >)$ is obtained from $(A, >)$ by adjoining an ideal element "∞" to A and defining $\infty>a$ for all $a\in A_\infty=A\cup\{\infty\}$. The author shows that the following four properties of a martingale $\{x_a, a\in A\}\subset L^p$, $1\leq p<\infty$, are equivalent: (i) the net $\{x_a, a\in A\}$ is uniformly integrable if $p=1$ or strongly bounded if $p>1$. (ii) the net $\{x_a, a\in A\}$ converges weakly in L^p; (iii) there is an $x_\infty\in L^p$ such that the net $\{x_a, a\in A_\infty\}$ is a martingale; and (iv) the net $\{x_a, a\in A\}$ converges strongly in L^p. (Received June 15, 1956.)

791. Samuel Karlin and J. L. McGregor (p): *Coincidence probabilities.* Preliminary report.

Let $P(t)=(P_{ij}(t))$, $t\geq 0$, be the transition matrix of a Markoff process whose state space is a finite or infinite set of consecutive integers. The path functions are assumed to be continuous, i.e., a transition from i to j cannot occur without visiting all intervening states. Let Δ denote the n-square determinant formed by the rows of $P(t)$ with indices $i_1<i_2<\cdots<i_n$ and the columns with indices $j_1<j_2<\cdots<j_n$. It is shown that Δ is the probability that if n particles start at $t=0$ at $i_1, \cdots, i_n$, then at time t they are at $j_1, \cdots, j_n$ without any two of them ever having been coincident. The analogous but more involved result when the path functions are not necessarily continuous is also obtained. (Received July 2, 1956.)

792. E. G. Kimme: *A general convergence theorem for sequences of stochastic processes.*

It has been shown that if $\{x_n(t, \omega), t\in[0, 1], n\geq 1\}$ is a sequence of separable stochastic processes with independent increments converging uniformly in distribution to separable stochastic process with independent increments and no fixed points of discontinuity $\{x(t, \omega), t \in [0, 1]\}$, then for a large class of functionals $F[\ ]$ defined on the sample functions of these processes, $F[x_n(\cdot, (\omega)]$ converges in distribution to $F[x(\cdot, \omega)]$. The present paper removes the conditions of independence from the stated result. (Received June 28, 1956.)

793. Klaus Krickeberg: *Stochastic convergence of semimartingales.*

Let $\mathfrak{B}$ be a Boolean sigma-algebra with unit E bearing a strictly positive and finite measure μ, and $\mathfrak{F}$ the space of all $\mathfrak{B}$-measurable extended real-valued functions on E (functionoids, "Ortsfunktionen), $\mathfrak{B}$ and $\mathfrak{F}$ being complete lattices under their order relations $\leqq$. Given a Moore-Smith sequence $(f_\sigma; \sigma \in \Theta)$ with $f_\sigma \in \mathfrak{F}$ its stochastic limit superior, $s \lim \sup_\sigma f_\sigma$, is defined as the infimum of all h in $\mathfrak{F}$ such that $\lim_\sigma \mu\{h' < f_\sigma\} = 0$ for every h' in $\mathfrak{F}$ with $h \leqq h'$ and $\{h' = h\} = \{h = +\infty\}$. Likewise $s \lim \inf_\sigma f_\sigma$ is obtained. Both limits are independent of μ and have the following properties: $s \lim \inf_\sigma f_\sigma \leqq s \lim \sup_\sigma f_\sigma$, and equality holds if and only if (f_σ) converges stochastically; if each f_σ is measurable relative to a fixed sigma-subalgebra $\mathfrak{A}$ of $\mathfrak{B}$, the stochastic limits are $\mathfrak{A}$-measurable, too; $s \lim \sup_\sigma f_\sigma \leqq \lim \sup_\sigma f_\sigma$, where lim sup denotes the "order" limit superior in $\mathfrak{F}$; Fatou's lemma $\lim \sup_\sigma \int_E f_\sigma d\mu \leqq \int_E (s \lim \sup_\sigma f_\sigma) d\mu$ holds if (f_σ) is terminally uniformly integrable. With the help of these stochastic limits inferior and superior it can be shown that each martingale $(f_\sigma; \sigma \in \Theta)$ relative to an increasing Moore-Smith sequence $(\mathfrak{B}_\sigma; \sigma \in \Theta)$ of sigma-subalgebras of $\mathfrak{B}$ converges stochastically if the positive parts f_σ^+ or the negative parts f_σ^- have bounded expectations. Under similar assumptions any semimartingale relative to $(\mathfrak{B}_\sigma)$ is stochastically convergent, too. (Received July 5, 1956.)

794. R. B. Leipnik: *Moment generating functions of quadratic forms in serially correlated normal variables.*

An integral equation method used by Kac in the study of Wiener functionals is successfully adapted to the problem of calculating in closed algebraic form joint moment generating functions of linear combinations of four special quadratic forms in normal variables. The class of gaussian processes for which this method works is characterized. The autocovariance function is either of the form $\sigma^2 \rho^{|t_2 - t_1|} + \rho^{t_1 + t_2} \cdot (A(t_1) + A(t_2))$ or $\sigma^2 \min(t_1, t_2) + A(t_1) + A(t_2)$. The integral equation is solved with the help of Mehler's identity. For the case of the Uhlenbeck-Ornstein process $(A(t) = 0)$ the moment generating functions are studied for various quadratic estimators of σ^2 and ρ. Asymptotic results as the number of observations $\rightarrow \infty$, and time between observations $\rightarrow 0$ are obtained. (Received June 28, 1956.)

795*t*. George Marsaglia: *Regression for independent variables.*

Consider random variables $x_1, \cdots, x_n, y_1, \cdots, y_n$ and assumptions of the type $E(y_i) = aE(x_i) + b$. For normal variables, this may be viewed as an extension of the general linear hypothesis, in that the vector of expected values is assumed to lie in the union of a collection of vector spaces, rather than in a single specified vector space. This weaker restriction on the expected values requires a more stringent condition on the covariance matrix, to ensure the existence of the maximum likelihood estimates. The principal result of this note provides such estimates for the case of independent x's and y's, allowing multiple determinations. An incidental result is that the estimates of a and b provide, in terms of line fitting, the line which minimizes the sum of the squares of the distances from the points to the line. (Received July 23, 1956.)

TOPOLOGY

796. L. W. Anderson: *On bounded subsets of topological lattices.*

A topological lattice is a Hausdorff space, L, together with a pair of continuous functions, $\wedge: L\times L\rightarrow L$ and $\vee: L\times L\rightarrow L$, which satisfy the usual conditions stipulated for a lattice. If L is a locally compact connected topological lattice which is homeomorphic with a subset of Euclidean 2-space and if C is a compact subset of L then there are elements, a and b, in L such that $a\leqq x\leqq b$ for all $x\in C$. (Received July 5, 1956.)

797. R. D. Anderson: *One-dimensional continuous curves and a homogeneity theorem.*

Let $\mathfrak{K}$ be the class of all one-dimensional locally connected compact metric continua up to topological equivalence. Let $\mathfrak{M}$ be the subclass of $\mathfrak{K}$ consisting of those elements of $\mathfrak{K}$ without local cut points. The following three theorems are proved. I. In order that an element M of $\mathfrak{M}$ be the universal curve it is necessary and sufficient that no open subset of M be imbeddable in the plane. II. The simple closed curve and the universal curve are the only homogeneous elements of $\mathfrak{K}$. III. In order that an element K of $\mathfrak{K}$ be the universal curve it is necessary and sufficient that K be cross-connected. A continuum X is said to be weakly cross-connected provided that for every two disjoint pairs (A, A') and (B, B') of points of X there exist two disjoint arcs in C containing $A+A'$ and $B+B'$ respectively. A locally connected continuum X is said to be cross-connected provided that for every connected open subset D of X for which Cl (D) is locally connected, Cl (D) is weakly cross-connected. (Received June 29, 1956.)

798t. R. D. Anderson: *Zero-dimensional compact groups of homeomorphisms.*

The author shows that each zero-dimensional compact metric topological transformation group can operate on a universal curve in a fixed point free manner, i.e. so that each element of the group other than the identity moves every point of the space (see the author's other abstract, this meeting, for some characterizations of the universal curve). Furthermore the space of orbits may be required to be a universal curve itself or, in case the group is infinite, the space of orbits may be required to be a regular curve. In the course of the proof of the above theorems it is also shown that for each finite group G there exists an orientable 2-manifold $M(G)$ such that G can operate on $M(G)$ in a fixed point free manner. (Received June 29, 1956.)

799. R. H. Bing: *A necessary and sufficient condition that a 3-manifold be S^3.*

It is shown that a compact 3-manifold M is topologically S^3 if each simple closed J in M lies in a topological cube K in M. The outline of the proof is as follows. First, M is triangulated so as to give it an affine structure. Next it is shown that M is topologically S^3 if the 1-skeleton R_1 of this triangulation lies on the interior of a polyhedral cube. Then it is shown that if J is a polyhedral simple closed curve in M, then J lies on the interior of a polyhedral cube in M. Finally a polyhedral simple closed curve J is knotted about R_1 in such a way that any polyhedral cube that contains J on its interior can be adjusted so as to contain R_1 on its interior. (Received June 29, 1956.)

800t. R. E. Chamberlin: *The fixed point property for certain continua.*

A tree is a finite, acyclic 1-dimensional complex. A tree-like continuum is a compact metric space with a cofinal sequence of open coverings whose nerves are trees. X has the fixed point property if for every continuous $F: X \to X$ there is a p with $F(p) = p$. Theorem: Every tree-like continuum has the fixed point property. If T is a tree and p and q are points of T let $[p, q]$ denote the smallest connected set containing p and q. Definition: If $f: T \to T'$, $g: T \to T'$ are two simplicial mappings of a tree T into a tree T', f dominates g if (a) f is onto and (b) for every $p_1, p_2 \in T$, there exist $q_1, q_2 \in T$ with $f(q_1) = g(p_1)$, $f(q_2) = g(p_2)$ and such that $f([q_1, q_2]) \subset g([p_1, p_2])$. The proof of the Theorem depends on the Lemma: Given simplicial mappings $f, g: T \to T'$ such that f dominates g. Then there is a p with $f(p) = g(p)$. Suppose $F: X \to X$ is continuous and $\{U_i\}$ and $\{V_j\}$ are finite open coverings of X with nerves T and T' respectively. If for each i there is a j with $F(U_i) \subset V_j$, the simplicial mapping $f: T \to T'$ defined by assigning the vertex corresponding to U_i to the vertex corresponding to some V_j with $F(U_i) \subset V_j$ is said to be induced by F. Let $F: X \to X$ be the identity and $G: X \to X$ be continuous. If X is tree-like, there is a cofinal sequence of open coverings with nerves $\{T_n\}$ and mappings $f_n, g_n: T_{n+1} \to T_n$ induced by F and G respectively with f_n dominating g_n. This fact, together with the Lemma proves the Theorem. (Received July 5, 1956.)

801. Haskell Cohen and L. I. Wade: *Clans with zero on an interval.*

A clan is a compact connected Hausdorff topological semigroup with unit. Let S be a clan with zero on an interval. It is known that the unit 1 is an endpoint. Let d be the other endpoint, L be the closed interval $[d, 0]$, and R be the closed interval $[0, 1]$. It can be shown that R is an abelian subclan, and its structure has been characterized in an unpublished work of P. S. Mostert and A. L. Shields. It is now shown that (1) RL and LR are contained in L, (2) either $L^2 \subset L$ or $L^2 \subset R$, (3) L is abelian. Also, under the additional hypothesis that $\{1 \mid 1 \in L \text{ and } 1^2 = d^2\} = d$, it follows that (4) S is abelian, (5) if $L^2 \subset L$ then L is a homomorphic image of an ideal in R, (6) if $L^2 \subset R$ then L is a copy of an ideal in R with its products in R as in the interval from $-1/2$ to 1 under ordinary multiplication. (Received July 6, 1956.)

802. Leonard Gillman: *Rings with Hausdorff structure space.* I.

Let $A = \{ \cdots, x, \cdots \}$ be semi-simple, $\mathfrak{P} = \{ \cdots, P, \cdots \}$ its primitive ideals, $\mathfrak{Q} = \{ \cdots, Q, \cdots \}$ its proper prime ideals, and let $\mathfrak{S} = \{ \cdots, S, \cdots \}$ satisfy $\mathfrak{P} \subset \mathfrak{S} \subset \mathfrak{Q}$. Define $\mathfrak{S}(x) = \{S: x \in S\}$; the sets $\mathfrak{S}(x)$ are a base for the closed sets in $\mathfrak{S}$. Define $N_S = \{x: \exists \text{nbhd } \mathfrak{U} \text{ of } S \text{ in } \mathfrak{S} \text{ such that } x \in \bigcap \mathfrak{U}\}$; N_S, is an ideal and $N_S \subset S$. 1. If $S \supset Q$, then $Q \supset N_S$. 2. (Due to C. W. Kohls.) Every N_S is an $\bigcap$ of Q's 3. Every $\mathfrak{S}(x)$ is open iff every $N_S = S$. 4. These are equivalent: (a) $\mathfrak{S}$ is H ($=$ Hausdorff); (b) given $S \neq S'$, $\exists x$ such that $x \in N_S$, $x \notin S'$; (c) each S is the unique S' such that $S' \supset N_S$. 5. If every $\mathfrak{S}(x)$ is open, then $\mathfrak{S}$ is H. 6. If $\mathfrak{S}$ is H, then each $Q \subset$ at most one S. 7. Corollary. If A is an integral domain with at least two P's, then $\mathfrak{P}$ is not H. 8. If $\mathfrak{S}$ is H, then these are equivalent: (a) $\mathfrak{Q} = \mathfrak{S}$; (b) every Q is an $\bigcap$ of S's; (c) every $Q \subset$ an S, and every $\mathfrak{S}(x)$ is open. 9. Corollary. $\mathfrak{Q}$ is H iff every $\mathfrak{Q}(x)$ is open. 10. Corollary. If A is biregular, then $\mathfrak{Q} = \mathfrak{P}$, and every $\mathfrak{P}(x)$ is open. 11. A is strongly regular iff (i) every A/P is a division ring, (ii) every $\mathfrak{P}(x)$ is open, and (iii) every $A/\bigcap \sim \mathfrak{P}(x)$ has an identity. (Received July 6, 1956.)

803*t*. Leonard Gillman: *Rings with Hausdorff structure space*. II.

Notation as in preceding abstract. 12. If A is a commutative ring with identity and $\mathfrak{S}$ is H, and if every finitely generated ideal is principal, then every $N_\mathfrak{s}$ is prime. 13. If A is a commutative ring with identity and $\mathfrak{P}$ is H, then these are equivalent: (a) A is regular; (b) every ideal is an $\bigcap$ of P's; (c) every Q is an $\bigcap$ of P's; (d) $\mathfrak{Q}=\mathfrak{P}$; (e) A is adequate: (f) every $\mathfrak{P}(x)$ is open. (The example $A=$ring of integers shows that these not be equivalent if $\mathfrak{P}$ is not H.) The proof of 12 makes use of the following general results. 14. For any family $\mathfrak{A}$ of prime ideals, these are equivalent: (a) for all $\mathfrak{B}\subset\mathfrak{A}$, $\bigcap\mathfrak{B}$ is prime; (b) for any $Q, Q'\in\mathfrak{A}$, $Q\cap Q'$ is prime; (c) $\mathfrak{A}$ is totally ordered under set inclusion. 15. Let A be a commutative ring with identity, let $\mathfrak{A}$ denote the family of all prime ideals contained in a given proper ideal I, and suppose that every finitely generated ideal contained in I is principal. Then $\mathfrak{A}$ is totally ordered under set inclusion. (Received July 6, 1956.)

804. Melvin Henriksen: *On minimal completely regular spaces associated with a given ring of continuous functions.*

Let $C(X)$ denote the ring of continuous real-valued functions on a completely regular space X. If X and Y are completely regular spaces, the rings $C(Y)$ and $C(X)$ are said to be ***strictly isomorphic*** if one is a dense subspace of the other, say Y is dense in X, and every $f\in C(Y)$ has a continuous extension over X. In an unpublished paper (see Bulletin Amer. Math. Soc. Abstract 62-4-545) L. J. Heider asked (in different but equivalent language) if it is possible to associate with each completely regular space X a subspace μX minimal with respect to the property that $C(\mu X)$ and $C(X)$ are strictly isomorphic. Let $\eta X=\{p\in X: C(X\sim\{p\})$ and $C(X)$ are ***not*** strictly isomorphic$\}$. ***Theorem.*** A minimal subspace μX exists if $C(\eta X)$ and $C(X)$ are strictly isomorphic, in which case $\mu X=\eta X$. Thus if μX exists, it is unique. Finally it is noted that if βN denotes the Stone-Čech compactification of the countable discrete space, then $\mu(\beta N)$ does not exist. (Received July 6, 1956.)

805*t*. R. P. Hunter: *Types of (n, k) adherence and indecomposability.*

The connexe M is said to contain an (n, k) adherent sequence of sets $P_1, P_2, \cdots, P_{n+1}$, if $M-P_1\supset M_1-P_2\supset\cdots\supset M_{n-1}-P_n\supset M_n\supset V\supset P_{n+1}$; where each M_i is a proper connexe subclosure of, and relatively closed in M_{i-1}, $M=M_0$, $P_i\cdot M_{i-1}\neq 0$ and V is open in M_k. Definitions of Swingle are used (Bull. Amer. Math. Soc. vol. 47 (1941) p. 796). The following hold in a Moore space l_3. The connexe M is indecomposable if and only if it contains no (1, 0) adherent sequence. Every proper connexe subclosure of M is indecomposable if and only if M contains no (2, 1) adherent sequence. If the compact connexe M contains a (1, 0) but no (2,1) adherent sequence it is nonclosed. The connexe M is n-indecomposable if and only if it contains n but not more than n points p_i $i=1, 2, \cdots, n$ such that p_i, p_j is a (1,0) adherent sequence. If the closed connexe M is the essential sum of n proper connexe subclosures and contains no $(n, n-1)$ adherent sequence it is n-indecomposable and indeed the essential sum of n non-compact hereditarily indecomposable continua. (Received July 2, 1956.)

806. V. L. Klee, Jr.: *Homogeneity of infinite-dimensional parallelotopes.*

A topological space X is said to be homogeneous with respect to a family S of its subsets if whenever $Y\in S$ and f is a homeomorphism of Y onto a set $fY\in S$, then f

can be extended to a homeomorphism of X onto X. It is known that Hilbert space is homogeneous with respect to its compact subsets, and the Hilbert parallelotope P with respect to its finite subsets [Trans. Amer. Math. Soc. vol. 78 (1955) pp. 30–45]. The scope of the latter result is extended by the fact that every infinite-dimensional compact convex subset of a normed linear space is homeomorphic with P [ibid.]. The present paper employs earlier methods, and especially the transformations of O. H. Keller [Math. Ann. vol. 105 (1931) pp. 748–758], to establish homogeneity of P with respect to its countable closed subsets. It is also proved that there is only one isotopy-class of homeomorphisms of P onto P, a result established earlier for Hilbert space [Trans. Amer. Math. Soc. vol. 74 (1953) pp. 10–43]. (Received July 3, 1956.)

807t. W. S. Massey: *Some cohomology invariants of higher kind.*

The object of this paper is to give some examples of cohomology operations which are homotopy type invariants of topological spaces, but are not universally defined. They are constructed by purely algebraic methods starting from a suitable (e.g. singular or Alexander-Spanier) cochain ring of the given space X with coefficients in an associative ring R. From an ordered abstract simplicial complex K and the given cochain ring one constructs (by a process which is too involved to describe here) a bi-complex. As usual, this bi-complex gives rise to two spectral sequences, only one of which is of interest. The first term E_1 of the interesting spectral sequence is a direct sum of tensor products of the cohomology ring $H^*(X, R)$ with itself, and the differential operator $d_1:E_1\rightarrow E_1$ may be expressed in terms of the products in $H^*(X, R)$. The succeeding differential operators, d_2, d_3, etc., are the desired cohomology operations of higher kind. For different choices of the abstract complex K, one obtains different spectral sequences. If K consists of a single m-simplex and all its faces, it is possible to make explicit computations and give examples where these higher operations distinguish between spaces of different homotopy types. (Received April 9, 1956.)

808. P. S. Mostert: *On a compact Lie group acting on a manifold.*

Let M be a manifold of dimension $n+1$ and G a compact Lie group acting on M in such a way that there is at least one n-dimensional orbit. Then the space of orbits M/G is homeomorphic to one of (1) a circle, (2) an open interval, (3) a half-open interval, or (4) a closed interval. Moreover, there exists a subgroup N of G such that, in (1) and (2), M is homeomorphic to $M/G\times G/N$; in case (3), there exists a subgroup $K\supset N$ such that K/N is an r-sphere for some $r\geq 0$ and such that M is homeomorphic to $(M/G\times G/N)/R$ where R is the relation identifying $p\times G/N$ to $p\times G/K$ over the end point p of M/G; in (4) a similar situation prevails with subgroups K_1 and K_2. (In this latter case, K_1/N and K_2/N may be of different dimensions). The action of G is equivalent to G acting differentiably. Applications are given in dimensions two and three. (Received June 28, 1956.)

809t. J. C. Oxtoby: *A category analogue of the zero-one law.*

Let X be the product of a sequence of topological spaces X_i, each of which has a countable base and in each of which every nonempty open set is of second category. Then X has the same two properties and every tail set in X that has the property of Baire is either of first category or residual. (A tail set is a set of the form $(P^{n}_{i=1}X_i)\times B_n$ for every n.) The proof is based on the Kuratowski-Ulam category analogue of the Fubini Theorem. (Received August 21, 1956.)

810*t*. C. D. Papakyriakopoulos: *On the ends of the fundamental groups of 3-manifolds with boundary.*

Let M be a finite 3-manifold closed ($=$without boundary) or with boundary whose components are not 2-spheres. *When is the number e of ends of $\pi_1(M)$, 0 or 1 or 2 or ∞?* This problem has been solved if M is closed, by E. Specker [Commt. Math. Helv. vol. 23 (1949) Satz VI]. If M has boundary the solution is as follows: (1) *If M is aspherical and the injection $\pi_1(N_i)\rightarrow\pi_1(M)$ is an isomorphism for every component N_i of N, then $e=1$.* (2) *If M is aspherical, connected and of genus one, and the injection $\pi_1(N)\rightarrow\pi_1(M)$ is not an isomorphism, then $e=2$.* (3) *In any other case, $e=\infty$.* The proof of (1)–(3) is based on the theorem contained in the author's abstract *On a lemma of Kneser* [Bull. Amer. Math. Soc. Abstract 62-2-279], on two theorems of the author [Ann. Math. vol. 62, no. 2 (1955) Theorems 1 and 2], and on a theorem of E. Specker [Satz VII, of the above paper]. (Received May 3, 1956.)

811. L. E. Pursell: *Abstract multi-derivatives on infinitely-differentiable manifolds.*

Let X be an ∞-differentiable manifold. An abstract derivative is a mapping $L: C^\infty(X)\rightarrow C^\infty(X)$ such that $L(af+bg)=aLf+bLg$ and $L(fg)=fLg+gLf$. An abstract multi-derivative of order r is a mapping $T_r: [C^\infty(X)]^r\rightarrow C^\infty(X)$ which is an abstract derivative in each of its independent variables when the other $r-1$ variables are held constant. It is shown that on a coordinate neighborhood where $x^{i_1}, \cdots, x^{i_r}$ are the coordinate functions $T_r(f_1, \cdots, f_r)=\sum_{i_1 \cdots i_r} (\partial f_1/\partial x^{i_1}) \cdots (\partial f_r/\partial x^{i_r}) T_r(x^{i_1}, \cdots, x^{i_r})$. Hence abstract multi-derivatives may be identified with contravariant tensor fields. (C. Chevalley, *Theory of Lie groups*, I, has a similar result for analytic manifolds.) For abstract multi-derivatives S_m, T_n we define the Kronecker or outer product S_mT_n by $(S_mT_n)(f_1, \cdots, f_{m+n})=S_m(f_1, \cdots, f_m)T_n(f_{m+1}, \cdots, f_{m+n})$. Consider sequences $\{S_r\}_{r=0}^\infty$ $\{T_r\}_{r=0}^\infty$ whose 0th element is a real number and whose rth element is a multi-derivative of order r. Define $\{S_r\}+\{T_r\}=\{S_r+T_r\}$ and $\{S_r\}\{T_r\}=\{S_0T_r+S_1T_{r-1}+ \cdots +S_rT_0\}$. These sequences form a ring which determines the space X if X is compact. For noncompact X, X is determined if one considers only multi-derivatives with compact supports. ($Z(T_r)=[x: T_r(f_1, \cdots, f_r)(x)=0$ for all $f_1, \cdots, f_r]$. Support of $T_r=\text{Cl}\ (X-Z(T_r))$.) (Received July 5, 1956.)

812*t*. G. O. Sabidussi: *On the minimum order of graphs with given automorphism group.*

Let G be a finite group. Define $\alpha(G)$ as min $\alpha_0(X)$, the minimum taken over all graphs X whose automorphism group is isomorphic to G. ($\alpha_0(X)$ is the number of vertices of X). Earlier estimates for $\alpha(G)$ are improved as follows: If G is a group of order g containing a set H of h generators such that every element of H is of order ≥ 6, then $\alpha(G)\leq g[(\log h)/(2 \log 2)+2]$. (The restriction on the order of the generators can be removed, the result being a more complicated formula). For the cyclical group C_n of order n the exact value of $\alpha(C_n)$ is determined. If G is isomorphic to a subgroup of the symmetric group S_n of order $n!$, then $\alpha(G)\leq n(n+1)(n+6)/2$. (Received June 4, 1956.)

813*t*. G. O. Sabidussi: *Graph properties independent of the automorphism group.*

Given a graph X, let P_i, $i=0, \cdots, 4$, stand for the following properties of

X: P_0:X is covariant (no automorphism of X leaves all vertices of X invariant); P_1:X is of connectivity n, $n \geq 1$; P_2:X is of chromatic number n, $n \geq 2$; P_3:X is regular of degree n, $n \geq 3$; P_4:X contains a Hamiltonian circuit and is spanned by a graph Y' homeomorphic to a given connected graph Y. The following theorem is proved: Given a finite group G and an integer i, $1 \leq i \leq 4$, there exist infinitely many nonisomorphic connected covariant graphs X such that (a) the automorphism group of X is isomorphic to G; (b) X has property P_i. (Received June 4, 1956.)

814. J. M. Slye: *Spaces with two dimensional rulings.*

If f is a mapping of a space B onto a space X, then G, the collection of inverse images of points of X, is called a ruling of B by k, a set of G, if each sequential limit set of sets of G that is not a point is homeomorphic to k. A point Q of B is called a simple point with respect to the ruling if there exists a connected open set D containing the point P, whose inverse image g contains Q, and $D-P$ is the sum of two mutually separated sets each of which contains a sequence of points whose inverse images have Q in their limit sets. A point of a set g of G is a singular point if it is a sequential limit point both of simple points and of nonsimple points of g. It is shown that a locally compact, locally connected unifoliate metric space B without local cut points, with a ruling G of regular curves with both k and X simplicial and no point that is a limit point of its singular points is a two manifold. (Received July 5, 1956.)

815t. Stephen Smale: *Regular curves on Riemannian manifolds.*

A regular curve on a Riemannian manifold is a curve with a continuously turning nontrivial tangent vector. A regular homotopy is a homotopy which at every stage is a regular curve and keeps end points and directions fixed. Let x_0 be a point of the unit tangent bundle T of a Riemannian manifold M. The space (under a certain topology) of all regular curves on M starting at the point and direction determined by x_0 is denoted by E. A map π from E onto T is defined by sending a curve into the tangent of its end point at its end point. It is proved that (E, π, T) has the covering homotopy property for polyhedra. Using this result and the fact that E is homotopically trivial one gets: There is a 1-1 correspondence between the set of classes (under regular homotopy) of regular curves on M which start and end at the point and direction determined by x_0, and $\pi_1(T, x_0)$. In a sense this is a generalization of the Whitney-Graustein Theorem (H. Whitney, *On regular closed curves in the plane*," Compositio Math. vol. 4 (1937) pp. 276–284). (Received June 25, 1956.)

816t. P. M. Swingle and R. P. Hunter: *Indecomposable trajectories in phase space.*

Let D be a connected domain in a separable Moore phase space, S_n, satisfying Axioms 1 and 2. An indecomposable trajectory is an arc-wise indecomposable connexe. The following exist densely in D: a set of n mutually exclusive indecomposable trajectories; an arc-wise trajectory n-indecomposable connexe (trajectory union). If an image point moves counter clock-wise densely in a torus, an arc-wise trajectory may or may not be indecomposable: it even may be locally connected. But, if an arc-wise trajectory is dense in a locally compact D of S_2, it is indecomposable. There exists a nonwidely connected hereditarily indecomposable connexe dense in D of S_2. (If, for N, H, K connected and $N=H+K$, both H and K have different closure than N, then N is decomposable: if N is not decomposable, it is indecomposable). (Received May 24, 1956.)

817t. G. T. Whyburn: *Dimension and nondensity preservation of mappings.*

The property of a mapping $f(X)=Y$ to preserve nondensity for compact subsets of X is characterized in terms of the lightness kernel L_f of f consisting of all $x\in X$ such that $f^{-1}f(x)$ is totally disconnected. These results are applied to show that if $w=f(z)$ is continuous in a region X of the z-plane and differentiable at all points of a dense set $f^{-1}(Y_0)$ in X which is the inverse of an open subset Y_0 of $Y=f(X)$, then $\dim f(K)\leqq 1$ for each compact set K in X of dimension $\leqq 1$. Under the same conditions it is shown that f is strongly quasi-open. (Received July 5, 1956.)

818. R. E. Zink: *A note concerning regular measures.*

Let X be a topological space and let (X, S, μ) be an associated measure space. Let $f(x)$ be non-negative and measurable (S). Define the measure ν on S by means of the formula: $\nu(E)=\int_E f(x)d\mu(x)$, for all E belonging to S. Using the ancient techniques of measure theory, one can easily establish the following remarks: (1) If μ is inner regular, so also is ν. (2) If μ is outer regular, then ν is outer regular if and only if for every set E on which f is integrable: (i) There exists a measurable open set containing E on which f is also integrable. (ii) There exists, for every pair of positive numbers ϵ, δ, a measurable open set U containing $E-N(f)$ such that $\mu(\{x: f(x)>\delta\}\cap U)<\epsilon$. (Received July 5, 1956.)

V. L. Klee, Jr.,
Associate Secretary

RESEARCH PROBLEMS

15. Richard Bellman: *Approximation theory.*

Consider the differential equation (1) $dx/dt=\Phi(x)$, $x(0)=c$ where $\Phi(x)$ is a continuous function of x satisfying additional conditions ensuring the existence of a unique solution over the interval $0\leqq t\leqq T$ for $a\leqq c\leqq b$.

Let (2) $dy/dt=\sum_{k=0}^{K} a_k(c, t)y^k$, $y(0)=c$, represent an approximation to (1) where the coefficients $a_k(c, t)$ are functions to be determined so as to minimize one of the following functionals: (3) (a) $J_1=\operatorname{Max}_{0\leqq t\leqq T}|x-y|$, (b) $J_2=\int_0^T(x-y)^2dt$, (c) $J_3=\operatorname{Max}_{a\leqq c\leqq b}\operatorname{Max}_{0\leqq t\leqq T}|x-y|$ (d) $J_4=\operatorname{Max}_{a\leqq c\leqq b}\int_0^T(x-y)^2dt$, (e) $J_5=\int_a^b\left[\int_0^T(x-y)^2dt\right]dc$ under the following alternatives: (4) (a) $a_k(c, t)$ depends only upon c, for $k=0, 1, \cdots, K$. (b) $a_k(c, t)$ are polynomials of degree M in t with coefficients dependent upon c. (The case $K=1$ is the most interesting.)

Consider the analogous problem for systems of the form (5) $dx_i/dt=\Phi_i(x_1, x_2, \cdots, x_n)$, $x_i(0)=c_i$, $i=1, 2, \cdots, n$, in the particular case where the approximating equation is (6) $dy_i/dt=a_i(c)+\sum_{j=1}^{n} b_{ij}(c)y_j(t)$, $y_i(0)=c_i$, and we wish to minimize the functional (7) $\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\left[\int_0^T\sum_{i=1}^{n}(x_i-y_i)^2dt\right]\Pi_i dc_i$. (Received August 17, 1956.)

REPORT OF THE TREASURER

The Treasurer this year again presents to the membership an abridged statement of the Society's financial position, set up in a semi-informal narrative style. A copy of the complete Treasurer's Report as submitted to the Trustees and the Council will be sent to any member requesting it from the Treasurer at the Providence office. Moreover, the Treasurer will be happy to answer any questions members may wish to put to him concerning the Society's financial affairs.

In general, it may be said that the Society's finances continue satisfactory, although the margin between income and expense is very narrow—indeed, our expenses exceeded our income by about $3400, an overexpenditure of about 1½% of income.

Returns on invested funds this year have been at the rate of 4.8% computed on book value, after deduction of custodial expense. This is the same as last year.

I

A Description of the Financial Position of the Society, as of May 31, 1956

The Society had Cash on deposit		
In the Rhode Island Hospital Trust Co.	$ 49,960.99	
In various interest-bearing savings accounts	68,943.21	
In petty cash and drawing accounts in Providence and New Haven	1,050.00	$119,954.20
It had reserves invested until needed in Government bonds		54,606.09
There was owing to it		
By the United States Government	$ 20,867.52	
By members, subscribers and others	15,025.51	35,893.03
It had in stamps and in the postage meter		912.79
And had temporarily advanced to the Investment and other special accounts		2,662.86
Making a total of CURRENT ASSETS of		$214,028.97
The Society also held cash for investment of		302.75
And investment securities valued at		349,129.00
(The market value, May 31, 1956 was $450,765.53)		
TOTAL ASSETS, therefore, were		$563,460.72

Offsetting these assets, the Society			
Owed members, subscribers and vendors	$3,287.35		
Held contributions designated for special purposes	810.29		
Making a total of Current Liabilities, money it might have to pay out on short notice, of		$ 4,097.64	
Held funds received from various special sources to support particular projects, such as the Summer Institute, the Register of Mathematicians, etc.		59,932.93	
Had advanced for recovery from future sales for various Society publications—Colloquium and Survey volumes, Birkhoff papers, Russian translations, etc.		29,728.35	
And held in its General Fund, the result of 68 years of prudent operations, the sum of		120,270.05	
Thus accounting for all the CURRENT FUNDS			$214,028.97
The Invested Funds represent the following:			
(1) The Endowment Fund, largely the gift of members about thirty years ago		$ 94,000.00	
(2) The Library Proceeds Fund, derived from the sale of the Society's Library in 1950		66,000.00	
(3) The Prize Funds—Bôcher, Cole, Moore		6,575.00	
(4) The Mathematical Reviews Fund, a gift of the Rockefeller Foundation to make possible the establishment of the Reviews in 1940		80,000.00	
(5) Reserves established by the Trustees to protect the life memberships and life subscriptions formerly available, and as a "hedge" against investment losses		72,380.11	
(6) Other funds, derived mainly from bequests to the Society by members which the Trustees were either required to invest or which they have invested at their option—the income being used for the general purposes of the Society		28,551.47	
A total of invested funds of		$347,506.58	
And this, together with cash due the current funds of		1,925.17	
Accounts for the total holding of investment securities of			$349,431.75
TOTAL LIABILITIES, therefore, were			$563,460.72

II

AN ACCOUNT OF THE FINANCIAL TRANSACTIONS OF THE SOCIETY DURING THE FISCAL YEAR 1955–1956

The Society has two types of receipts—funds for special purposes and projects, and the General Fund, from which are met the general operating expenses of the organization, including the publication of the Bulletin, the Proceedings, and the Transactions. Income from sales of and subscriptions to these journals is placed in the General Fund, but in practice is allocated to the expenses of the journals themselves, as though such income were in fact special. It is so treated in the following presentation.

To meet its GENERAL obligations, the Society RECEIVED:

From dues and contributions of individual members	$59,963.00		
Less enlisted men's special discount	403.00	$59,560.00	
From dues of institutional members		25,212.50	
From investment and trusts		22,696.31	
From contracts, in payment of indirect costs		16,109.67	
From miscellaneous sources		2,686.46	
Total General Receipts			$126,264.94

These funds were EXPENDED

For general administrative and meeting expenses		$58,646.93	
To meet deficits in Society publications:			
Bulletin: (Total expense, $20,530.61)	$14,590.10		
Proceedings: (Total expense, $25,567.02)	19,043.47		
Transactions: (Total expense, $34,275.84)	13,478.34		
Mathematical Reviews: (Total expense, $92,598.35)	16,776.54	$63,888.45	
In subsidies to non-Society publications		2,556.57	
To cover the Society's share of the cost of such joint projects: the Policy Committee, the Combined List of Members, the Employment Register		4,284.10	
For miscellaneous expenses		284.11	
Total General Expenses			$129,660.16
Leaving an EXCESS of EXPENSES over INCOME of			$ 3,395.22

Which was taken from moneys saved from previous years.

Respectfully submitted,

ALBERT E. MEDER, JR.
Treasurer

September 30, 1956

BOOK REVIEWS

Homological algebra. By Henri Cartan and Samuel Eilenberg. Princeton, The Princeton University Press, 1956. 15+390 pp. $7.50.

At last this vigorous and influential book is at hand. It took nearly three years from completed manuscript to bound book; Princeton is penalized 15 yards for holding.

Homological algebra deals both with the homology of algebraic systems and with the algebraic aspects of homology theory. The first topic includes the homology and cohomology theories of groups, of associative algebras, and of Lie algebras. The second topic includes the care and feeding of exact sequences and spectral sequences, as well as the manipulation of functors of chain complexes. For example, the Künneth problem reads: Given the homology of complexes K and L, what is the homology of $K \otimes L$? Again, the universal coefficient problem reads: Given a group G and the homology of a complex K, what is the homology of the complexes $K \otimes G$ and Hom (K, G)? These problems and these two functors, tensor product and Hom, are treated not just for groups, but in proper generality for left modules over an arbitrary ring Λ. Explicitly, if A and G are such modules, Hom_Λ (A, G) denotes the group of Λ-module homomorphisms of A into B. When G is a right Λ-module and A a left module—a situation denoted neatly as $(G_\Lambda, {}_\Lambda A)$—the tensor product taken over Λ is written as $G \otimes_\Lambda A$. A Λ-complex K is a graded differential left Λ-module; its homology $H(K)$ is the usual graded module $H(K) = \sum H_n(K)$, it has the usual definition and an unusual definition (Chap. IV), dual to the usual one.

The various aspects of homological algebra all meet in the notion of a projective resolution (Chap. V). A left module P is *projective* (Chap. I) if any homomorphism of P into a quotient module B/C can be "lifted" into a homomorphism of P into B. (Thus a free module is projective, but not necessarily vice versa.) A projective resolution of A is an exact sequence $\cdots \to X_n \to X_{n-1} \to \cdots \to X_0 \to A \to 0$ composed of A and a complex X which consists of projective modules X_n, $n = 0, 1, \cdots$. Given two such resolutions X and X' for the same A, the familiar method of climbing up one dimension at a time provides a chain transformation of X into X' and proves X and X' chain equivalent. Given any module G_Λ the homology groups $H(G \otimes_\Lambda X)$ are therefore independent of the choice of the resolution of A and depend only on G and A; they are called the torsion products and are denoted (Chap. VI)

$$\operatorname{Tor}_n^{\Lambda}(G, A) = H_n(G \otimes_{\Lambda} X), \qquad n = 0, 1, \cdots;$$

in particular, $\operatorname{Tor}_0(G, A) = G \otimes_{\Lambda} A$. Similarly, given any ${}_{\Lambda}G$ the cohomology groups $H(\operatorname{Hom}_{\Lambda}(X, G))$ depend only on A and G and are written

$$\operatorname{Ext}_{\Lambda}^n(A, G) = H^n(\operatorname{Hom}_{\Lambda}(X, G)), \qquad n = 0, 1, \cdots;$$

in particular $\operatorname{Ext}^0(A, G)$ is $\operatorname{Hom}_{\Lambda}(A, G)$. Any module homomorphism $\gamma: A \to B$ extends to a map of a resolution of A to a resolution of B; hence each Tor_n is a covariant functor of its arguments G and A, while Ext^n is contravariant in A and covariant in G.

These two functors now apply to the typical problems of homological algebra. Let Λ be a hereditary ring (i.e., every left or right ideal is projective as a module). In the Künneth problem, if K and L are complexes of projective modules, there is an exact sequence

$$(1) \quad 0 \to H(K) \otimes_{\Lambda} H(L) \xrightarrow{\alpha} H(K \otimes_{\Lambda} L) \xrightarrow{\beta} \operatorname{Tor}_1^{\Lambda}(H(K), H(L)) \to 0,$$

the sequence splits, and the natural homomorphisms α and β have degrees 0 and 1, respectively. The same sequence, with L a module (differentiation zero), solves the universal coefficient problem for homology, under somewhat weaker conditions, while for the coefficient problem in cohomology one has the exact sequence

$$0 \to \operatorname{Ext}_{\Lambda}^1(H(K), G) \xrightarrow{\alpha'} H(\operatorname{Hom}_{\Lambda}(K, G)) \xrightarrow{\beta'} \operatorname{Hom}_{\Lambda}(H(K), G) \to 0,$$

valid under similar conditions, with maps α' and β' of degrees 1 and 0, respectively. This sequence is a case of a "Cokünneth" Theorem (Theorem 3.1.a of Chap. VI) for $\operatorname{Hom}_{\Lambda}(K, L)$; this theorem, insufficiently emphasized by the authors, contains in particular a homotopy classification of maps of K onto L.

Exact sequences can be grown. If $0 \to A \to B \to C \to 0$ is a given exact sequence of left modules, then for each right module G the derived sequence

$$0 \to G \otimes_{\Lambda} A \to G \otimes_{\Lambda} B \to G \otimes_{\Lambda} C \to 0$$

in general fails to be exact at $G \otimes_{\Lambda} A$. Exactness returns when the left-hand zero is replaced by $\operatorname{Tor}_1(G, C)$, and the sequence is continued to the left as

$$(2) \quad \cdots \to \operatorname{Tor}_{n+1}^{\Lambda}(G, C) \xrightarrow{\theta} \operatorname{Tor}_n^{\Lambda}(G, A) \to \operatorname{Tor}_n^{\Lambda}(G, B) \to \operatorname{Tor}_n^{\Lambda}(G, C) \to$$

where θ is a suitable "connecting homomorphism." Similarly, given ${}_{\Lambda}G$, the derived sequence

$$0 \rightarrow \mathrm{Hom}_\Lambda (C, G) \rightarrow \mathrm{Hom}_\Lambda (B, G) \rightarrow \mathrm{Hom}_\Lambda (A, G) \rightarrow 0$$

becomes exact when the right-hand zero is replaced by $\mathrm{Ext}^1_\Lambda (C, G)$, and the sequence is continued to the right with the higher Ext^n functors.

The homology of various algebraic systems can be written with torsion products. To get the homology of a group Π (Chap. X), take Λ to be the integral group ring $Z(\Pi)$ and regard the ring Z of integers as the left $Z(\Pi)$-module for which $xm = m$ for each integer m and each $x \in \Pi$. For modules $G_{Z(\Pi)}$ and $_{Z(\Pi)}A$ the homology and cohomology groups of Π are then defined by

$$(3) \qquad H_n(\Pi, G) = \mathrm{Tor}_n^{Z(\Pi)}(G, Z), \qquad H^n(\Pi, A) = \mathrm{Ext}^n_{Z(\Pi)}(Z, A).$$

The latter are the cohomology groups of Π originally defined by Eilenberg-Mac Lane in terms of a standard complex. This complex can be viewed as a particular projective resolution of the $Z(\Pi)$-module Z; for special Π the use of other resolutions expedites the calculation of H^n (e.g., for Π cyclic see Chap. XII, §7).

To get the homology of an associative algebra Γ over a commutative ring K (Chap. IX), the authors first turn bimodules into left modules. More exactly, let Γ^* be the algebra anti-isomorphic to Γ and set $\Gamma^e = \Gamma \otimes_K \Gamma^*$; any two-sided Γ-module A can then be interpreted as a left Γ^e-module. The homology and cohomology of Γ is now defined as

$$(4) \qquad H_n(\Gamma, A) = \mathrm{Tor}_n^{\Gamma^e}(A, \Gamma), \qquad H^n(\Gamma, A) = \mathrm{Ext}^n_{\Gamma^e}(\Gamma, A).$$

These are the groups defined originally by Hochschild, who used a standard complex which may be regarded as a projective resolution of the Γ^e-module Γ.

To get the homology of a Lie algebra L over a commutative ring K (Chap. XIII), first take the enveloping associative algebra L^e, defined as the tensor algebra T of the K-module L modulo the ideal spanned by all $a \otimes b - b \otimes a - [a, b]$ for $a, b \in L$ and $[a, b]$ the bracket product in L. The augmentation $\epsilon: T \rightarrow K$ of the tensor algebra (identity on scalars, zero on elements of L) induces an algebra homomorphism $\epsilon: L^e \rightarrow K$, and K becomes a left L^e-module with operators $xk = \epsilon(x)k$ for $x \in L^e$ and $k \in K$. The rôle of L^e is that each left representation module of the Lie algebra L is a left L^e-module, and vice versa. For modules in the situation $(A_{L^e}, {}_{L^e}C)$ one then defines the homology and cohomology groups of L as

$$(5) \qquad H_n(L, A) = \mathrm{Tor}_n^{L^e}(A, K), \qquad H^n(L, C) = \mathrm{Ext}^n_{L^e}(K, C).$$

In case L is free as a K-module, the Birkhoff-Witt theorem shows that these groups agree with those originally defined by Chevalley-Eilenberg, using a more complicated complex constructed directly from L.

There is a striking parallel between the formulas (3), (4), and (5); this the authors exploit by a new notion of an augmented ring (of which more later). In the first and third cases the ring $\Lambda = Z(\Pi)$ or $\Lambda = L^e$ is an algebra over Z or K respectively. These two cases are unified in terms of supplemented algebras. A *supplemented algebra* (Chap. X) is an algebra Λ over a commutative ring K together with a K-algebra homomorphism $\epsilon : \Lambda \to K$. This ϵ induces on K a left Λ-module structure; the homology groups of the supplemented algebra Λ are defined in terms of this structure for each A_Λ as Tor_*^Λ (A, K). In case Λ is K-free, these groups are isomorphic (Theorem 2.1) to the Hochschild homology groups of Λ, with A regarded, through ϵ, as a bimodule. By this token the homology either of groups or of (K-free) Lie algebras can be regarded as special cases of the Hochschild homology.

This brief outline of homological algebra does not adequately represent the generality of the treatment in *Homological algebra*. For example, much of the discussion is carried out for arbitrary covariant functors $T(A)$ which are *additive* (Chap. II) in the sense that $T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2)$ for any sum of module homomorphisms α_1 and α_2 (a condition which implies that $T(A+B) = T(A) + T(B)$ for a direct sum of modules A and B). Any such functor has a left and a right satellite functor (Chap. III). For example, the left satellite $S_1T(A)$ is found from any exact sequence $0 \to M \to P \to A \to 0$, P projective, as the kernel of $T(M) \to T(P)$, and is independent of the choice of that sequence, while the right satellite is defined dually (reverse all arrows). For any exact sequence $0 \to A \to B \to C \to 0$ there is a connecting homomorphism $S_1T(C) \to T(A)$. The functor T is called half exact if the related sequence $T(A) \to T(B) \to T(C)$ is always exact at $T(B)$; in this case the connecting homomorphisms yield an infinite exact sequence of iterated satellites in the form

$$\cdots \to S_2T(C) \to S_1T(A) \to S_1T(B) \to S_1T(C) \to T(A) \to \cdots .$$

The functors Tor_* are the iterated left satellites of $\otimes_\Lambda$, while Ext^* are the right satellites of Hom_Λ. There is an axiomatic description of satellites (Theorem 5.1; see also the elegant characterization of Tor given in the introduction).

Homological methods applied to arbitrary functors are more powerful. For a contravariant additive functor $T(A)$ the nth right derived functor is defined as the nth homology group of the complex $T(X)$, where X is a projective resolution of A, and similarly for covariant functors and functors of several arguments. Chapter V discusses these notions, relates the derived functors to satellites and describes the "balanced" covariant functors $T(A, G)$ for which resolution of either factor yields the same derived functor—the one relevant example being the functor $G\otimes_\Lambda A$. One also employs injective resolutions (dual to projective resolutions). For example (p. 107) the nth right derived functor of $\mathrm{Hom}^n_\Lambda\ (A,\ C)$ is $\mathrm{Ext}^n_\Lambda\ (A,\ C)$; it may be obtained as the nth homology group of any one of $\mathrm{Hom}_\Lambda\ (X,\ C)$, $\mathrm{Hom}_\Lambda\ (A,\ Y)$ or $\mathrm{Hom}_\Lambda\ (X,\ Y)$, where X is a projective resolution of A and Y an injective one of C.

Chapter VI introduces dimension concepts. The *projective dimension* of a Λ-module A is the least non-negative inreger n such that there is a projective resolution X of A with $X_k=0$ for all $k>n$. The *left global dimension* of a ring Λ is the least non-negative integer m such that all left Λ-modules A have projective dimension $\leqq m$, or, equivalently, such that $\mathrm{Ext}^{m+1}_\Lambda=0$. In the exercises (of which most chapters have many, giving additional results) one finds also a weak dimension. Another interesting result is Rose's theorem (Chapter IX, Proposition 7.4) on the dimension of the tensor product of algebras. Hilbert's Theorem on chains of syzygies is comprised in the beautiful Theorem 6.5 of Chapter VIII: if Λ is the polynomial ring in n indeterminates over a field K, then the projective dimension of any Λ-module is at most n and that of any ideal of Λ is at most $n-1$. The demonstration uses a projective resolution due to Koszul; there is a similar theorem (Theorem 6.5′) for formal or convergent power series rings.

Chapter XI proliferates products. Given left modules A and C over the K-algebra Γ and similar modules over Γ' there is an obvious natural homomorphism

$$\mathrm{Hom}_\Gamma\,(A, C)\otimes \mathrm{Hom}_{\Gamma'}\,(A', C')\to \mathrm{Hom}_{\Gamma\otimes\Gamma'}\,(A\otimes A', C\otimes C'),$$

where all tensor products are taken over K. Upon replacing the arguments A and A' by resolutions one obtains an "external" product

$$\vee:\mathrm{Ext}^p_\Gamma\,(A, C)\otimes \mathrm{Ext}^q_{\Gamma'}\,(A', C')\to \mathrm{Ext}^{p+q}_{\Gamma\otimes\Gamma'}\,(A\otimes A', C\otimes C');$$

it is a homomorphism which reduces to the previous one when $p=q=0$. Given a "diagonal map" $D:\Gamma\to\Gamma\otimes\Gamma$ and $\Gamma'=\Gamma$ one induces a corresponding internal product

$$\cup:\mathrm{Ext}^p_\Gamma\,(A, C)\otimes \mathrm{Ext}^q_\Gamma\,(A', C')\to \mathrm{Ext}^{p+q}_\Gamma\,(A\otimes A', C\otimes C').$$

This yields the usual cup products in the cohomology of algebras or of groups, and can there be expressed by the usual Čech-Alexander formulas in the standard complexes. There are three more external products, each with an internal version, which spring as above from the three natural homomorphisms

$$(C \otimes A) \otimes (C' \otimes A') \to (C \otimes C') \otimes (A \otimes A'),$$
$$\operatorname{Hom} (A \otimes A', \operatorname{Hom} (C, C')) \to \operatorname{Hom} (C \otimes A, \operatorname{Hom} (A', C')),$$
$$\operatorname{Hom} (C, C') \otimes (A \otimes A') \to \operatorname{Hom} (\operatorname{Hom} (A, C), C' \otimes A').$$

The third one gives "cap" products. After handling these products expeditiously, with effective use of the unifying concept of a supplemented algebra, the chapter closes with a generalization of the Eilenberg-Mac Lane cup-product reduction theorem.

Chapter XII, one of the most interesting of the book, presents hitherto unpublished results of Tate and Artin-Tate on the homology and cohomology of finite groups. For a left module A over the group ring of a finite group Π one has the norm homomorphism $a \to \sum xa$, the sum being taken over all $x \in \Pi$. This induces a norm homomorphism $H_0(\Pi, A) \to H^0(\Pi, A)$ and hence makes it possible to combine the homology and cohomology groups of Π with coefficients in A into a single (doubly infinite) sequence. To calculate these groups one can use a suitably designed "complete" resolution; there is a (cup) product which works for the whole sequence. For certain finite groups Π there is known to be a period q; that is, an integer such that there is an isomorphism $H^n(\Pi, A) \cong H^{n+q}(\Pi, A)$ given for all n by a cup product. The chapter ends with the beautiful theorem that Π has such a period if and only if every abelian subgroup of Π is cyclic.

Homological algebra arose from extension problems: $\operatorname{Ext}_1^Z (A, C)$ is the group of abelian group extensions of C by A; $H^2(\Pi, G)$ is the group of extensions of the Π-group G by the group Π, and similarly for algebras and Lie algebras. These matters are treated systematically, elegantly, and belatedly in Chapter XIV. The most interesting known results in these directions (three-dimensional cohomology classes as obstructions to extension problems) are omitted.

The next chapter sets up the formalism of spectral sequences. The exposition is clear; there is a good explanation (§7) of how spectral sequences arise from relative homology; there is (finally!) a decent notation for a filtration. Unhappily the authors continue the conspiracy of silence according to which the rectangular diagrams, used by all the experts, never appear in print. As the authors note, spectral sequences were discovered by Leray (1945); as they do not note, they were independently discovered by Lyndon (Harvard thesis, May, 1946; cf. also Duke Math. J. vol. 15 (1948) pp. 271–292), who was the first to consider the subject matter of these chapters—spectral sequences applied to algebraic problems. The authors improperly credit Lyndon's spectral sequences to a much later paper by Hochschild-Serre.

Chapter XVI applies spectral sequences to various problems; associativity formulas for torsion products, the relation between homology groups of an algebra and those of an invariant subalgebra, and topological spaces with operators. The next and last chapter discusses the projective resolution of a complex K by a double complex X (i.e., a bigraded module with two differentiations). Given also a functor T, the double complex $T(X)$ has homology independent of the choice of the resolution X, has two filtrations and hence two spectral sequences. The resulting maze of objects constitute the *hyperhomological* invariants of K and T. The chapter ends triumphantly with a return to first problems; the Künneth sequence (1) for complexes K and L over a hereditary ring, previously known to be exact for K and L projective, is shown by spectral sequences to be exact provided only that $H(\mathrm{Tor}_1^A (K, L)) = 0$. This resounding success leads the authors to assert that the hyperhomological invariants of $K \otimes L$ may be regarded as a general solution of the Künneth problem. This assertion appears to be a case of trimming the problem to fit the technique, since a regard uncorrupted by spectres would formulate the Künneth problem as that of finding a formula for the homology of $K \otimes L$ in terms of a sufficient set of invariants (hyperhomologies, Bocksteins, or what have you) of K and L.

In spite of the delay in its publication, widespread acquaintance with the manuscript and with the ideas of this book has already played an important role in the development of this lively subject. For example, Eckmann and Hilton (unpublished) have investigated the interesting functors derived from Hom (A, B) by replacing the contravariant argument A by an injective resolution or the covariant B by a projective. Serre and Auslander-Buchsbaum (Proc. Nat. Acad. Sci. U.S.A. vol. 42 (1956) pp. 36–38) have achieved decisive results on the dimensions of local rings. More significant for the general presentation of the subject is the fact that the whole mechanism of projective resolutions works not just in the category of left modules over a ring, but equally well for right modules, for bimodules, or in many other categories. In an appendix to the book Buchsbaum sets forth these ideas, together with the necessary axioms on the *additive categories* (he calls them "exact" categories) in which this theory works. Subsequent unpublished work by Grothendieck indicates that this point of view will be convenient also for the homology of a space with coefficients in a sheaf. Hence it seems likely that a future presentation of homological algebra will operate in a suitable category, provided at least that someone concocts a convenient method of chasing diagrams without chasing elements.

The authors' approach in this book can best be described in philosophical terms and as monistic: everything is unified. Consider for instance the homology of groups; in view of its application to class field theory and to topology this topic is central in homological algebra. In this book the homology of groups appears as a special case of the homology of monoids (monoid = associative multiplicative system with identity), which in turn is a special case of the homology of supplemented algebras, again a case of the homology of augmented algebras, which is an instance of a torsion product, which at your choice is an instance of a derived functor or an iterated satellite functor.

Historically, each monistic doctrine is resolved by a subsequent pluralism. So it was here. When the authors started to write, it *was* true that all known cases of homology of algebraic systems (groups, algebras, and Lie algebras) could be neatly subsumed under the resolution, Tor, and Ext pattern. When they finished writing this was no longer so—and this because of the authors' own separate efforts elsewhere! The Eilenberg-MacLane homology of abelian groups (Trans. Amer. Math. Soc. vol. 71 (1951) pp. 294–330) has not yet been expressed by torsion products. The Eilenberg-Mac Lane bar construction (Annals of Math. vol. 58 (1953) pp. 55–106) is a standard construction more general than those produced by standard resolutions. Cartan's beautiful and powerful theory of constructions (Séminaire École Normale Supérieure, 1954/1955) is an extension of the idea of a projective resolution beyond the terms of this book. Still more recently, the as yet unpublished homology theories of Dixmier for Lie rings and of MacLane for rings are other examples of homology of algebraic systems not (at least as yet) obtainable by resolutions.

Perhaps Mathematics now moves so fast—and in part because of vigorous unifying contributions such as that of this book—that no unification of Mathematics can be up to date. The reviewer might also add his strictly personal opinion that the authors have not kept sufficiently in mind the distinction between a research paper and a book: a good research paper presents a promising new idea when it is hot—and when nobody knows for sure that it will turn out to be really useful; a good research book presents ideas (still warm) after their utility has been established in the hands of several workers.

This book contains too large a proportion of shiny new ideas which have nothing to recommend them but their heat and promise: satellites (these appear in Chapter III and then gradually disappear in later chapters), derived functors of anything but Hom and ⊗ (the reviewer watched in vain for other examples), semi-hereditary rings, functors derived simultaneously in several variables, supplemented algebras, and the homology of monoids. The same remark applies to spectral sequences. These sequences *have* proved their worth in topology but have not yet reached decisive results in the homology of algebraic systems: the result is that the uninitiated reader can hardly hope to understand what spectral sequences are all about by reading the three chapters devoted to them in this book. The reviewer is not claiming that spectral sequences and these other notions will not later have significant algebraic uses: some of them will, but until that time comes their presence clutters up the book.

Another danger of shiny new notions is that sometimes the shine proves illusory. For example, the authors define an *augmented ring* as the triple consisting of a ring Λ, a left Λ-module Q and a left Λ-module epimorphism $\epsilon:\Lambda\rightarrow Q$. Now Q and ϵ are determined up to isomorphism by the kernel of ϵ, which is a left ideal in Λ. Hence "augmented ring" is a new name for "left ideal in a ring." The authors have introduced these augmented rings because the homology of an augmented ring includes the homology of algebras, groups, and Lie algebras. The homology groups of an augmented ring $\epsilon:\Lambda\rightarrow Q$ are defined to be the groups $\mathrm{Tor}_n^{\Lambda}(A, Q)$. This definition does include the three desired cases as already displayed above, but a moment's reflection reveals that the definition has nothing to do with the epimorphism ϵ, and would work for *any* left Λ-module Q. The ϵ really occurs only in some wholly routine calculations of low dimensional homology groups. The only consequential theorem about the homology of augmented rings is a mapping theorem (VIII, 3.1) which gives the machinery for changing from one ring Λ to another. This theorem has nothing to do with a map ϵ; in fact the theorem is clarified by the recognition that it deals with a homomorphism of the system consisting of a ring Λ and a left Λ-module Q into a ring Λ' and a left Λ'-module Q'. The rest of the authors' discussion of "augmented rings" consists of theorems about left ideals.

The authors' treatment of the literature is off-hand. Künneth formulas appear, but no references to Künneth. Torsion products abound, with no credit to early discoverers (example: Čech, in Fundamenta Mathematica vol. 25 (1935) pp. 33–44) had the torsion product for abelian groups, defined essentially as a satellite). The authors have discovered that the Birkhoff-Witt theorem was known to Poincaré before either Birkhoff or Witt was born; they proceed to call it the Poincaré-Witt theorem. The facts of the matter are these. The theorem asserts that if the Lie algebra L over K is free as a K-module, then the natural map of L into its enveloping associative algebra L^e has kernel zero. Birkhoff and Witt both found proofs of this theorem in 1936; Birkhoff's was received by the editors 29 days earlier than Witt's, but there is every reason to suppose that the two were independent and that the theorem was then "in the air." Birkhoff makes a (partial) reference to Poincaré; Witt does not. Birkhoff defines the enveloping algebra, not as a quotient of the tensor algebra, but by "straightening" elements of that tensor algebra. His proof is carelessly done, but with some little trouble (which the reviewer has taken) his proof can be made complete and correct. The world would be happy to honor Poincaré, who was well ahead of his time on this, but not at the expense of a manifest injustice to Birkhoff. Since a three-handled theorem is clumsy, it will doubtless remain Birkhoff-Witt.

This book is very carefully prepared and well proof-read; the reviewer noted only one troublesome misprint: on page 185 the η in the top row of the square diagram and in the next line of the text should be ρ (notation from p. 168). The letter Λ is overworked; it appears variously as a ring, as an augmented ring, or as an algebra. More application of the usual (unexpressed) conventions about different letters for different notions would have helped the reader. The index might have the following additions: complete resolution 240, derivation 168, direct family of homomorphisms 4, direct product 4, direct sum 4, exterior ring 146, free ring 146, homology of differential module 54, image 3, kernel 3, Lie algebra 266, negative graded module 58, normal map 349, normalized standard complex 176, polynomial ring 146, positive graded module 58, 60, $\emptyset$-projective module 30, standard complex 175, syzygies 157.

SAUNDERS MACLANE

Methods in numerical analysis. By K. L. Nielsen. Macmillan, New York, 1956. 13+382 pp. $6.90.

Introduction to numerical analysis. By F. B. Hildebrand. New York, McGraw-Hill, 1956.+10 511 pp. $8.50.

Numerical analysis, with emphasis on the application of numerical techniques to problems of infinitesimal calculus in single variables. By Z. Kopal. New York, Wiley, 1955. 14+556 pp. $12.00.

The demand for computer personnel has been made abundantly clear—in the advertisement pages of our newspapers, and at formal conferences.[1] Universities have a hard time finding enough capable teachers for courses planned to produce mathematicians qualified in this area; there has also been some difficulty about suitable text material. The second difficulty may be the easier to overcome, for the present three volumes are undoubtedly the forerunners of more.

None of these show much trace of the influence of automatic computers; this, however, is no defect for we have found that the discipline of a spell with desk machines is an excellent preparation for all phases of work with automatic computers, and courses on numerical analysis without access to computers are for the expert not for the novice. Moreover, it was apparent at the conference and has become clearer since, that the need for computer personnel with a broad training in mathematics is more urgent than for those with less mathematics and a narrow training in numerical analysis. This is because there has been an admirable tendency to use the computer to help itself in certain more routine phases of problems, and because more powerful computers bring deeper problems into range.

The books are listed above in advancing order. Nielsen's is admittedly an elementary textbook for the practical man. Hildebrand's is for an introductory course in the subject and Kopal's aims to provide an advanced undergraduate textbook as well as a research handbook on certain topics. The last two are based on courses given at M.I.T.

All three books cover the basic topics: Difference Operators, Interpolation, Differentiation and Integration, Differential Equations, Trigonometrical Approximation. In the first two there is some account of least squares and the solution of systems of linear equations, and of nonlinear equations. There is a chapter on integral and integro-differential equations in the third. All have collections of examples; that of Hildebrand is most impressive. All could well have been supplemented by a detailed discussion of a rigorous "digital" calculation, like the evaluation of a square root or a polynomial, such as have been given by Householder, as an introduction to an essential part of modern numerical analysis.

In Nielsen's book, worked examples, naturally, play a large part and there is a detailed account of schemes for the solution of least-square problems, due to the author and L. Goldstein. This book includes a collection of tables. There are some theoretical uncertainties, e.g. the usual one about the θ in an error term of the form $f^{n}(a+\theta h)$, not depending on h, and the determination of the error term in the Lagrangian interpolation is at least notationally obscure.

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[1] E.g., A. W. Jacobson, editor, *Proceedings of the First Conference on training personnel for the computing machine field*, Detroit, Wayne University Press, 1955.

In both Hildebrand and Kopal there seems to be an overemphasis on approximate quadratures. Kopal has interesting historical notes, a detailed account of the choice of interval in numerical differentiation and some discussion of boundary value problems (in the one-dimensional case) which was not readily accessible in English. The mathematician reading Kopal will be disturbed from time to time by such phrases as "In general, we may expect . . . ," which he can, often, rightly question.

Certainly the second two books will be of value to the teacher and research worker, but they seem to be far too extensive for the ordinary student, whom they may discourage. There is no doubt that there are many principles in numerical analysis which can be incorporated at various places in regular courses and teachers can find suitable material in all three volumes. For instance, an efficient method for solving linear equations, with checking devices, can be discussed at the beginning of algebra courses. The concept of differences and Lagrangian ideas can be introduced when polynomials are being studied. Then, when calculus is being started, the ideas of numerical differentiation and integration can be added. When differential operators are being discussed in connection with differential equations with constant coefficients, an account of difference operations can be added. There are many opportunities in courses on matrix theory, to introduce ideas useful in numerical analysis. In analytical geometry, too, the extremal properties of the principal axes of conics leads to the ideas of Rayleigh, Ritz, and others. All this can be covered with the use of a desk calculator or two, and a few books of tables as a source for examples. These principles could be later consolidated in a short course, and there would seem to be a need for an appropriate text, of a fraction of the size of any of the three. A shortened version of Hildebrand could be very useful. This would be supplemented in due course by a similar account of the principles of programming—as soon as these are established—and of what we can call "modern" numerical analysis, the type of numerical analysis relevant in the use of the high speed automatic computers.

JOHN TODD

Algèbre. By R. Ballieu and F. Simonart. Louvain, Librarie Universitaire and Paris, Gauthier-Villars, 1955. xvi+355+1 pages.

In this book, classical algebra is treated, to quote the senior author, Simonart, "en langage moderne." The central themes are those aspects of linear algebra which pertain most directly to the solution of simultaneous linear equations over a field, and the study of polynomials and their roots, the theory of equations. This second theme is developed in the best expository tradition of the field. One cannot, unfortunately, say the same for the first. In a preliminary chapter, containing introductory material on such topics as groups, rings, fields and formal polynomials, matrices are introduced as arrays, and matric representations are given for both the complex numbers and the quaternions. Chapter II on vector spaces and linear systems, Chapter III on determinants, Chapter IV on simultaneous linear equations and Chapter V on quadratic forms contain, in addition to the broad subjects of their titles, such diverse topics as orthogonal matrices, centers of polynomials and Vandermonde's determinant. Many of the sections in this portion of the book, for instance, §18 on the evaluation of determinants and §27 on invariants and covariants, are very well written. The disappointing feature of the first five chapters is that the authors, in places, manage to leave the impression that the introduction of modern language into this old field is really unnecessary. The core of linear algebra, the linear transformation, enters, as it should, repeatedly, but its basic property of linearity plays only a minor role in this exposition, if it is ever really mentioned. Another omission of this type occurs in §18, 5. Here, the characteristic polynomial is introduced by fiat, and all connections with vital aspects of linear transformations are suppressed. Likewise in §9, 4, equivalence and similarity of matrices are defined with almost no indication that these concepts involve change of bases in suitable vector spaces.

Chapter VI on polynomials covers standard material on roots, interpolation, resultants and multiplicities. The treatment of the resultant is particularly comprehensive, but one could wish that its introduction had been better motivated. In §31, 2, Sylvester's determinant appears without explanation, and it becomes clear only later why this particular determinant is the key to the theory. The discriminant appears in the next chapter on symmetric functions.

The last two chapters are easily the best of the book. In Chapter VIII on transformations of polynomials, the effect of the rational substitution $y=P(x)/Q(x)$ on the polynomial $f(x)$ is studied by forming the resultant of f with $yQ-P$. The transformations are simplified in steps so that, as an end result, the method of diminishing the roots of f by a fixed constant arises naturally. Considerations of polynomials in which the roots are bound to each other by algebraic equations lead to a long section on the reciprocal equation. Wherever convenient, the authors apply the results of the chapter on symmetric functions to this development. Binomial equations come next with just enough group theory to organize the material. For instance, the standard properties of the Euler totient appear here as group-theoretic corollaries. The chapter ends with an adequate treatment of the cubic and quartic. The last chapter on the location of roots, both real and complex, is especially detailed. Convergence questions for both Newton's process and for the regula falsi are treated. Contemporary results (with references to live authors) are included in §48 where zeros of complex polynomials are discussed in such a way that the reader is not left with the all too common impression that the subject is embalmed. Although there are no exercises, the text abounds with well chosen examples. The few errors which appear seem to be caught in the errata at the end.

FRANKLIN HAIMO

Handbuch der Laplace-Transformation, Vol. II. *Anwendungen der Laplace-Transformation*. Part 1. By Gustav Doetsch. Birkhäuser, Basel and Stuttgart, 1955. 434 pp. 56.15 Swiss francs.

In this volume the author covers a wide range of applications, many of which might be considered as being applications of the Laplace transform, rather than properties of the Laplace transform, only by fiat. He begins with a collection of results from the first volume (cf. Bull. Amer. Math. Soc. vol. 58 (1952) p. 670), the "rules" for operating with the Laplace transform. Then he takes up a wide variety of connections between the asymptotic expansions of a pair of transforms. Ordinary Abelian and Tauberian theorems were dealt with in the first volume; here it is a question of what can be deduced about one member of a pair of transforms when an asymptotic expansion for the other is given. Much of this material is discussed here for the first time in systematic form, and many old isolated results now appear as special cases of the general theory. Numerous illustrative examples are taken up, for example Stirling's series for log $\Gamma(s)$, Bessel functions, theta functions, and the wave function for the hydrogen atom.

The second part of the volume takes up the connection between Laplace transforms and factorial series (which the author thinks deserve more attention than they have been getting). A number of miscellaneous convergent expansions are also discussed.

The third part deals exhaustively with the use of the Laplace transform to solve ordinary differential equations, first those with constant coefficients on a half line, then those with constant coefficients on a whole line, and finally those with variable coefficients, in so far as the method is applicable to them. Problems about servomechanisms, and electric networks and filters, are used as illustrations and discussed in great detail. The author was a pioneer in the systematic use of the Laplace transform for solving differential equations; although he admits that everything that can be done for a single equation can also be done without the Laplace transform, he defends the technique ably and claims that it is indispensable when systems of differential equations are to be solved. There is probably no better place to read about this technique than in this book: mathematical accuracy is always respected and possible pitfalls are pointed out, while at the same time close contact is maintained with the requirements (and terminology) of the engineer.

Partial differential equations, difference equations, and integral equations are to be dealt with in the third volume.

R. P. Boas, Jr.

Combinatorial topology. Vol. 1. By P. S. Aleksandrov. Trans. by Horace Komm. Rochester, Graylock Press, 1956. 16+225 pp. $4.95.

This is a translation of the first third of Aleksandrov's *Kombinatornaya topologiya* (Moscow-Leningrad, OGIZ, 1947). It consists of the first six chapters preliminary to the formal homology theory, and an appendix. This translated part can be understood by a reader without any specific prerequisite. However a certain amount of mathematical maturity is required of the reader.

Chapter I is a survey of the fundamentals of general topology. Many theorems in this chapter are not used in the sequel. Proofs of such theorems are often omitted, but references are given.

Chapters II, III and V are devoted to a rigorous treatment of geometrically intuitive material. Chapter II presents E. Schmidt's proof of the Jordan curve theorem. The topology of surfaces is developed in Chapter III, where one finds a detailed account of Alexander's derivation of the normal forms of closed surfaces. This chapter on surfaces leads naturally to geometric complexes and related notions, which are introduced in Chapter IV. Chapter V deals with Sperner's lemma and its applications to the Pflastersatz, Brouwer's theorem on the invariance of domain, and the Brouwer fixed point theorem. These chapters II, III and V acquaint the reader with several elementary but important topological facts, and thereby provide an excellent background experience in topology.

The final Chapter VI, of more abstract nature, is an introduction to that part of dimension theory which makes no use of homology theory. Only compact metric spaces are considered. Based on the covering definition of dimension, the chapter begins with the theorem on ϵ-mappings, Menger-Nöbeling's imbedding theorem, the theorem on essential mappings, and the sum theorem. Then it proceeds to the decomposition theorem, from which the equality of covering dimension with inductive dimension, and the characterization of dimension in terms of mappings from the Cantor discontinuum are quickly derived. For a reader interested in the subject, the author's masterly exposition *The present status of the theory of dimension* (Amer. Math. Soc. Translations, Ser. 2, vol. 1, 1955, pp. 1–26) should be a good supplement to this chapter.

The book concludes with an appendix on n-dimensional analytic geometry, which is substantially the same as Anhang II in Alexandroff-Hopf's *Topologie* I (Berlin, Springer, 1935).

In agreement with the common use, the term "bicompact" adopted in Alexandroff-Hopf's *Topologie* I is replaced by "compact" in the present book. But a new name "bicompactum" is proposed for a compact Hausdorff space. As in the older book, quotient space appears under the name of "decomposition space."

Although the material in this book is quite standard, it is well organized and carefully expounded. The book proceeds at a leisurely pace, especially so in the 76 pages of Chapters II and III. Except in Chapter I, the proofs are given in great detail. There is a large number of illuminating remarks, examples and figures; but unfortunately only a few exercises. The translation reads smoothly. As a whole, this book provides uniformly interesting reading, and can be highly recommended to every beginning graduate student of mathematics.

KY FAN

ERRATA, VOLUME 61

E. J. Akutowicz, *A qualitative characterization of Blaschke products in a half-plane.* Abstract 61-6-685.

Line 4 of the abstract: For "Consider . . . and prove" read "The author considers . . . and proves."

Line 6 of the abstract: For "*bounded*" read "*bounded in modulus by unit.*"

Line 7 of the abstract: Delete "$M\cdot$."

ERRATA, VOLUME 62

W. S. Loud, *Nonlinear systems with limiting and large-amplitude forcing.* Abstract 62-1-102.

Line 2 of the abstract: For "$\phi(u)=1$ if $u\leq -1$, u if $|u|\leq 1$, and -1 if $u\geq 1$," read "$\phi(u)=u$ if $|u|\leq 1$ and $\phi(u)=\operatorname{sgn} u$ if $|u|\geq 1$."

Line 8 of the abstract. For "$2r\dot{y}$" read "$2b\dot{y}$." Delete the minus sign before the k at the end of the line.

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Committee on Society Visiting Lectureship

Samuel Eilenberg, P. R. Halmos (Chairman), J. M. Thomas.

Visiting Lecturers:

1927–1928—Constantin Carathéodory, University of Munich.
1928–1929—Hermann Weyl, Zurich Technical School.
1929–1930—Enrico Bompiani, University of Rome.
1930–1931—Wilhelm Blaschke, University of Hamburg.
1931–1932—R. L. Moore, University of Texas.
1936–1937—Thirukkannapuram Vijayaraghavan, University of Dacca.
1948–1949—Casimir Kuratowski, University of Warsaw.
1951–1952—Marshall H. Stone, University of Chicago.

Committees to Select Hour Speakers

For Annual and Summer Meetings: E. G. Begle (Chairman), G. P. Hochschild, D. H. Lehmer.

For Eastern Sectional Meetings: R. D. Schafer (Chairman), Lipman Bers, E. R. Kolchin.

For Western Sectional Meetings: J. W. T. Youngs (Chairman), P. V. Reichelderfer, E. H. Spanier.

For Far Western Sectional Meetings: V. L. Klee (Chairman), Arthur Erdélyi, Ivan Niven.

For Southeastern Sectional Meetings: J. H. Roberts (Chairman), E. E. Floyd, G. B. Huff.

Committee on Places of Meetings

W. M. Whyburn (Chairman), C. B. Morrey, R. M. Thrall.

Policy Committee for Mathematics

Representatives of American Mathematical Society: A. A. Albert, Richard Brauer, R. L. Wilder, Leo Zippin.

Representative of Association for Symbolic Logic: H. B. Curry.

Representative of Institute of Mathematical Statistics: J. F. Daly.

Representatives of Mathematical Association of America: W. L. Duren, H. M. Gehman, D. E. Richmond (Chairman).

Representative of National Council of Teachers of Mathematics: J. R. Mayor.

Secretary: E. G. Begle.

Committee on Applied Mathematics

R. E. Bellman, Garrett Birkhoff, R. S. Burington, A. E. Heins, A. S. Householder, Shizuo Kakutani, F. J. Murray (Chairman).

Committee on Translations from Russian and Other Foreign Languages

R. E. Bellman, R. P. Boas, Jr., J. L. Doob (Chairman), Irving Kaplansky, Hans Samelson.

FORMER PRESIDENTS

J. H. Van Amringe, 1889–1890
Emory McClintock, 1891–1894
G. W. Hill, 1895–1896
Simon Newcomb, 1897–1898
R. S. Woodward, 1899–1900
E. H. Moore, 1901–1902
T. S. Fiske, 1903–1904
W. F. Osgood, 1905–1906
H. S. White, 1907–1908
Maxime Bôcher, 1909–1910
H. B. Fine, 1911–1912
E. B. Van Vleck, 1913–1914
E. W. Brown, 1915–1916
L. E. Dickson, 1917–1918
Frank Morley, 1919–1920
G. A. Bliss, 1921–1922
Oswald Veblen, 1923–1924
G. D. Birkhoff, 1925–1926
Virgil Snyder, 1927–1928
E. R. Hedrick, 1929–1930
L. P. Eisenhart, 1931–1932
A. B. Coble, 1933–1934
Solomon Lefschetz, 1935–1936
R. L. Moore, 1937–1938
G. C. Evans, 1939–1940
Marston Morse, 1941–1942
M. H. Stone, 1943–1944
T. H. Hildebrandt, 1945–1946
Einar Hille, 1947–1948
J. L. Walsh, 1949–1950
John von Neumann, 1951–1952
G. T. Whyburn, 1953–1954
R. L. Wilder, 1955–1956

ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $94,000, a considerable proportion was contributed by members of the Society. Under the terms of the will of the late Robert Henderson, for many years a Trustee of the Society, the Society receives approximately $4,000 yearly. Upon the death of the other legatees, the Society will receive the entire principal of the estate for its Endowment Fund.

SPECIAL FUNDS

($500 or more)

The Bôcher Memorial Prize

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases*, and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space*, and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

Sixth Award, 1943: To Jesse Douglas, for his memoirs *Green's functions and the problem of Plateau, The most general form of the problem of Plateau, and Solution of the inverse problem of the calculus of variations.*

Seventh Award, 1948: To A. C. Schaeffer and D. C. Spencer for their memoir *Coefficients of schlicht functions.*

Eighth Award, 1953: To Norman Levinson for his contributions to the theory of linear, nonlinear, ordinary, and partial differential equations contained in his papers of recent years.

The Frank Nelson Cole Prize in Algebra
The Frank Nelson Cole Prize in the Theory of Numbers

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and Editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebra und ihre Zahlentheorie*, Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem*, which appeared in volume 31 of the Transactions.

Third Award, 1939: To A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes*, published in volume 41 of the Annals of Mathematics.

Fifth Award, 1944: To Oscar Zariski, for four papers on algebraic varieties, published in volumes 61 and 62 of the American Journal of Mathematics and volumes 40 and 41 of the Annals of Mathematics.

Sixth Award, 1946: to H. B. Mann for his paper entitled *A proof of the fundamental theorem on the density of sums of sets of positive integers*, published in volume 43 of the Annals of Mathematics.

Seventh Award, 1949: To Richard Brauer, for his memoir *On Artin's L-series with* *general group characters*, published in volume 48 of the Annals of Mathematics.

Eighth Award, 1951: To Paul Erdős, for his many papers in the Theory of Numbers, and in particular for his paper, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, published in the Proceedings of the National Academy, volume 35, pp. 374–385, July, 1949.

Ninth Award, 1954: To Harish-Chandra, for his papers on representations of semisimple Lie algebras and groups, and particularly for his paper *On some applications of the universal enveloping algebra of a semisimple Lie algebra*, published in volume 70 of the Transactions of the American Mathematical Society.

The Eliakim Hastings Moore Fund

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meetings of the Chicago Section of the American Mathematical Society. The income from the fund of $2,575 is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. The principal of this fund is $23,650.

The Ernest William Brown Fund

From the estate of Professor Ernest William Brown, a fund of $1,000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund

Upon his retirement from the Board of Trustees on December 31, 1940, Dr. Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

The James K. Whittemore Fund

From the estate of Professor James K. Whittemore, a fund of $1,000 is available for use in the interest of mathematical research.

The Genevra B. Hutchinson Fund

A fund of $1,000 from the estate of the late Genevra B. Hutchinson, in memory of her husband Professor J. I. Hutchinson, for use, at the discretion of the Council and Trustees, in the advancement of mathematical research.

The Helen A. Merrill Fund

From the estate of the late Professor Helen A. Merrill, a fund of $650 is available for use in the interest of mathematical research.

Library Proceeds Fund

A fund of $66,000 established by the Trustees in 1951 from the proceeds of the sale of the Society's Library.

BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

Officers

Section 1. There shall be a President, a President Elect (during the even-numbered years only), an Ex-president (during the odd-numbered years only), three Vice Presidents, a Secretary, four Associate Secretaries, and a Treasurer.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

ARTICLE II

Board of Trustees

Section 1. There shall be a Board of Trustees consisting of seven trustees, five trustees elected by the Society in accordance with Article VII, together with the President and Treasurer of the Society ex officio. The Board of Trustees shall designate its own presiding officer and secretary.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the financial business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

Publications Committees

Section 1. There shall be eight Publications Committees, which shall be the seven Editorial Committees specified in Section 2 of this Article and the Committee on Printing and Publishing specified in Section 3 of this Article.

Section 2. There shall be seven Editorial Committees as follows: committees for the *Bulletin*, for the *Proceedings*, for the *Colloquium Publications*, for *Mathematical Surveys*, for *Mathematical Reviews*, a joint committee for the *Transactions* and the *Memoirs* and a committee consisting of the representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

Section 3. There shall be a Committee on Printing and Publishing.

Section 4. The size of each Publication Committee shall be determined by the Council.

ARTICLE IV

Council

Section 1. The Council shall consist of fifteen members at large and the following ex officio members: the officers of the Society specified in Article 1, the members of the Publications Committees specified in Article III, any former Secretary for a period of two years following his term of office, and members of the Executive Committee who remain on the Council by the operation of Article VII, Section 4.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 4. All members of the Council shall be voting members. The method for settling matters before the Council at any meeting shall be by majority vote of the members present. If the result of a vote is challenged, it shall be the duty of the presiding officer to determine the true vote by a roll call. In a roll call vote, each Council member shall vote only once (although he may be a member of the Council in several capacities), and he shall state before the vote in which capacity he votes. The group consisting of the four Associate Secretaries shall have one vote, and it shall be divided equally among those who vote as Associate Secretaries. Each of the eight Publications Committees shall have one vote, and it shall be divided equally among those who vote as members of the respective Publications Committees. All other members of the Council shall have one vote each. Fractional votes shall be counted.

Section 5. Any group of members of the Council who have a total of five votes as defined in Section 4 of this Article shall constitute a quorum for the transaction of business at any meeting of the Council.

Section 6. Between meetings of the Council, business may be transacted by a mail vote. Votes shall be counted as in the case of a roll call vote as specified in Section 4 of this Article, "members present" being replaced by "members voting." An affirmative vote by mail on any proposal shall be declared if and only if (a) more than half of the total number of possible votes is received by the time announced for the closing of the polls, and (b) at least three-quarters of the votes received by then are affirmative. If members who have a total of five or more votes request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless either (1) at the discretion of the Secretary, the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated or (2) the Council places the matter at issue before the Executive Committee for action.

Section 7. The Council may delegate to the Executive Committee (Article V) certain of its duties and powers. Between meetings of the Council the Executive Committee shall act for the Council on such matters and in such ways as the Council may specify. Nothing herein contained shall be construed as empowering the Council to divest itself of responsibility for formulating and administering the scientific policies of the Society.

Section 8. The Council shall also have power to speak in the name of the Society with respect to matters affecting the status of mathematics or mathematicians, such as proposed or enacted Federal or State Legislation, conditions of employment in universities, colleges, or business, research or industrial organizations, regulations, policies or acts of governmental agencies or instrumentalities, and other items which tend to affect the dignity and effective position of mathematics.

With the exception noted in the next paragraph, a favorable vote of two-thirds of the entire membership of the Council, taken in accordance with the provisions of Section 4 of this Article, shall be necessary to authorize any statement in the name of the Society with respect to such matters. With the exception noted in the next paragraph, such a vote may be taken only if written notice shall have been given to the Secretary by the proposer of any such resolution not later than one month prior to the Council meeting at which the matter is to be presented, and the vote shall be taken not earlier than one month after the resolution has been discussed by the Council.

If, at a meeting of the Council, there are present members having a total of twelve votes, as defined in Section 4 of this Article, then the prior notification to the Secretary may be waived by unanimous consent. In such a case, a unanimous favorable vote by those present shall empower the Council to speak in the name of the Society.

The Council may also refer the matter to a referendum mail vote of the entire membership of the Society, and shall make such reference if a referendum is requested, prior to final action by the Council, by two hundred or more members. The taking of a referendum shall act as a stay upon Council action until the votes have been canvassed, and thereafter no action may be taken by the Council except in accordance with a plurality of the votes cast in the referendum.

ARTICLE V

Executive Committee

Section 1. There shall be an Executive Committee of the Council, consisting of four elected members and the following ex officio members: the President, the Secretary, the President Elect (during even-numbered years), and the Ex-president (during odd-numbered years).

Section 2. The Executive Committee of the Council shall be empowered to act for the Council on matters which have been delegated to the Executive Committee by the Council. If three members of the Executive Committee request that any matter be referred to the Council, the matter shall be so referred. The Executive Committee shall be responsible to the Council and shall report its actions to the Council. It may consider the agenda for meetings of the Council and may make recommendations to the Council.

Section 3. Each member of the Executive Committee shall have one vote. An affirmative vote on any proposal before the Executive Committee shall be declared if and only if at least four affirmative votes are cast for the proposal. A vote on any proposal may be determined at a meeting of the Executive Committee, but it shall not be necessary to hold a meeting to determine a vote.

ARTICLE VI

Executive Director

Section 1. There shall be an Executive Director who shall be a paid employee of the Society. He shall have charge of the central office of the Society, and he shall be responsible for the general administration of the affairs of the Society in accordance with the policies that are set by the Board of Trustees and by the Council.

Section 2. The Executive Director shall be elected by the Council for a specified term and approved by the Board of Trustees. The terms and conditions of his employment shall be fixed by the Board of Trustees.

Section 3. The Executive Director shall work under the immediate direction of the Board of Trustees and of the Council, and in cooperation with the Secretary. He shall attend meetings of the Board of Trustees, of the Council, and of the Executive Committee, but he shall not be a member of any of these bodies.

ARTICLE VII

Election and Terms of Officers

Section 1. The term of office shall be one year in the case of the President Elect and the Ex-president, five years in the case of the Trustees, two years in the case of the President, Vice Presidents, the Secretary, the Associate Secretaries and the Treasurer. The term of office in case of members of the Publications Committees shall be three years, except that when the size of a Publication Committee is changed the Council may authorize the election of a member for a term of less than three years. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. The term of office for elected members of the Executive Committee shall be two years, two of the elected members retiring annually. In every case, however, the officials specified in Articles I, II, III, IV, and V, with the exception of the President Elect and the Ex-president, shall continue to serve until their successors have been duly elected and qualified.

Section 2. Election of officers specified in Article I (with the exception of the President and the Ex-president), Trustees, members of the Publications Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member of the Society by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot among the members having the highest number of votes.

Section 3. At the end of his term of office, the President Elect shall become the President. At the end of his term of office, the President shall become the Ex-president.

Section 4. Within fourteen days after the election of officers at the Annual Meeting the Secretary shall send to all members of the Council for a mail vote a ballot containing two names for each place to be filled on the Executive Committee. The nominees shall be chosen by a committee appointed by the President. Members of the Council may vote for persons not nominated. Any member of the Council who is not an ex officio member of the Executive Committee (see Article V, Section 1) shall be eligible for election to the Executive Committee. In case a member is elected to the Executive Committee for a term extending beyond his regular term on the Council, he shall automatically continue as a member of the Council during his term on the Executive Committee.

Section 5. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an ex officio member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 6. If the President of the Society die or resign while a President Elect is in office, the President Elect shall serve as President for the remainder of the year and thereafter shall serve his regular two-year term. If the President of the Society die or resign when no President Elect is in office, the Council, with the approval of the Board of Trustees, shall designate one of the Vice Presidents to serve as President for the balance of the regular presidential term. If the President Elect of the Society die or resign before becoming President, his office shall remain vacant until the next regular election of a President Elect, and the Society shall, at the next Annual Meeting, elect a President for a two-year term. If the Ex-president die or resign before the expiration of his term of office, the Council, with the approval of the Board of Trustees, shall designate a former President of the Society to serve as Ex-president during the remainder of the regular term of the Ex-presidency. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, and the members of the Publications Committees shall be filled by the Council with the approval of the Board of Trustees. If a member of a Publications Committee takes temporary leave from his duties, that Committee shall nominate a substitute for consideration by the Council. The Council shall then elect a substitute. The Council shall fill from its own membership any vacancy in the elected membership of the Executive Committee.

Section 7. If any Trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 8. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

Section 9. The following provisions shall govern the election of Trustees in 1955 and 1956. The Trustees in office on January 1, 1955, shall serve until their respective terms expire. At the annual meeting in 1955, one Trustee shall be elected for a term of five years and one for a term of two years. At the annual meeting of 1956, one Trustee shall be elected for a term of five years, one for a term of three years, and one for a term of two years. This section shall be repealed without further action on January 1, 1957.

ARTICLE VIII

Members and Their Election

Section 1. Election of members shall be by vote of the Council or of its Executive Committee.

Section 2. There shall be four classes of members, namely ordinary, contributing, corporate, and institutional.

Section 3. Application for admission to ordinary membership shall be made by the applicant on a blank provided by the Secretary, and shall be approved either by two ordinary members of the Society or by the institutional member whose nominee he is (see Article IX, Section 6). Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council approved by the Board of Trustees.

Section 4. An ordinary member may become a contributing member by paying the dues for such membership. (See Article IX, Section 3.)

Section 5. A university or college, or a firm, corporation, or association interested in the support of mathematics may be elected a corporate or an institutional member.

ARTICLE IX

Dues and Privileges of Members

Section 1. Any applicant shall be admitted to ordinary membership immediately upon election by the Council (Article VIII) and the discharge within sixty days of election of his first annual dues. Dues may be discharged by payment or by remission when the provision of Section 7 of his Article is applicable. The first annual dues shall apply to the year of election, except that if any applicant is elected after August 15 of any year he may elect to have his first annual dues apply to the following year.

Section 2. The annual dues of an ordinary member of the Society shall be fourteen dollars, except that the Council, with the approval of the Board of Trustees, may establish special rates in exceptional cases and for members of an organization with which the Society has a reciprocity agreement.

Section 3. The minimum dues for a contributing member shall be twenty dollars per year. Members may, upon their own initiative, pay larger dues.

Section 4. The minimum dues of an institutional member shall depend on the amount of published material credited to that member in certain journals during a specific period. The formula for computing these dues shall be established from time to time by the Council subject to approval by the Board of Trustees. Institutions may pay larger dues than the computed minimum.

Section 5. The minimum dues of a corporate member shall be one hundred dollars.

Section 6. The privileges of a corporate or an institutional member shall depend on its dues in a manner to be determined by the Council, subject to approval by the Board of Trustees. These privileges shall be in terms of Society publications to be received by the institution and of the number of persons it may nominate for ordinary membership in the Society.

Section 7. The dues of an ordinary member of the Society shall be remitted for any years during which he is the nominee of an institutional member.

Section 8. After retirement from active service on account of age, any ordinary or contributing member who is not in arrears of dues and with membership extending over at least twenty years may, by giving proper notification to the Secretary, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings but not the *Bulletin* and *Proceedings*.

Section 9. An ordinary or contributing member shall receive the *Bulletin* and *Proceedings* as privileges of membership during each year for which his dues have been discharged, except that a member may substitute for the *Proceedings*, the *Transactions* or *Mathematical Reviews* by paying a premium established by the Trustees for this privilege.

Section 10. Ten dollars of the annual dues of those who receive the *Bulletin* and the *Proceedings*, or the *Transactions* or *Mathematical Reviews* in substitution for the *Proceedings*, under the provisions of this Article shall be allocated in payment therefor. Additionally, in the case of substitution of the *Transactions* or *Mathematical Reviews* for the *Proceedings*, the entire amount of the premium referred to in Section 9 shall also be so allocated.

Section 11. The annual dues of ordinary, contributing, and corporate members shall be payable after October first of the preceding year and due by January first of the year to which they apply. The Society shall submit bills for dues. If the annual dues of any member remain undischarged beyond what the Board of Trustees deem to be a reasonable time, his name shall, after due notice, be removed from the list of members. If a member wishes to discontinue his membership at any time, he shall submit his resignation in writing to the Society.

Section 12. Any member who became a life member before October 25, 1941, by the payment of a sum determined in accordance with actuarial principles shall have for life the status and privileges of an ordinary member without further payment of dues. No additional applications for life memberships will be accepted.

ARTICLE X

Meetings

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. There shall be a business meeting of the Society at the Annual Meeting and at the Summer Meeting. A business meeting of the Society shall take final action only on business accepted by unanimous consent, or business notified to the full membership of the Society in the call for the meeting. Such notification shall be made only when so directed by a previous business meeting of the Society or by the Council.

Section 2. Meetings of the Executive Committee may be called by the President; he shall call a meeting at any time upon the written request of two of its members.

Section 3. The Council shall meet at the Annual Meeting of the Society. Special meetings of the Council may be called by the President; he shall call a special meeting at any time upon the written request of five of its members. No special meeting of the Council shall be held unless written notice of it shall have been sent to all members of the Council at least ten days before the day set for the meeting.

Section 4. The Board of Trustees shall hold its regular Annual Meeting each year between December 15 and January 20, but not before the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days'notice of such meeting mailed to the last known post office address of each Trustee. He should call a meeting upon the receipt of a written request of two of the Trustees. Meetings for the transaction of business may also be held by common consent of all the Trustees.

Section 5. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE XI

Publications

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish two journals, known as the *Transactions of the American Mathematical Society* and the *Proceedings of the American Mathematical Society*. It shall publish a series of mathematical papers known as the *Memoirs of the American Mathematical Society*. The object of the *Transactions*, *Proceedings*, and *Memoirs* is to make known important mathematical researches. It shall publish a periodical called *Mathematical Reviews*, containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall publish a series of monographs called *Mathematical Surveys* which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the publications of the Society listed in Section 1 of this article, and the participation of the Society in the editorial management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 2.

Section 3. The Committee on Printing and Publishing shall advise the Council and Board of Trustees on business and non-editorial matters concerning the publications of the Society, and shall perform such other functions and shall exercise such powers as are properly assigned or delegated to it by the Council or the Board of Trustees.

ARTICLE XII

Amendments

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

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[1] Where the title of the address is not given, the reference is to the Report of the Meeting at which the address was given.